T, Q and periods in $SU(3)$ $\mathcal{N} = 2$ SYM

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Main Motivation: Alexei Zamolodchikov's unpublished paper "Generalized Mathieu equation and Liouville TBA"-2000

Our result can be thought of as a natural extension of Alexei Zamolodchikov's conjecture relating Floquet exponent of Mathieu equation to Baxter's T function in $c = 25$ Liouville CFT.

& **Very helpful papers:**

R. Poghossian JHEP 1104:033,2011 generalizing the equation that defines the Seiberg-Witten curve for Ω background in NS limit.

F.Fucito, J. F. Morales, R.Poghossian, D. Ricci Pacifici JHEP 1105:098,2011 From a finite difference equation, a linear differential equation was obtained.

D.Fioravanti & D.Gregory arXiv:1908.08030 where the implication of this conjecture for the period on the A-cycle of (effective) $SU(2)$ gauge theory has been highlighted and used.

Patrick Dorey, Roberto Tateo Nucl.Phys. B571 (2000) 583-606 and **V.Bazhanov, N.Hibberd, M.Khoroshkin** Nucl.Phys.B 622:475-547, 2002 The third order linear differential equation for W_3 'minimal' case was studied and TQ, QQ equations were obtained.

Based on: *D. Fioravanti, H.Poghosyan and R. Poghossian arXiv:1909.11100*

By using the third order differential equation derived from the deformed Seiberg-Witten differential for pure $SU(3)$ $\mathcal{N}=2$ SYM in NS limit of Ω-background we derive the corresponding QQ and related TQ functional relations. We show how numerical integration of the differential equation along imaginary direction with standard boundary conditions allows one to find the monodromy matrix and corresponding Floquet exponents, which in the context of gauge theory, coincide with the A-cycle periods $a_{1,2,3}$. We have convincingly demonstrated the correctness of this identities trough comparison with instanton computation. But the main value of this method is that it makes accessible also the region of large coupling constants, which is beyond the reach of instanton calculus. Eventually, we suggest a simple relation between Baxter's T-function and A-cycle periods $a_{1,2,3}$ of $SU(3)$ theory.

Nekrasov partition function and the VEVs $\langle \mathbf{tr}\, \phi^J \rangle$

Consider pure $SU(N)$ theory without hypers in Ω-background. The instanton part of partition function is given by [Nekrasov: arXiv:hep-th/0206161]

$$Z_{inst}(\mathbf{a}, \epsilon_1, \epsilon_2, q) = \sum_{\vec{Y}} Z_{\vec{Y}} \left((-)^N q\right)^{|\vec{Y}|} , \tag{0.1}$$

where sum runs over all N-tuples of Young diagrams $\vec{Y} = (Y_1, \cdots, Y_N)$, $|\vec{Y}|$ is the total number all boxes, $\mathbf{a} = (a_1, a_2, \cdots, a_N)$ are VEV's of adjoint scalar from $\mathcal{N} = 2$ vector multiplet, ϵ_1, ϵ_2, parametrize the Ω-background and the instanton counting parameter $q = \exp 2\pi i \tau$, with $\tau = \frac{i}{g^2} + \frac{\theta}{2\pi}$ being the (complexified) coupling constant. The coefficients $Z_{\vec{Y}}$ are factorized as [R. Flume, R. Poghossian: hep-th/0208176]

$$Z_{\vec{Y}} = \prod_{u,v=1}^{N} \frac{1}{P(Y_u, a_u | Y_v, a_v)} , \tag{0.2}$$

where the factors $P(\lambda, a|\mu, b)$ for arbitrary pair of Young diagrams λ, μ and associated VEV parameters a, b, are given explicitly by the formula

$$P(\lambda, a|\mu, b) = \tag{0.3}$$
$$\prod_{s\in\lambda}(a - b + \epsilon_1(1 + L_\mu(s)) - \epsilon_2 A_\lambda(s)) \prod_{s\in\mu}(a - b - \epsilon_1 L_\lambda(s) + (1 + \epsilon_2 A_\lambda(s)))$$

