On complexity in holography

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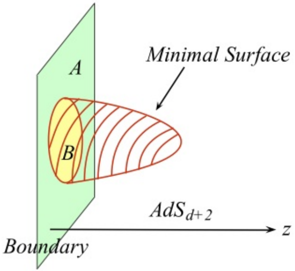
The AdS/CFT correspondence provides a nice theoretical lab to explore relations between quantum information and gravity

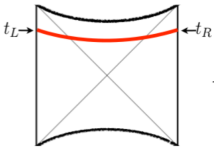


The entanglement entropy of a subregion A is proportional to the area of the minimal surface anchored to the subregion A (Ryu-Takayanagi, 2006)

Thermofield double state in AdS

AdS eternal Black Holes are dual to Thermofield double state

$$|\Psi_{TFD}\rangle \propto \sum_n e^{-E_n\beta/2 - iE_n(t_L+t_R)}|E_n\rangle_R |E_n\rangle_L \,.$$



The Einstein-Rosen bridge grows linearly with time

It grows with time far after the black hole reaches thermal equilibrium!
Entanglement is not enough, because is saturates at the thermalization time [Susskind 2014]

Classical computational complexity

Example: a system composed of K classical bits

- Simple state: $(00000000\ldots)$
- Generic state: $(0010111001\ldots)$
- Simple operation: flip a single bit $(0 \leftrightarrow 1)$

Computational complexity is the minimal number of operations needed to obtain a generic final state from a reference state.

Classical quantities:

- Maximum entropy $S = K \log 2$
- Thermalization time $t_{\mathrm{therm}} \sim K^p$
- Maximum complexity $C = K/2$
- Time to get maximally complex $t_{\mathrm{compl}} \sim K^p$

Quantum complexity

Example: a system of K qubits

- Simple state $|0\rangle = |00000\dots\rangle$
- Generic state $|\psi\rangle = \sum_{i=1}^{2^K} \alpha_i |i\rangle$
- Simple operation: act on 2 qubits

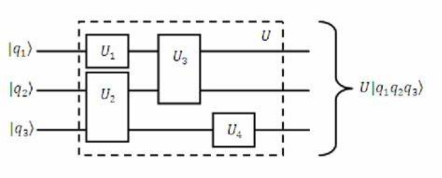
Complexity is the minimum number of unitary operarations required to transform a generic state in the reference state.

Quantum physical quantities:

- Maximum entropy $S = K \log 2$
- Thermalization time $t_{\mathrm{therm}} \sim K^p$
- Maximum complexity $C = e^K$
- Time to get maximally complex $t_{\mathrm{compl}} \sim e^K$

Complexity in QFT

In QM (finite numbers of freedom):
Approach by Nielsen, Geodesics on the space of unitary evolutions



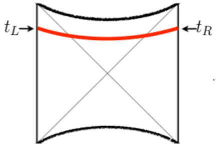
In QFT:

- it has been recently studied for Gaussian states in free field theories by several authors, e.g. Myers and collaborators
- attempts in $d = 2$ CFTs using Liouville action, Caputa, Kundu, Miyaji, Takayanagi and Watanabe, [arXiv:1706.07056 [hep-th]].

Complexity=Volume conjecture

Complexity is proportional to the spatial volume V of the Einstein-Rosen bridge anchored at the boundary:

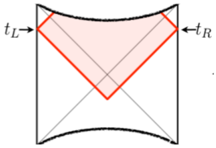
$$C_V \sim \frac{\mathrm{Max}(V)}{Gl}$$



- Extensive and proportional to the number of degrees of freedom of the system $\frac{dC}{dt} \sim TS$
- Extremal black holes are ground states and therefore static: they have vanishing complexity rate

The complexity of the boundary state is proportional to the classical action I computed in the Wheeler-de Witt patch associated to a boundary section:

$$C_A = \frac{I}{\pi \hbar}$$



- Similar behaviour to CV proposal for late times
- For intermediate times, Volume and Action conjectures give different results

$$I = I_{\mathcal{V}} + I_{\rm GHY} + I_{NB} + I_{\mathcal{J}} + I_{\rm ct} \, ,$$

[Lehner, Myers, Poisson and Sorkin, arXiv:1609.00207]

