The fate of the circular Wilson Loop in $\mathcal{N} = 4$ defect theory

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based on "*The fate of circular Wilson loops in N=4 SYM with defect: phase transitions, double scaling limit and OPE expansion*"

S.B, Silvia Davoli, Luca Griguolo, Domenico Seminara arXiv[1910.xxxx]

Outline

- Motivations
- Description of the set-up
- Results on the gravity side (strong coupling)
  1. Boundary conditions
  2. Parameters space
  3. Structure of the solutions
- Conclusions

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Motivations for Defect Conformal Field Theories

- Introducing a defect reduces the amount of symmetry in QFT
- dCFTs with holographic duals constitute an interesting new arena for precision tests of the AdS/CFT correspondence
- Non-vanishing one-point functions already at tree level
- Interesting applications to integrability

dCFT: Field theory picture

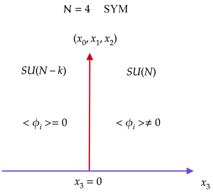
Defect version of $\mathcal{N} = 4$ SYM theory

[DeWolfe,Freedman,Ooguri, 2003; Gaiotto,Witten, 2008; Buhl-Mortensen et al. 2017]

- A **codimension one defect** is inserted at $x_3 = 0$, separating two vacua of $\mathcal{N} = 4$ SYM:
- **Higgsing**: 3 scalars acquire an x_3-dependent VEV: $(i = 1, 2, 3)$

$$\langle \phi_i(x) \rangle_{cl} = -\frac{1}{x_3} t_i \oplus 0_{(N-k)\times(N-k)} \quad x_3 > 0$$

t_i : k-dimensional irr. repr. of the $SU(2)$ algebra



- The VEV originates from the b.c. on the defect preserving 1/2 of the original supersymmetry
- The superconformal symmetry $PSU(2,2|4)$ of $\mathcal{N} = 4$ SYM is broken down to its subgroup $OSp(4|4)$. In particular the original bosonic sector $SO(4,2) \times SO(6)$ resuces to

$$\underbrace{SO(3,2)}_{\text{Res. Conf. symm.}} \times \underbrace{SO(3) \times SO(3)}_{\text{R-symmetry}}$$

dCFT Holographic dual: String theory picture

D3-D5 brane configuration

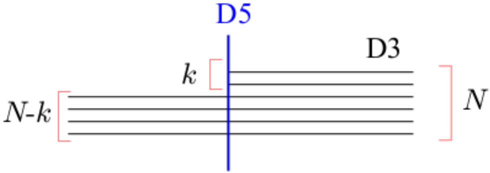
[Karch, Randall, 2001; Gaiotto,Witten, 2008; Nagasaki, Tanida, Yamaguchi, 2012]

- D5 $\rightarrow$ probe brane in $AdS_5 \times S^5$

	0	1	2	3	4	5	6	7	8	9
D3	○	○	○	○	×	×	×	×	×	×
D5	○	○	○	×	○	○	○	×	×	×

- The **D5** has a profile that spans $AdS_4 \times S^2$ in the presence of a background flux of k units through the S^2

  $\Rightarrow$ k out of the N D3 branes get dissolved in the D5 brane:



Double-scaling limit:

An (unexpected) window on the weak coupling regime

[Nagasaki, Tanida, Yamaguchi, 2012]

- Compared to the usual AdS/CFT scenario, in this theory we have an extra parameter k that controls the VEV of the scalar fields
- one can consider the **double scaling limit**: $\frac{\lambda}{k^2}$
  - sugra computations (valid for large λ) $\rightarrow$ considered for **large k** in such a way that λ/k^2 is kept small
  - the results **on both side** of the correspondence are found to be expressible in powers of λ/k^2