The instanton part of (deformed) prepotential is given by

$$F_{inst}(\mathbf{a}, q) = -\epsilon_1\epsilon_2 \log Z_{inst} \,. \tag{0.4}$$

Instanton calculus allows one to obtain also the VEV's $\langle \mathbf{tr}\, \phi^J \rangle$, ϕ being the adjoint scalar of vector multiplet:

$$\langle \mathbf{tr}\, \phi^J \rangle = \sum_{i=1}^{N} a_u^J + Z_{inst}^{-1} \sum_{\vec{Y}} Z_{\vec{Y}} \mathcal{O}_{\vec{Y}}^J q^{|\vec{Y}|} \,, \tag{0.5}$$

where $Z_{\vec{Y}}$ is already defined by (0.2), (0.3), and [A. S. Losev, A. Marshakov, and N. A. Nekrasov], [R. Flume, F. Fucito, J. F. Morales, and R. Poghossian hep-th/0403057]

$$\mathcal{O}_{\vec{Y}}^J = \sum_{u=1}^{N} \sum_{(i,j)\in Y_u} \Big((a_u + \epsilon_1 i + \epsilon_2 (j-1))^J + (a_u + \epsilon_1 (i-1) + \epsilon_2 j)^J$$
$$- (a_u + \epsilon_1 (i-1) + \epsilon_2 (j-1))^J - (a_u + \epsilon_1 i + \epsilon_2 j)^J \Big) \,. \tag{0.6}$$

Baxter's difference equation and deformed Seiberg-Witten 'curve'

In the NS limit the sum [0.1] is dominated by a single term corresponding to a unique array of Young diagrams $\vec{Y}^{(cr)}$. By using this fact one can define a entire function Y(z) the zeros of which are determined by the column length of $\vec{Y}^{(cr)}$ that will satisfy the following difference equation

$$Y(z+\epsilon_1)+\frac{q}{\epsilon_1^{2N}}Y(z-\epsilon_1)=\epsilon_1^{-N}P_N(z+\epsilon_1)\,Y(z)\,, \qquad (0.7)$$

where in particular when $N=3$ we have

$$P_3(z)=z^3-\frac{u_2}{2}\,z-\frac{u_3}{3}\ . \qquad (0.8)$$

Now, let us briefly recall how the difference equation (0.7) is related to the Seiberg-Witten curve. Introducing the function

$$y(z)=\epsilon_1^N\ \frac{Y(z)}{Y(z-\epsilon_1)}$$

one can rewrite (0.7) as

$$y(z) + \frac{q}{y(z-\epsilon_1)} = P_N(z)\,. \tag{0.9}$$

Notice that setting $\epsilon_1 = 0$ in (0.9) one obtains an equation of hyperelliptic curve, which is just the Seiberg-Witten curve. When $\epsilon_1 \neq 0$, everything goes surprisingly similar to the original Seiberg-Witten theory. For example the rôle of Seiberg-Witten differential is played anew by the quantity

$$\lambda_{SW} = z\frac{d}{dz}\log y(z)\,,$$

and, as in the undeformed theory, the expectation values u_J are given by the contour integrals

$$\langle \mathbf{tr}\,\phi^J\rangle = \oint_{\mathcal{C}} \frac{dz}{2\pi i} z^J \partial_z \log y(z)\,, \tag{0.10}$$

where $\mathcal{C}$ is a large contour, enclosing all zeros and poles of $y(z)$.

