Supersymmetric localization as a proving ground

TFI 2019: Theories of Fundamental Interactions 2019 - Torino, Italy

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October 22, 2019

Introduction

Localization in the past few years (a very very partial list)

- 4d Coulomb branch operators [Baggio, Niarchos, and Papadodimas(2014), Gerchkovitz, Gomis, Ishtiaque,Karasik, Komargodski, Pufu (2016)]
- Chiral algebras [Peelaers and Pan (2017)], topological algebras [Dedushenko, Pufu, Yacoby (2016)]
- 3d Coulomb branches [Bullimore, Dimofte, and Gaiotto (2015), Okuda and Yoshida (2019), Assel, Cremonesi, and Renwick (2019)]
- Black hole entropy [Benini, Hristov, Zaffaroni (2015), Cabo-Bizet, Cassani, Martelli, and Murthy (2018), ...]
- Higher dimensional A-twists [Closset, Kim, and Willett (2017)] and "Pestunization" [Festuccia, Qiu, Winding, and Zabzine (2018)]

The following rule of thumb seems to hold: if an effect is a direct consequence of supersymmetry, it should admit an explanation in terms of localization. However...

The Witten index

A complete derivation using localization of the Witten index (the $\mathbb{T}^d$ partition function) for theories without a continuous R-symmetry, e.g.

- 3d $\mathcal{N} = 1$ SYM with matter and/or Chern-Simons terms,

$$\mathcal{I}_{\text{CSYM}}(k,n) = \frac{1}{(n-1)!} \prod_{j=-n/2+1}^{n/2-1} (k-j)$$

- 4d $\mathcal{N} = 1$ SYM

$$\mathcal{I}_{SU(n)} = n,$$

and some theories with matter,

is still missing.

What about the gaugino condensate in SYM? Can we compute using localization

$$\text{tr}\lambda\lambda \propto \Lambda^3?$$

The Affleck-Dine-Seiberg superpotential

I am not aware of any 4d $\mathcal{N}=1$ localization calculation that involves instantons.

Can we derive the ADS superpotential

$$W_{\text{quantum}} = C_{N,F}\left(\frac{\Lambda^{3N-F}}{\text{det}M}\right)^{\frac{1}{N-F}}$$

from a localization calculation? If not, why not?

How about supersymmetry breaking in chiral gauge theories? There has been progress, but not using localization.

What's next?

At this point, I think it is safe to say that people know what they are doing when performing localization

- very few assumptions,
- derivation of results simultaneously for large classes of theories,
- detailed checks (of dualities etc)
- when surprises occur, they should be taken seriously.

It is worthwhile asking whether we can get more mileage out of localization. I will argue for a broader viewpoint: if you want to motivate a new construction in quantum field theory, try to embed it in a localization calculation.

An example: local defect operators in 4d gauge theory.

Defect operators

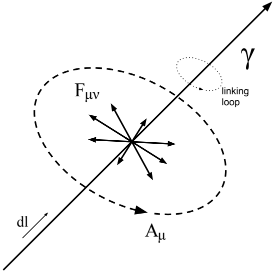
A **defect operator** is an operator whose insertion alters the space of fundamental fields in the path integral so as to have a prescribed singularity [I rely on Kapustin (2005), but many previous examples]

- can create "topological disorder" if the space of fields in the punctured neighborhood of the singularity is disconnected.
- can carry quantum numbers not realized by ordinary operators.
- expectations values can serve as order parameters.
- interesting connection with entanglement entropy.
- play a crucial role in duality operator maps.
- can be very difficult to study perturbatively, but supersymmetric versions exist and can be incorporated in localization calculations.

Vortex line operators

A co-dimension 2 defect operator

- Specify [Witten (1988), Seiberg Moore (1989)]
  - the worldline of the operator: a line (or loop) γ
  - an element $\beta \in \mathfrak{g}$ modulo large gauge transformations
- Define a singular connection A
  - A has holonomy β around a small linking loop.
  - equivalently $F_A = \beta \star [\gamma]$ (a delta function on the loop)
- If $\pi_0 (G)$ vanishes, the configuration for A is continuously connected to the zero connection, i.e. not topologically protected.



Local vortex operators

In 2d, the co-dimension 2 vortex operator is a local defect operator

- Sometimes called "Wilson point operators". [Blau and Thompson (1993), Nekrasov (2017)]
- Supersymmetric version studied using localization. [Hosomichi, Lee, and Okuda (2017)]
- The lower dimensional version of the surface operators studied by Gukov and Witten. [Gukov and Witten (2006,2008)]
- The vortex line operator also has a supersymmetric version. [Drukker, Okuda, and Passerini (2012), Kapustin, Willett, and IY (2012)]

There is surprisingly little work on the significance of these operators for 2d gauge theories, apart from some discussions in the context of mirror symmetry.