    $\Rightarrow$ weak/strong computations are comparable

Circular Wilson loop in $\mathcal{N} = 4$ defect theory

- We consider a **circular Wilson Loop** of **radius R** placed on a plane parallel to the defect at a **distance L** from it: [Aguilera-Damia, Correa, Giraldo-Rivera, 2017]

$$W(C) = \mathrm{Tr}\, P \exp \left\{ \oint_C d\tau \, \left(i A_\mu \dot{x}^\mu - |\dot{x}| \left(\phi_3 \sin\chi + \phi_6 \cos\chi \right) \right) \right\}$$

$$x^\mu = (0, R\cos\tau, R\sin\tau, L) \qquad \chi \in [0, \frac{\pi}{2}]$$

- $\chi = 0$ **BPS point**, the operator + the defect preserve **1/4** of the supercharges

- conformal invariance $\rightarrow$ $\langle W \rangle$ depends on R and L only through the ratio R/L

- In this talk we will explore the interaction of the WL with the defect in the **strong coupling limit** $\rightarrow$ non-perturbative computations in the string theory side

String Theory setting:

- **$\mathbf{AdS_5 \times S^5}$ metric** (Poincaré patch):

$$ds^2 = \underbrace{\frac{1}{y^2}\left(-dt^2 + dr^2 + r^2 d\phi^2 + dx_3^2 + dy^2\right)}_{\mathbf{AdS_5}} + \underbrace{\left(d\theta^2 + \sin^2\theta d\Omega_{(1)}^2 + \cos^2\theta\, d\tilde{\Omega}_{(2)}^2\right)}_{\mathbf{S^5}}$$

$d\Omega_{(i)}^2$ $(i = 1, 2)$ represents two spheres inside S^5.

- The $D5-$brane wraps the first of the two S^2 and has the form:

$$y = \frac{1}{\kappa} x_3 \qquad \theta = \frac{\pi}{2} \qquad \tilde{\theta} = \tilde{\theta}_0 \qquad \tilde{\phi} = \tilde{\phi}_0$$

$(\tilde{\phi}_0, \tilde{\theta}_0)$ fixed point in the second S^2; $\theta = \frac{\pi}{2}$ is the equator of the S^5.

- **Two competing classical string solutions** for the circular WL parallel to the defect:
  - *spherical dome*: dominant for $\frac{L}{R} >> 1$, it does not move on the S^5

$$y(\sigma)^2 + r(\sigma)^2 = R^2 \qquad \phi = \tau$$

  - minimal surface describing a *fundamental string stretching from the boundary to the $D5-$brane* $\rightarrow$ dominant for $\frac{L}{R} << 1$

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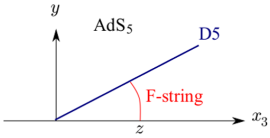
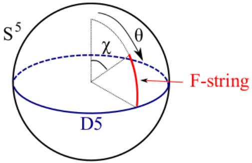
Connected surface

- **Ansatz:** The solution moves both in AdS_5 and S^5 $[0 \le \sigma \le \tilde{\sigma} \quad 0 \le \tau < 2\pi]$:

$$y = y(\sigma) \qquad r = r(\sigma) \qquad x_3 = x_3(\sigma) \qquad \phi = \tau \qquad \theta = \theta(\sigma)$$

- **Boundary Conditions:** Fundamental string $\rightarrow$ stretched from the boundary $(\sigma = 0)$ to the D5 $(\sigma = \tilde{\sigma})$





- **boundary conditions** in $\sigma = 0$:

$$r(0) = R \qquad y(0) = 0$$

$$x_3(0) = L \qquad \theta(0) = \chi$$

- **boundary conditions** in $\tilde{\sigma}$:

$$C_1 \equiv y(\tilde{\sigma}) - \frac{1}{\kappa} x_3(\tilde{\sigma}) = 0 \qquad \theta(\tilde{\sigma}) = \frac{\pi}{2}$$

$$C_2 \equiv y'(\tilde{\sigma}) + \kappa\, x_3'(\tilde{\sigma}) = 0 \qquad C_3 \equiv r'(\tilde{\sigma}) = 0$$