To keep expressions simple, from now on we will set $\epsilon_1 = 1$. In fact, at any stage the ϵ_1 dependence can be easily restored on dimensional grounds. Taking the results of previous subsection, the difference equation for $N = 3$ case (0.7) can be rewritten as

$$Y(z) - \left(z^3 - \frac{u_2}{2} z - \frac{u_3}{3}\right) Y(z-1) + q\, Y(z-2) = 0\,, \tag{0.11}$$

By means of inverse Fourier transform we can derive a third order linear differential equation for the function

$$f(x) = \sum_{z \in \mathbb{Z}+a} e^{x(z+1)} Y(z)\,. \tag{0.12}$$

At least when $|q|$ is sufficiently small, it is expected that the series is convergent for finite x, provided a takes one of the three possible values a_1, a_2 or a_3. Taking into account the difference relation (0.11), one can easily check that the function (0.12) solves the differential equation

$$-f'''(x) + \frac{u_2}{2} f'(x) + \left(e^{-x} + q\, e^{x} + \frac{u_3}{3}\right) f(x) = 0\,. \tag{0.13}$$

Denoting

$$q = \Lambda^6 \ \text{ and shifting the variable } x \to x - \log \Lambda^3$$

the differential equation (0.13) may be cast into a more symmetric form.

Solutions at $x \to \pm\infty$

It is convenient to introduce parameters p_1, p_2, p_3 satisfying $p_1 + p_2 + p_3 = 0$

$$-f'''(x) + \frac{u_2}{2} f'(x) + \left(\Lambda^3(e^x + e^{-x}) + \frac{u_3}{3}\right) f(x) = 0\,. \tag{0.14}$$

where

$$u_2 = \langle \mathbf{tr}\,\phi^2 \rangle = p_1^2 + p_2^2 + p_3^2 = 2(p_1^2 + p_2^2 + p_1 p_2); \tag{0.15}$$
$$u_3 = \langle \mathbf{tr}\,\phi^3 \rangle = p_1^3 + p_2^3 + p_3^3 = -3p_1 p_2 (p_1 + p_2)\,. \tag{0.16}$$

We define $\Lambda \equiv \exp\theta$. At large positive values $x \gg 3\ln\Lambda$ the term $\Lambda^3 e^{-x}$ in (0.14) can be neglected. In this region the differential equation can be solved in terms of hypergeometric function ${}_0F_2(a, b; z)$ defined by the power series

$${}_0F_2(a, b; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!(a)_k (b)_k}\,, \tag{0.17}$$

where

$$(x)_k = x(x+1)\cdots(x+k-1) \tag{0.18}$$

is the Pochhammer symbol. Three linearly independent solutions can be chosen as

$$U_i(x) \approx e^{(x+3\theta)p_i}\; {}_0F_2(1 + p_i - p_j, 1 + p_i - p_k; e^{x+3\theta})\,, \tag{0.19}$$

and the indices (i, j, k) are cyclic permutations of $(1, 2, 3)$.

In the end, we must verify that the Wronskian of the three solutions (0.19) (below and later on, for brevity, we use the notation $p_{ij} \equiv p_i - p_j$)

$$Wr[U_1(x), U_2(x), U_3(x)] \equiv \det \begin{pmatrix} U_1(x) & U_2(x) & U_3(x) \\ U_1^{'}(x) & U_2^{'}(x) & U_3^{'}(x) \\ U_1^{''}(x) & U_2^{''}(x) & U_3^{''}(x) \end{pmatrix} = p_{12}p_{23}p_{31} \qquad (0.20)$$

is not zero provided the parameters p_i are pairwise different. Thus, we have confirmed that generically the $U_i(x)$ are linearly independent and constitute a basis in the space of all solutions.

Similarly in region $x \ll -3\theta$ the term $\Lambda^3 e^x$ of (0.14) becomes negligible and one can write down the three linear independent solutions

$$V_i(x) \approx e^{(x-3\theta)p_i} \ {}_0F_2(1-p_i+p_j, 1-p_i+p_k; -e^{-x+3\theta}) \ . \qquad (0.21)$$

In fact, we obtain the same result for the Wronskian

$$Wr[V_1(x), V_2(x), V_3(x)] = p_{12}p_{23}p_{31} \, . \qquad (0.22)$$

The QQ and TQ relations

All three solutions $V_i(x)$ grow very fast at $x \to -\infty$, but there is a special linear combination (unique, up to a common constant factor) which vanishes in this limit. If it is the fastest one (as we suspect), this solution is usually referred to as subdominant. Using formula for asymptotics of ${}_0F_2$ we are able to establish that the correct combination is

$$\chi(x) = \tfrac{\Gamma(p_{12})\Gamma(p_{13})}{4\pi^2}\, V_1(x) + \tfrac{\Gamma(p_{23})\Gamma(p_{21})}{4\pi^2}\, V_2(x) + \tfrac{\Gamma(p_{31})\Gamma(p_{32})}{4\pi^2}\, V_3(x)\,. \qquad (0.23)$$