Monopole operators I

A co-dimension 3 "local 't Hooft operator"

- Related to the "dual photon" of $U(1)$ gauge theories, and hence to the Polyakov confinement mechanism.
- Vanishing of the expectation value can be used as a criterion for screening of magnetic flux. [Witten (1999)]
- For a gauge group G, $\pi_1(G)$ defines an 't Hooft charge
  - if the charge vanishes, the configuration is continuously connected to the zero connection.
  - if there is a $U(1)$ factor, there is a topological current

$$J = \star F.$$

It is possible to study the quantum numbers at e.g. large N_f [Borokhov, Kapustin, and Wu (2002), Pufu (2013), Dyer, Mezei, and Pufu (2013)], using the ϵ expansion [Chester, Mezei, Pufu, and IY (2015)] ...

Monopole operators II

The general definition for a **bare monopole operator** is

- Specify [Borokhov, Kapustin, and Wu (2002), Borokhov (2003)]
  - a position for the operator $\vec{x} \in \mathbb{R}^3$.
  - a GNO charge or an element $H \in \Gamma_G$ of the magnetic weight lattice of G.
- Define a singular connection A such that on a small sphere around x

$$A^N \approx H\left(1 - \cos\theta\right) d\varphi, \quad A^S \approx H\left(-1 - \cos\theta\right) d\varphi$$

  - The A's are generically unstable solutions of the Yang-Mills equations on $\mathbb{S}^2$.

In the presence of a Chern-Simons term, the monopole must be "dressed" to be gauge invariant.

Supersymmetric monopole operators

Add a profile for a scalar in the vector multiplet [Borokhov, Kapustin, and Wu (2002), Borokhov (2003)]

$$\Phi \approx \frac{H}{|\vec{x}|},$$

and the monopole operator becomes BPS, a singular solution to the Bogomolnyi equation

$$\star F = D\Phi.$$

The existence of the monopole operator clarifies a number of aspects of supersymmetric theories in 3d

- Acts as a creation operator for the BPS vortex soliton.
- Elucidates the operator map in 3d mirror symmetry.
- Realizes currents for (super)symmetry enhancement in many models.

Instanton operators

A co-dimension 5 local operator [Lambert, Papageorgakis, and Schmidt-Sommerfeld (2014)]

- based on $\mathbb{S}^4$ configurations with non-vanishing c_2.
- supersymmetric versions exist. [Lambert, Papageorgakis, and Schmidt-Sommerfeld (2014), Rodriguez-Gomez and Schmude (2015)]
- responsible for (super)symmetry enhancement in certain models. [Tachikawa (2015)]
- Topologically protected due to the existence of a topological current

$$J = \star \mathrm{Tr}\left(F \wedge F\right).$$

3d vs 4d

Comparison of some relevant aspects of 3d and 4d gauge theories

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- confinement and global symmetry breaking ✓
- global symmetry and supersymmetry enhancement ✓
- Mirror symmetry $\leftrightarrow$ S duality ✓
- Wilson and vortex lines $\leftrightarrow$ Wilson and 't Hooft lines (and mixed operators) ✓
- BPS vortex solitons $\leftrightarrow$ BPS 't Hooft-Polyakov monopoles ✓
- monopole operators $\leftrightarrow$?

Should we expect local defects in 4d?

The space of gauge fields on $\mathbb{S}^3$ is always connected

$$\pi_2(G) = \emptyset$$

- no interesting characteristic classes,
- no topologically protected states,
- hidden (discrete) global symmetries?

Under some additional assumptions, there are no local defect operators in 4d gauge theories, [Kapustin (2005)]

- scale invariant singularity.
- compatibility with the equations of motion.

Supersymmetric local defects in 4d?

Supersymmetric monopole and instanton operators are captured by the superconformal index

- in 3d, as moduli solving an equation $\star F = \sigma \wedge dt$. [Kim (2009)]
- in 5d, contributions from the 5d Nekrasov partition function. [Kim, Kim, and Lee (2012)]

No such contributions to the 4d superconformal index. A number of explanations are possible

- the operators do not exist.
- they exist but are not BPS.
- they are BPS but do not preserve enough (or the right type of) supersymmetry.
- the superconformal index has additional contributions not taken into account in previous calculations.
- a different index is needed in order to capture the effect of the BPS defect operators.

Pure gauge theory

At minimum, a configuration representing a local defect should be gauge invariant (solve the equation of motion for A on $\mathbb{S}^3$). Are there any candidate configurations at all? Yes! There are solutions (merons) of the equations of motion for a pure gauge theory on $\mathbb{S}^3$. They

- exist only for non-abelian groups.
- do not require introducing a dimensionful parameter.
- are scalars - a rotation of $\mathbb{S}^3$ can be compensated by a gauge transformation

$$\mathcal{L}_v A = D\Phi\left(v\right).$$

Unfortunately, all of the solutions I know of are unstable - however, the minimal solution has only a single unstable mode!