Equations of motion

Minimizing the Polyakov action we get the following equations:

- Equations of motion for $\theta(\sigma)$ and $x_3(\sigma)$:

$$x_3'(\sigma) = -cy^2(\sigma) \qquad \theta'(\sigma) = j$$

- Equations of motion for $r(\sigma)$ and $y(\sigma)$:

$$yy'' + r'^2 + r^2 - y'^2 + c^2y^4 = 0 \qquad yr'' - 2r'y' - yr = 0.$$

- and in addition the VC constraint

$$\mathcal{V}(\sigma) \equiv \frac{r^2 - y'^2 - r'^2}{y^2} - c^2y^2 = j^2$$

Here j and c are integration constants.

General solution of the e.o.m.

The general solution of the e.o.m. can be given in terms of only one unknown function $g(\sigma) \equiv \frac{r(\sigma)}{y(\sigma)}$

$$y(\sigma) = \frac{\sqrt{\epsilon_0}}{c} \frac{1}{\sqrt{1+g^2(\sigma)}} \text{sech}[v(\sigma) - \eta] \qquad r(\sigma) = \frac{\sqrt{\epsilon_0}}{c} \frac{g(\sigma)}{\sqrt{1+g^2(\sigma)}} \text{sech}[v(\sigma) - \eta]$$

$$x_3(\sigma) = x_0 - \frac{\sqrt{\epsilon_0}}{c} \tanh[v(\sigma) - \eta] \qquad \theta(\sigma) = j\sigma + \theta_0$$

Here $v(\sigma)$ is defined by $v'(\sigma) = \frac{\sqrt{\varepsilon_0}}{1+g^2(\sigma)}$ with the b.c. $v(\sigma) = 0$, while $g(\sigma)$ obeys

$$g'(\sigma)^2 + (j^2 - 1)g(\sigma)^2 - g(\sigma)^4 = -\varepsilon_0 - j^2,$$

where $\varepsilon_0 \geq 0, x_0, \theta_0$ and η are new integration constants. In AdS_5 the solution draws a sub-manifold

$$(x_3 - x_0)^2 + y^2 + r^2 = \frac{\varepsilon_0}{c^2}$$

Imposing boundary conditions:

The boundary conditions in $\sigma = 0$ allows us to determine the integration constants c, x_0 and θ_0

$$x_3(0) = L \;\Rightarrow\; x_0 = L - R\sinh\eta$$

$$\theta(0) = \chi \;\Rightarrow\; \theta_0 = \chi$$

$$r(0) = 0 \;\Rightarrow\; c = \frac{\sqrt{\epsilon_0}}{R}\operatorname{sech}\eta$$

The boundary conditions at $\sigma = \tilde{\sigma}$ allows us to determine the maximal value $\tilde{\sigma}$ of the world-sheet coordinate σ

$$\theta(\tilde{\sigma}) = \frac{\pi}{2} \;\Rightarrow\; \tilde{\sigma} = \frac{1}{j}\left(\frac{\pi}{2} - \chi\right)$$

A suitable combination of the remaining three b.cs. fixes η in terms of L/R

$$\eta = \operatorname{arcsinh}\frac{L}{R}$$

Remaining boundary conditions at $\tilde{\sigma}$

- We are left with two independent boundary conditions to impose

$$C_1 : \operatorname{arcsinh}\frac{L}{R} = v(\tilde{\sigma}) + \operatorname{arctanh}\left(-\frac{1}{\sqrt{\epsilon_0}}\frac{g'(\tilde{\sigma})}{g(\tilde{\sigma})}\right)$$

$$C_2 : \qquad \kappa = -\frac{g'(\tilde{\sigma})}{\sqrt{j^2+\epsilon_0-g^2(\tilde{\sigma})}} \qquad\qquad \kappa \equiv \frac{\pi k}{\sqrt{\lambda}}$$

- Explicit form for $g(\sigma)$

$$g(\sigma) = \sqrt{\frac{j^2-1}{m+1}}\,\mathrm{ns}\left(\sqrt{\frac{j^2-1}{m+1}}\sigma, m\right) \qquad m \equiv \frac{j^2-1-\sqrt{(j^2+1)^2+4\varepsilon_0}}{j^2-1+\sqrt{(j^2+1)^2+4\varepsilon_0}}$$

The range for the modulus m is either $-1 \leq m \leq 0$ if $j^2 \geq 1$ or $m \leq -1$ if $0 \leq j^2 \leq 1$.