Its asymptotic expansion at $x \to -\infty$ is given by

$$\chi(x) = \frac{v^{-\frac{1}{3}} e^{-3v^{1/3}}}{2\pi\sqrt{3}} \left(1 - \left(\frac{1}{9} - \frac{u_2}{2}\right) v^{-\frac{1}{3}} + \left(\frac{u_2^2}{8} - \frac{5u_2}{36} + \frac{u_3}{6} + \frac{2}{81}\right) v^{-\frac{2}{3}} \right.$$
$$\left. - \left(-\frac{u_2^3}{48} + \frac{u_2^2}{18} - \frac{u_3 u_2}{12} - \frac{13u_2}{324} + \frac{7u_3}{54} + \frac{14}{2187}\right) v^{-1} + O\left(v^{-\frac{4}{3}}\right)\right), \qquad (0.24)$$

where we denoted

$$v = \exp(3\theta - x)$$

and u_2, u_3 are defined in terms of p_i in (0.16).

Since $U_i(x)$ constitute a complete set of solutions one can represent $\chi(x)$ as a linear combination

$$\chi(x,\theta) = \sum_{n=1}^{3} Q_n(\theta)\Gamma(p_{nj})\Gamma(p_{nk})e^{-3p_n\theta}U_n(x,\theta), \qquad (0.25)$$

where the important quantities $Q_n(\theta)$, based on general theory of linear differential equations, are expected to be entire functions of θ (and also of parameters $\mathbf{p}$ dependence on which will be displayed explicitly only if necessary). The following, easy to check property plays an essential role in further discussion. Namely the Wronskian of any two solutions $f(x)$, $g(x)$ of the differential equation (0.14)

$$W[f(x),g(x)] \equiv f(x)g'(x) - g(x)f'(x)$$

satisfies the *adjoint* equation, i.e. the one obtained by reversing the signs $\mathbf{p} \to -\mathbf{p}$ and $\Lambda^3 \to -\Lambda^3$. Taking inspiration from this property, it is then possible to show exactly that

$$Wr\left[\chi(x,\theta+\frac{i\pi}{3}),\chi(x,\theta-\frac{i\pi}{3})\right] = -\frac{i}{2\pi}\bar{\chi}(x,\theta)\,, \qquad (0.26)$$

where $\bar{\chi}(\theta) = \chi(\theta,-\mathbf{p})$.

Let us investigate the $x \to \infty$ limit of (0.26). Taking into account (0.25) we obtain the functional relations

$$\frac{\sin(\pi p_{jk})}{2i\pi^2}\,\bar{Q}_n(\theta) = Q_j\left(\theta + \frac{i\pi}{3}\right)Q_k\left(\theta - \frac{i\pi}{3}\right) - Q_j\left(\theta - \frac{i\pi}{3}\right)Q_k\left(\theta + \frac{i\pi}{3}\right), \quad (0.27)$$

where again, the bar on Q_n indicates the sign change $\mathbf{p} \to -\mathbf{p}$

$$\bar{Q}_n(\theta, \mathbf{p}) \equiv Q_n(\theta, -\mathbf{p})$$

and (n, j, k) is a permutations of $(1, 2, 3)$.

And finally let us establish the $\theta \to -\infty$ asymptotics of $Q_k(\theta)$ and $\bar{Q}_k(\theta)$. Obviously, in this case both (0.19) and (0.21) are approximate solutions of (0.14) at $x \sim 0$. Thus, comparison of (0.23) with (0.25) ensures that for $\theta \ll 0$

$$Q_k(\theta) \sim \frac{\exp(-3\theta p_k)}{4\pi^2}\,; \qquad \bar{Q}_k(\theta) \sim \frac{\exp(3\theta p_k)}{4\pi^2}\,. \qquad (0.28)$$

It is easy to see that above asymptotic behavior is fully consistent with functional relations (0.27).