- $I_{\mathcal{V}}$, bulk, Einstein-Hilbert
- Gibbons-Hawking-York term, $I_{\rm GHY} = \frac{\varepsilon}{8\pi G} \int_{\mathcal{B}} d^2x \, \sqrt{|h|} \, K$
- null boundaries, $I_{NB} = \frac{1}{8\pi G} \int_{\mathcal{B}} d\lambda dS \, \tilde{\kappa}$
- joints $I_{\mathcal{J}} = \frac{1}{8\pi G} \int_{\Sigma} d\theta \sqrt{\sigma} \, \mathfrak{a} \, ,$
- Counterterm $I_{\rm ct} = \frac{1}{8\pi G} \int d\theta \, d\lambda \, \sqrt{\sigma} \, \Theta \log |\tilde{L} \Theta| \, ,$

An ambiguous scale in the action

In order to make the action reparameterization-independent:

$$I_{\rm ct} = \frac{1}{8\pi G} \int d\theta \, d\lambda \, \sqrt{\sigma} \, \Theta \log |\tilde{L}\Theta| \,,$$

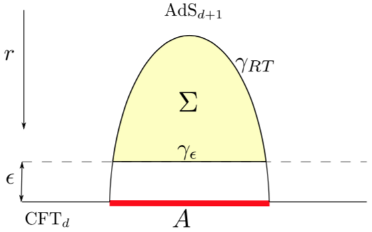
Θ: expansion of null geodesic at null boundary

- This term does not affect late time complexity rates, and has a small impact for the finite part of complexity at finite time
- It affects drastically the divergent part of the complexity, in particular subregion

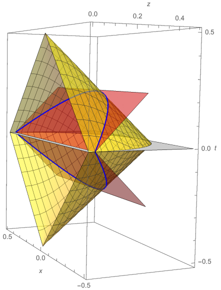
Question :
what's the physical meaning of the counterterm scale $\tilde{L}$?

Subregion complexity: volume

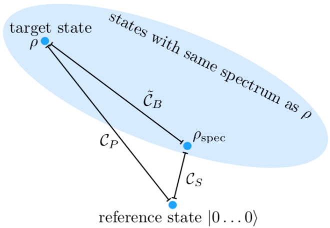
For CV, it was conjectured by Alishahiha that mixed state complexity is dual to the volume of the codimension-1 extremal slice in the bulk attached to the boundary subregion and its RT surface.



In CA, mixed state complexity should be dual to the action of the intersection of the WDW patch and the entanglement wedge associated to the given spatial subregion. Carmi, Myers and Rath, [arXiv:1612.00433 [hep-th]].



What is the appropriate def of Subregion complexity?



- Purification complexity: the minimal number of gates to transform the initial pure state into a purification of the mixed state ρ;
- spectrum complexity $\mathcal{C}_S$, w the minimal number of operations needed to prepare a mixed state ρ_{spec} with the same spectrum as ρ;
- basis complexity $\mathcal{C}_B$, the minimum number of gates needed to prepare ρ from ρ_{spec}.

Agon, Headrick and Swingle, [arXiv:1804.01561 [hep-th]].

Roadmap

- How holographic complexity conjecture behaves in situations without Lorentz Invariance?
  The case of **Warped AdS Black holes**:
  - Time dependence of CV and CA are similar as in AdS
  - Subregion complexity has a different behaviour
- Which, among the several proposal for subregion complexity, better fits the holographic conjectures?
  Role of the scale $\tilde{L}$?
  This motivates a more systematic investigation of holographic subregion complexity in AdS:
  - Subregion CV in AdS_3 in out of equilibrium situation: Vaidya metric
  - Subregion CA at equilibrium in AdS_3

Complexity in WAdS

Black holes in (spacelike) Warped AdS_3

A deformation of AdS_3 with different UV asymptotics

$$ds^2 = L_{AdS}^2 \left(dt^2 + \frac{dr^2}{r^2(\nu^2+3)} + 2\nu r \, dt d\theta + r^2 \frac{3(\nu^2-1)}{4} d\theta^2 \right)$$

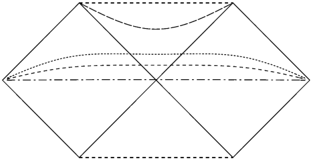
- dual to a class of non Lorentz-invariant CFTs (WCFTs)
- Black hole solutions [Anninos, Padi, Song, Strominger]