Allowed regions for the parameters

We find convenient to use m as an independent integration constant instead of ε_0.

- Positivity of ε_0 + allowed ranges for m select two regions in the (j, m) plane:

REGION (A): $-1 \leq m \leq 0$ and $j^2 \geq -\dfrac{1}{m}$ REGION (B): $m \leq -1$ and $j^2 \leq -\dfrac{1}{m}$.

Our goal is now to solve the boundary conditions C_1 for the distance L and C_2 for the flux κ to determine

> the last two integration constants (j, m) as functions of κ, $\frac{L}{R}$ and χ

Instead of j^2 we prefer to use the auxiliary variable $x = \sqrt{\frac{j^2-1}{j^2(m+1)}}$. We shall solve the b.c. for the flux to determine x as function of m, χ, κ

- **Positivity of the flux** $\kappa > 0$ + positivity of $g(\sigma)$ $\Rightarrow$ constraints on the range of x

$$\text{REGION (A): } 1 \leq x \leq \text{Min}\left(\frac{1}{\sqrt{1+m}}, \frac{\mathbb{K}(m)}{\left(\frac{\pi}{2}-\chi\right)}\right)$$

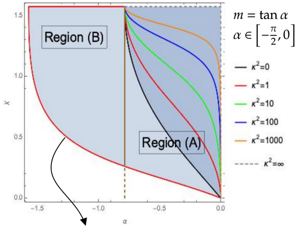
$$\text{REGION (B): } 1 \leq x \leq \frac{\mathbb{K}(m)}{\left(\frac{\pi}{2}-\chi\right)}$$

Allowed regions for the parameters

- The requirement that the intervals for x are not empty $\Rightarrow$ the region bounded by the **red curve**
- Moreover the equation for the flux

$$\kappa = -\frac{g'(\tilde{\sigma})}{\sqrt{j^2 + \epsilon_0 - g^2(\tilde{\sigma})}}$$

cannot be solved for a generic choice of the parameters in the region (A)



$m_0 \equiv m = -1/j^2 \Rightarrow c = 0$

Our family of solutions coincides with the class of exact solutions discussed by Correa et al.

$m_0 \Rightarrow \chi = \frac{\pi}{2} - \mathbb{K}(m_0)$

- Fixed χ and κ, there exists a critical value m_c such that $m \geq m_c \Rightarrow$ **no solution**

> **Geometrical interpretation of m_c:**
> the distance L/R vanishes as $m \to m_c$
> $\Rightarrow$ The **WL touches the defect**

- The set of coloured curves $\Rightarrow$ the value of m_c as function of the angle χ for different values of κ^2
- Allowed region for m for fixed κ^2: on the left of the relevant coloured curve

Behavior of the distance with m

The final step is to determine m as a function of χ, κ and L/R by exploiting the boundary condition for the distance

$$\eta = \text{arcsinh}\frac{L}{R} = v(\tilde{\sigma}) + \text{arctanh}\left(-\frac{1}{\sqrt{\epsilon_0}}\frac{g'(\tilde{\sigma})}{g(\tilde{\sigma})}\right)$$

- For fixed χ and κ, m can span the interval $[m_0, m_c]$: in this interval we can uniquely solve m in terms of L/R only if the r.h.s. is a monotonic function of m.