The functional relations (0.27) suggest the following $SU(3)$ analog of Baxter's TQ equations:

$$T(\theta)Q_j\left(\theta-\frac{\pi i}{6}\right)\bar{Q}_k\left(\theta+\frac{\pi i}{6}\right)=Q_j(\theta-\frac{5\pi i}{6})\bar{Q}_k\left(\theta+\frac{\pi i}{6}\right) \quad (0.29)$$

$$+Q_j\left(\theta+\frac{\pi i}{2}\right)\bar{Q}_k\left(\theta-\frac{\pi i}{2}\right)+Q_j\left(\theta-\frac{\pi i}{6}\right)\bar{Q}_k\left(\theta+\frac{5\pi i}{6}\right)$$

for $j,k\in\{1,2,3\}$ with $j\neq k$. To uncover the essence of this construction, notice that for a fixed pair of indices (i,j) (0.29) can be thought as definition of function $T(\theta)$ in terms of Q's. Then the nontrivial question is "do other choices of (j,k) lead to the same T?" Fortunately, elementary algebraic manipulations with the help of (0.27) ensure that the answer is positive. As mentioned earlier, $Q_i(\theta)$ are entire functions. A thorough analysis shows that due to (0.27) all potential poles of $T(\theta)$ have zero residue. Thus $T(\theta)$ is an entire function too.

The Bethe ansatz equations can be represented as

$$\frac{Q_j(\theta_\ell - \frac{2\pi i}{3})\bar{Q}_k\left(\theta_\ell + \frac{\pi i}{3}\right)}{Q_j\left(\theta_\ell + \frac{2\pi i}{3}\right)\bar{Q}_k\left(\theta_\ell - \frac{\pi i}{3}\right)} = -1\,, \tag{0.30}$$

where θ_ℓ are the zeroes of $Q_j(\theta)$.
Functional relations similar to (0.27) and (0.29) emerge also in the context of ODE/IM for 'minimal' 2d CFT with extra spin 3 current (W_3 symmetry) P. Dorey, R. Tateo Nucl.Phys. B571 (2000) 583-606, V. Bazhanov, A. Hibberd, S. Khoroshkin Nucl.Phys.B622:475-547,2002. From there we can extrapolate that our case might correspond to the special choice of Virasoro central charge $c = 98$ for Toda CFT. In fact, this value of the central charge lies outside the region discussed in above references.

The Floquet-Bloch monodromy matrix

Consider the basis of solutions $f_1(x)$, $f_2(x)$, $f_3(x)$ of

$$-f'''(x) + \frac{u_2}{2} f'(x) + \left(\Lambda^3(e^x + e^{-x}) + \frac{u_3}{3}\right) f(x) = 0\,. \quad (0.31)$$

with standard initial conditions ($n, k \in \{1,2,3\}$)

$$f_n^{(k-1)}(x)\Big|_{x=0} = \delta_{k,n}\,. \quad (0.32)$$

Since the functions $f_n(x + 2\pi i)$ are solutions too, we can define the monodromy matrix $M_{k,n}$ as

$$f_n(x + 2\pi i) = \sum_{k=1}^{3} f_k(x) M_{k,n}\,. \quad (0.33)$$

Clearly

$$M_{k,n} = f_n^{(k-1)}(2\pi i)\,.$$

The solutions

$$f(x) = \sum_{z \in \mathbb{Z}+a} e^{x(z+1)} Y(z)\,. \tag{0.34}$$

with $a \in \{a_1, a_2, a_3\}$ have diagonal monodromies and can be represented as certain linear combinations of $f_n(x)$. In other words the eigenvalues of the monodromy matrix $M_{k,n}$ must be identified with $\exp(2\pi i a_k)$, with $k = 1, 2, 3$:

$$Spec(M_{k,n}) = \{\exp(2\pi i a_1), \exp(2\pi i a_2), \exp(2\pi i a_3)\}\,. \tag{0.35}$$