Warping parameter ν:

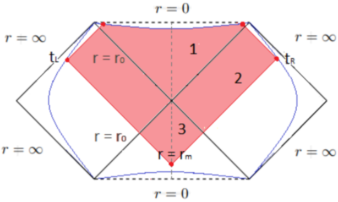
- If $\nu = 1$ we recover the BTZ black hole in AdS spacetime
- If $\nu^2 < 1$, closed timelike curves, so we take $\nu^2 > 1$
- In Einstein gravity ν is related to central charges

$$c_L = c_R = \frac{12 l \nu^2}{G(\nu^2+3)^{3/2}} \, .$$

Complexity growth: CV and CA



$$\dot{V} = 4\pi G\, L_{AdS}\, \frac{2}{\sqrt{3+\nu^2}}\, TS\,.$$



$$\dot{I} = TS\,.$$

Subregion action complexity, AdS

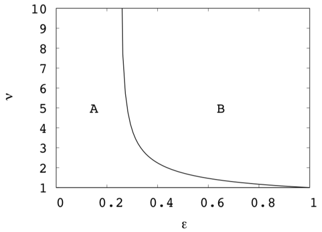
General arguments of Agon, Headrick and Swingle, [arXiv:1804.01561 [hep-th]] :

- Conjecture: C_P should be subadditive for the left L and right R factors of the thermofield double state TD. An analog guess was made about superadditivity of C_B.
- C_B decreases with temperature T and approaches zero for large T, while C_P should not have strong dependence on T

- Depending on the choice of $\tilde{L}$ for the AdS neutral black hole, one can get either that subregion CA is superadditive or subadditive for the L,R sides. For $\tilde{L} > L_{AdS}$, superadditive.
- for the AdS neutral black hole the behaviour of subsystem CA as a function of temperature also depends on $\tilde{L}$. It decreases with T for $\tilde{L} > L_{AdS}$

- new divergences: besides the linear term in the cutoff δ, an additional $\log \delta$ divergence arises.
- subregion C_A is always superadditive for the L, R sides
- subregion C_A has a temperature dependence which is also $\tilde{L}$-independent and that is correlated with specific heat (decreases with temperature for positive specific heat C_J)

Similar properties as for AdS for $\tilde{L} > L_{AdS}$!



$$\epsilon = r_- / r_+$$

Subregion Complexity in AdS_3 at equilibrium

Subregion CV in BTZ black hole

For AdS_3, subregion CV for a segment is independent of temperature:

$$\mathcal{C}_V^{\mathrm{AdS}} = \mathcal{C}_V^{\mathrm{BTZ}} = \frac{2\,c}{3}\left(\frac{l}{\varepsilon} - \pi\right),$$

Protection by Gauss-Bonnet theorem, see Abt et al. [1710.01327]

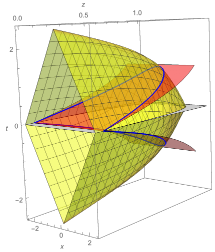
For generic number of segments:

$$\mathcal{C}_V^{\mathrm{AdS}} = \mathcal{C}_V^{\mathrm{BTZ}} = \frac{2\,c}{3}\left(\frac{l_{tot}}{\varepsilon} + \kappa\right),$$

$$\kappa = -2\pi\chi + \frac{\pi}{2}\,m\,,$$

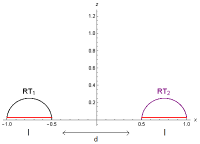
χ is the Euler characteristic ($\chi = 1$ for a disk) and m is the number of ninety degrees junctions

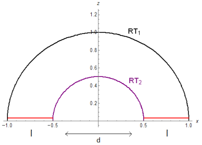
Subregion CA in BTZ black hole

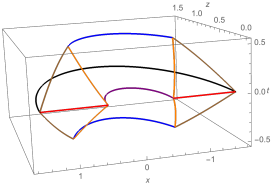


$$\mathcal{C}_A^{\mathrm{BTZ}} = \frac{l}{\varepsilon}\frac{c}{6\pi^2}\log\left(\frac{\tilde{L}}{L_{AdS}}\right) - \log\left(\frac{2\tilde{L}}{L_{AdS}}\right)\frac{S^{\mathrm{BTZ}}}{\pi^2} + \frac{1}{24}c\,.$$