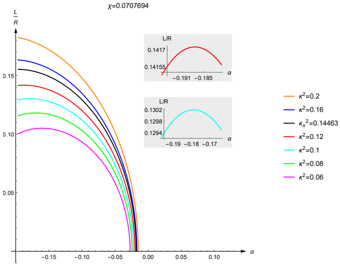
  We study the behavior of $\left.\frac{\partial\eta}{\partial m}\right|_{\chi,\kappa}$ for $m \to m_c$ and $m \to m_0$

  - $\left.\frac{\partial\eta}{\partial m}\right|_{m=m_c} = -\frac{c_0}{\sqrt{m_c - m}} + O(m - m_c)$
  - $\left.\frac{\partial\eta}{\partial m}\right|_{m=m_0} \Rightarrow$ finite term function of κ^2 and m_0

Behavior of the distance with m

- Exist a **critical angle** $\chi_s \simeq 0.331147$ that separates two dinstinct phases:

$$\mathbf{0 < \chi < \chi_s}$$



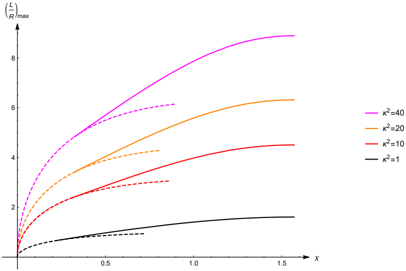
- $\left.\frac{\partial \eta}{\partial m}\right|_{m=m_0}$: always negative unless $\Rightarrow \quad \kappa^2 < \kappa_s^2$
- $\boldsymbol{\kappa^2 \geq \kappa_s^2} \Rightarrow$ the distance is a **monotonic function** of m
- $\boldsymbol{\kappa^2 < \kappa_s^2} \Rightarrow$ the distance is **not monotonic** in $m \Rightarrow$ the same behavior holds for $\boldsymbol{\chi_s \leq \chi \leq \frac{\pi}{2}}$
- Presence of a non-monotonic behavior (for a certain range of parameters) $\Rightarrow$ **existence of different branches of solutions**

Maximal distance

- In both regions determined by χ_s there is a **maximal distance** after which the connected solution does not exist
- $L_{\max}$ $\Rightarrow$ determined **analitically** when $\mathbf{0 < \chi < \chi_s}$ and $\boldsymbol{\kappa^2 \geq \kappa_s^2}$

$$L_{\max} = R\sqrt{\frac{\kappa^2 m_0}{m_0 - 1}}$$

- For the other values of χ and κ we determined $L_{\max}$ **numerically**



- **Dashed curves** $\Rightarrow$ maximal distance determined analitically
- **Continuous curves** $\Rightarrow$ maximal distance determined numerically
- The maximal distance grows both with χ and κ^2

Regularized area of the connected extremal surfaces

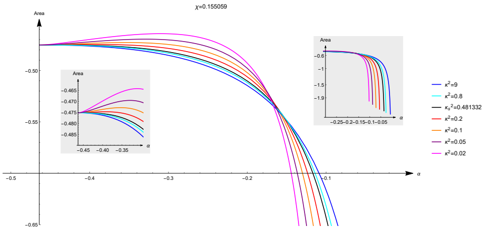
The **renormalized area** is given in terms of incomplete elliptic integral of the second kind $[E(\Phi,\tilde{m})]$

$$S_{\text{ren.}} = \sqrt{\lambda n}\left(\sqrt{n}\tilde{\sigma} - E\left(\text{am}\left(\sqrt{n}\tilde{\sigma}|m\right)|m\right) - \frac{\text{cn}\left(\sqrt{n}\tilde{\sigma}|m\right)\text{dn}\left(\sqrt{n}\tilde{\sigma}|m\right)}{\text{sn}\left(\sqrt{n}\tilde{\sigma}|m\right)}\right) \equiv \sqrt{\lambda}\hat{S}_{\text{ren}}$$

Since $\left.\frac{\partial\hat{S}_{\text{ren.}}}{\partial m}\right|_{\kappa,\chi} = \sqrt{-(n+1)(mn+1)}\left.\frac{\partial\eta}{\partial m}\right|_{\kappa,\chi}$, the area and the distance possess a similar behavior as functions of m for fixed κ and χ.