For any fixed values of parameters Λ, $\mathbf{p}$, it is easy to integrate numerically the differential equation (0.14) with boundary conditions (0.32), find the matrix $M_{k,n}$ and then its eigenvalues $\exp(2\pi i a_n)$. Taking into account Matone relation of Ω-background,

$$u_2 \equiv \langle \mathbf{tr}\, \phi^2 \rangle = \sum_{n=1}^{3} a_n^2 + 2q\partial_q F_{inst}(q, \mathbf{a})\,, \qquad (0.36)$$

we can access the deformed prepotential for any value of the coupling constant.

Comparison of the instanton counting against numerical results

It is straightforward to calculate $\langle \mathbf{tr}\, \phi^2 \rangle$ or $\langle \mathbf{tr}\, \phi^3 \rangle$ as a power series in q. Here are the 3-instanton results (it is assumed that $a_1 + a_2 + a_3 = 0$ and by definition $a_{jk} \equiv a_j - a_k$)

$$\langle \mathbf{tr}\, \phi^2 \rangle = \sum_{k=1}^{3} a_k^2 - \frac{12(1-h_2)q}{\prod_{j<k}(a_{jk}^2 - 1)} + \frac{P_{2,2}q^2}{\prod_{j<k}(a_{jk}^2-1)^3(a_{jk}^2-4)} + O(q)^4 \quad (0.37)$$

$$\begin{aligned}\langle \mathbf{tr}\, \phi^3 \rangle &= \sum_{k=1}^{3} a_k^3 + \frac{54 h_3 q}{\prod_{j<k}(a_{jk}^2 - 1)} + \frac{P_{3,2}q^2}{\prod_{j<k}(a_{jk}^2-1)^3(a_{jk}^2-4)} \\ &- \frac{P_{3,3}q^3}{\prod_{j<k}(a_{jk}^2-1)^5(a_{jk}^2-4)(a_{jk}^2-9)} + O(q)^4 \,, \end{aligned} \quad (0.38)$$

where

$$h_2 = \frac{a_1^2 + a_2^2 + a_3^2}{2}\,; \qquad h_3 = -a_1 a_2 a_3 \,, \quad (0.39)$$

and

$$\begin{aligned} P_{2,2} = 36(&220 - 1027h_2 + 1659h_2^2 - 698h_2^3 - 958h_2^4 + 1257h_2^5 - 521h_2^6 \\ &+68h_2^7 - 13959h_3^2 + 33804h_2h_3^2 - 25434h_2^2h_3^2 + 5292h_2^3h_3^2 \\ &+297h_2^4h_3^2 + 13851h_3^4 - 5103h_2h_3^4) \end{aligned} \quad (0.40)$$

By means of numerical integration of the differential equation we have computed the eigenvalues of monodromy matrix (0.34) for several values of the instanton parameter $q = \Lambda^6$, namely for the values

$$\Lambda = \exp\left(\frac{k-1}{20} - 5\right), \qquad k = 1, 2, \cdots, 120\,, \tag{0.41}$$

and fixed values of parameters

$$p_1 = 0.12\,; \qquad p_2 = 0.17\,; \qquad p_3 = -0.29\,,$$

Due to identification (0.35) this allows to find the corresponding A-cycle periods a_1, a_2, a_3. Inserting the values of a_k, Λ in (0.37), (0.38) we can calculate $\langle \mathbf{tr}\,\phi^2 \rangle$ and $\langle \mathbf{tr}\,\phi^3 \rangle$. The consistency requires that at small values of q one should always obtain $\langle \mathbf{tr}\,\phi^2 \rangle = p_1^2 + p_2^2 + p_3^2 = 0.1274$ and $\langle \mathbf{tr}\,\phi^3 \rangle = p_1^3 + p_2^3 + p_3^3 = -0.017748$.