Instability

The instability is a big problem for defining an operator. There are some options for trying to fix this

- turn on additional fields (probably scalars), without ruining gauge invariance.
- consider a different action for the gauge fields - this can result from integrating out other fields, or modes of the gauge field itself.
- consider a different action for the gauge fields due to turning on background fields.

The situation in the pure gauge theory is similar to the one for GNO monopoles. Unfortunately, there is no 4d version of the 't Hooft charge.

Why these solutions and not others?

Consider how the defects fit together: suppose the Lie algebra element defining a monopole operator is improperly quantized, $H \notin \Gamma_G$

- the configuration on the boundary of the northern hemisphere defines a (non-trivial) vortex,
- the same cutting relationship holds true for the (rotationally symmetric) instanton operator and the $\mathbb{S}^3$ configurations on the previous slides,
- and again for the $\mathbb{S}^3$ configurations and the GNO monopole.

In fact, the relationship seems to continue past 5d, with a proper ansatz for the gauge field. This seems like a good indication that these configurations are special. Specifically, they have the right properties to terminate an 't Hooft line (with vanishing 't Hooft charge).

What about supersymmetry?

BPS configurations solve first order equations, but the simplest first order equation on $\mathbb{S}^3$, the Bogomolnyi equation

$$\star F = D\Phi$$

can only be solved for $A = 0$ (pure gauge). We can try a few things

- consider a different equation: the 3d versions of the non-abelian Seiberg-Witten monopole equations or the Kapustin-Witten equations?
- rotational invariance $\mathcal{L}_v A = D\Phi\,(v)$ is itself a BPS equation in the Ω background.
- consider a trivial equation like the 2d equation $\star F = D$.

The last option may seem silly, but solutions to these equations show up in supersymmetric partition functions. [Benini and Zaffaroni (2015), Bershtein, Bonelli, Ronzani, and Tanzini (2015)].

Supersymmetry II

BPS equations usually imply stability. Can unstable configurations contribute to localization calculations?

Yes, GNO monopoles in 2d Yang-Mills on a compact Riemann surface contribute [Witten (1992)]

$$Z_g^{SU(2)}\left(g_{\mathsf{YM}}\right) = \ldots + f_g\left(g_{\mathsf{YM}}\right) \sum_{m\in\mathbb{Z}} \exp\left(-\frac{(2\pi m)^2}{g_{\mathsf{YM}}^2}\right),$$

- the result of **non-abelian localization**.
- vortex operators appear naturally.
- attempts have been made to transfer this calculation to four dimensions, but not very successfully.

Perhaps the lesson here is just to try and find the right TQFT?

Analogy with 2d

$\mathbb{S}^3$ solutions have interesting properties, seem very similar to 2d vortex operators

- the "BPS equation" $\star F = D$ is the one relevant for vortex operators,

$$\delta\lambda = (\star F - D)\,\epsilon + \ldots$$

- the 4d singularities associated with $\mathbb{S}^3$ configurations are a local non-integer source for the topological density (hence "merons")

$$\frac{1}{2\pi}F = \beta\,\delta^{(2)}\left(\vec{x}\right) \quad \rightarrow \quad \frac{1}{8\pi^2}\mathrm{tr}F \wedge F = \frac{1}{2}\delta^{(4)}\left(\vec{x}\right)$$

- the result of a "singular gauge transformation"

$$A = e^{-\beta\phi}de^{\beta\phi} \quad \rightarrow \quad A = \frac{1}{2}g^{-1}dg,$$

for a singular g with winding number 1 on $\mathbb{S}^3$.

Where to next?

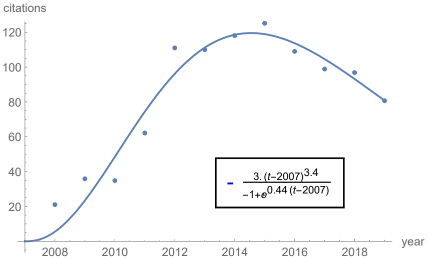
Lots of conjectures, but what could produce a smoking gun for an **interesting** defect? Localization of course! (in progress)

- Find the right amount of supersymmetry.
- Define the correct index.
- Figure out the quantum numbers.
- Any connection with duality?
- ...

If a configuration contributes to a localization calculation in a supersymmetric gauge theory, it is interesting for gauge theory in general.

Thank you

Citations of Pestun's *"Localization of gauge theory on a four-sphere and supersymmetric Wilson loops"*... not quite Planck's law.



- citations in 2020: 68
- cumulative citations at last proton decay: 1324

Thank you for listening