Behavior of the area with m

$$\mathbf{0 < \chi < \chi_s}$$

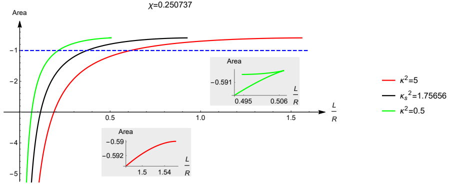


- $\boldsymbol{\kappa^2 \geq \kappa_s^2}$ The area, as the distance, monotonically increases when m is lowered from m_c to m_0
- $\boldsymbol{\kappa^2 < \kappa_s^2}$ the curve displays a maximum for the same value of m of the distance $\Rightarrow$ the same behavior holds for $\boldsymbol{\chi_s \leq \chi \leq \frac{\pi}{2}}$
- Close to m_c the area diverges for all values of κ^2
- Independently of κ^2 all the curves terminate on the same point

Transition: connected solution vs dome

- To understand when the **connected solution** becomes dominant with respect to the **dome** $\Rightarrow$ plot the area as a function of the distance from the brane

$$0 < \chi < \chi_s$$



- $\kappa^2 \geq \kappa_s^2$: the area is a monotonic function of the distance
- $\kappa^2 < \kappa_s^2$: **two families of extremal surfaces** when the distance is decreased from its maximal value, **the upper branch is always subdominant** $\Rightarrow$ the same behavior holds for $\chi_s \leq \chi \leq \pi/2$
- There is a **critical distance** for which the area of the dome is equal to the area of the connected solution
- The connected solution becomes dominant below the critical distance $\Rightarrow$

phase transition of Gross-Ooguri type

- The transition is of the **first order** since the area is continuous but not its first derivative

Double-scaling limit

- We want to **match the string computation with the field theory result:** $\rightarrow$ possibile because of $k \Rightarrow$ we can organize the expression for S_{ren} as a series in $\frac{\lambda}{k^2}$
- **Strong coupling regime**: expand our classical solution in power of $\frac{1}{\kappa^2} \Rightarrow$ large value of the flux
- We require that the **distance** L/R of the Wilson loop from the defect remains **finite**
- First two terms in the expansion

$$S_{\text{ren}} \simeq -\frac{\pi k R}{L}\left[\sin\chi + \frac{\lambda}{4\pi^2 k^2}\,\frac{1}{\cos^3\chi}\left(\frac{\pi}{2} - \chi - \frac{\sin 2\chi}{2}\right)\left(\sin^2\chi + \left(\frac{L}{R}\right)^2\right) + O\left(\frac{\lambda^2}{\pi^4 k^4}\right)\right]$$

- perfect agreement with the perturbative computation

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BPS Configuration

$$\chi = 0$$

- The admissible region for m shrinks to a point $\Rightarrow m = 0$
- The solution collapses to a point and **no regular connected solution exists for the BPS configuration**
- Weak coupling analysis $\Rightarrow$ the first non-trivial BPS perturbative contribution is evaluated in terms of hypergeometric functions
- Its large k expansion does not scale in a way to match the string solution $\Rightarrow$ not possible to recover the large k limit from the equivalent asymptotic expansion of the $\chi \neq 0$ case

Conclusions

- We analyzed the Circular Wilson loop operator in the $\mathcal{N} = 4$ SYM theory with the insertion of a defect
- **String Theory side:**
  - we solved a non-trivial boundary conditions problem
  - we are left with three independent parameters χ, κ, m and we analyzed their allowed region of variation
  - we have studied the possible structure of the connected solution
  - we have shown that taking the $\kappa \to \infty$ limit, we recover the perturbative computation for the expectation value of the Wilson loop for any value of the angle χ and the distance $\frac{L}{R}$

Thank you for the attention!!