Λ	a_1	a_2
0.00822974704902	0.1200000000131	0.169999999982
0.0223707718562	0.1200000053049	0.169999992932
0.0608100626252	0.1200021402877	0.169997148430
0.165298888222	0.1208841761521	0.168828966405
0.246596963942	0.1349151981823	0.151933010167
0.272531793034	0.142136769453 - 0.019455438633 i	0.142136769453 + 0.019455438633 i
0.449328964117	0.092117229441 - 0.135924390553 i	0.092117229441 + 0.135924390553 i
0.740818220682	0.003727137475 - 0.568756791077 i	0.003727137475 + 0.568756791077 i
1.22140275816	0.000899023180 - 1.071594057757 i	0.000899023180 + 1.071594057757 i
2.01375270747	0.00036203460 - 1.78605985179 i	0.00036203460 + 1.78605985179 i
3.32011692274	0.00013130957 - 2.96965962318	0.00013132399 + 2.96965962932 i

Table: The values a_1, a_2 obtained through numerical integration of the differential equation (0.14) with initial conditions (0.32) for $p_1 = 0.12$, $p_2 = 0.28$.

Λ	$\langle \mathbf{tr}\, \phi^2 \rangle$	$\langle \mathbf{tr}\, \phi^3 \rangle$
0.00822974704902	0.1274000000000	-0.0177480000000
0.0223707718562	0.1274000000000	-0.0177480000000
0.0608100626252	0.1274000000000	-0.0177480000000
0.165298888222	0.1274000000000	-0.0177480000000
0.246596963942	0.1273999999998	-0.0177480000000
0.272531793034	0.1273999999922	-0.0177479999994
0.449328964117	0.1273774046391	-0.0177462190257
0.740818220682	0.1313057536866	-0.0178774876030

Table: The values $\langle \mathbf{tr}\, \phi^2 \rangle$, $\langle \mathbf{tr}\, \phi^3 \rangle$ obtained by inserting the values of a_1, a_2 from Table 1 into (0.37), (0.38) supplemented by q^4 and q^5 corrections. To be compared with (by definition) $\langle \mathbf{tr}\, \phi^2 \rangle = p_1^2 + p_2^2 + p_3^2 = 0.1274$ and $\langle \mathbf{tr}\, \phi^3 \rangle = p_1^3 + p_2^3 + p_3^3 = -0.017748$.

Extension of Zamolodcikov's conjecture to $SU(3)$

The simpler case of the gauge group $SU(2)$ has been analyzed recently in D. Fioravanti & D. Gregory arXiv:1908.08030. In this case one has to deal with the Mathieu equation. Corresponding TQ relation was investigated in Alexei Zamolodchikov's unpublished paper "Generalized Mathieu equation and Liouville TBA"-2000, where it was conjectured (and demonstrated numerically) an elegant relationship between T-function and Floquet exponent ν of Mathieu equation:

$$T = \cos(2\pi\nu)\,. \tag{0.42}$$

Here we suggest a natural extension of Zamolodchikov's conjecture for $SU(3)$ case:

$$T(\theta) = \sum_{n=1}^{3} e^{2\pi i a_n}\,. \tag{0.43}$$

Notice, that at $\theta \ll 0$ the asymptotic (0.28) leads to $T(\theta) \sim \sum_{n=1}^{3} e^{2\pi i p_n}$, which is consistent with (0.43), since for $\theta \ll 0$ instanton corrections disappear and a_k coincides with p_k.

Conclusions

It would be very interesting to have a TBA for our case and check our conjecture (0.43) as it was done by Al. Zamolodchikov in *unpublished paper "Generalized Mathieu equation and Liouville TBA"-2000*. Actually, even relevant would be a Y-system and a gauge TBA that may shed light on the dual B-cycle periods $\mathbf{a_D}$ along the route presented in D. Fioravanti & D. Gregory arXiv:1908.08030 and A.Grassi, J.Gu, and M.Marino for the $SU(2)$ case.
Of course, it is very plausible that the imaginable generalizations of our results, and in particular of (0.43), might hold for arbitrary $SU(N)$ gauge groups.

THANKS