Subregion CA in BTZ black hole: 2 segments







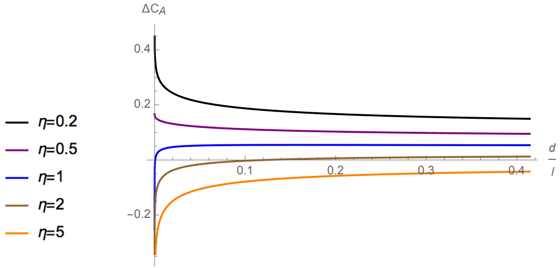
$$\mathcal{C}_A = \frac{c}{3\pi^2}\left\{ \log\left(\frac{\tilde{L}}{L}\right)\frac{l}{\varepsilon} - \log\left(\frac{2\tilde{L}}{L}\right)\log\left(\frac{d(d+2l)}{\varepsilon^2}\right) - \frac{\pi^2}{4} \right.$$

$$+\left[\log\left(\frac{\tilde{L}}{L}\right) + \log\left(\frac{2(d+l)}{\sqrt{d(d+2l)}}\right)\right]\log\left(\frac{(d+l+\sqrt{d(d+2l)})^2}{l^2}\right)$$

$$\left. +\mathrm{Li}_2\left(\frac{\sqrt{d(d+2l)}}{d+l}\right) - \mathrm{Li}_2\left(-\frac{\sqrt{d(d+2l)}}{d+l}\right)\right\} .$$

Mutual information: $I(A|B) = S(A) + S(B) - S(A \cup B)$ is positive, because entanglement entropy is subadditive
By analogy, Mutual Complexity:

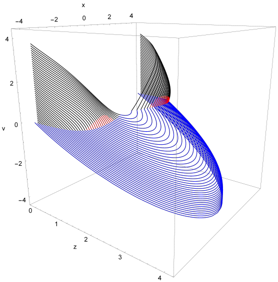
$$\Delta\mathcal{C} = \mathcal{C}(\hat{\rho}_A) + \mathcal{C}(\hat{\rho}_B) - \mathcal{C}(\hat{\rho}_{A\cup B})\,.$$

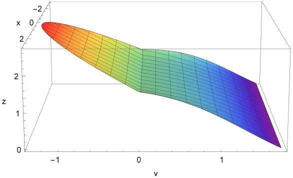


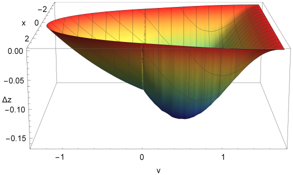
Mutual complexity $\Delta\mathcal{C}_A$ for several values of $\eta = \tilde{L}/L_{AdS}$ as a function of $\frac{d}{l} \in [0, \frac{d_0}{l} = \sqrt{2} - 1]$. $c = 1$.

Subregion Complexity out of equilibrium

Complexity in AdS_3 Vaidya

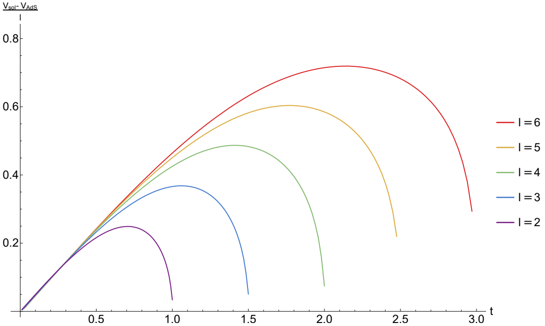






$l = 6,\ r_h = 1$ and $t = 1.75.$

Time dependence of complexity



The initial slope reproduces the result for complexity in Vaidya

Conclusions

- Time dependence of complexity in WAdS is very similar as in AdS, $\dot{C} \propto TS$. Differently from AdS, properties of subregion complexity do not depend on $\tilde{L}$
- At equilibrium (in the BTZ) there is an elegant relation between subregion action complexity of a segment and its entanglement entropy. The expression for two segments is more complicated. The sign of mutual complexity depends on geometry and $\tilde{L}$
- We studied CV out of equilibrium. A similar study for CA is desirable.

Questions:

- A more precise definition of complexity in field theory side
- Physical meaning of the parameter $\tilde{L}$ in the Action conjecture
- Field theory dual of holographic subregion complexity?

Thank you!