Membranes and Domain Walls in $\mathcal{N}=1$, D=4 SYM

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Main goal - to reveal how $\mathbf{N=1}$, $\mathbf{D=4}$ SYM domain walls look like

- $\mathbf{N=1}$, $\mathbf{D=4}$ SYM was constructed in 1974
  
  (Wess & Zumino, Ferrara & Zumino, Salam & Strathdee)

- Studied intensively over 45 years

- In particular, since early 80s it is known that pure $SU(\mathbf{N})$ SYM has $\mathbf{N}$ degenerate susy vacua distinguished by different vevs of the gluino condensate (related by $Z_N$ R-symmetry transformations)

  \[
  \langle \text{Tr} \lambda^\alpha \lambda_\alpha \rangle = \Lambda^3 e^{2\pi i \frac{n}{N}}, \quad n = 0, 1, \ldots N - 1
  \]

- There should exist BPS domain walls interpolating between different vacua and having the following tension (Dvali & Shifman ‘97)

  \[
  T_{DW} = \frac{\mathbf{N}}{8\pi^2} | \langle \lambda \lambda \rangle_n - \langle \lambda \lambda \rangle_l |
  \]
Since the 90s, domain walls in $\mathbb{N}=1$, $D=4$ SYM and SQCD have been intensively studied with the use of different approaches.

- For a recent review and latest developments see

Solitonic solutions of low-energy EFT describing the pure SYM domain walls have not been found until recently

- **Reason:** SYM domain walls are not smooth solitonic field configurations. Their existence requires the presence of a source which has been lacking in pure $N=1$ SYM
  – dynamical membranes

- **Aim of this talk** to show
  – how to couple the membrane to $N=1$ SYM and its Veneziano-Yankielowicz effective formulation
  – how the membrane creates BPS domain walls and what is their shape
Review of pure $\mathbb{N}=1$ SU(N) SYM

- Field content - adjoint vector supermultiplet

$$A^I_m, \lambda^I_\alpha, \bar{\lambda}^I_{\dot{\alpha}}, D^I, \quad I \in \text{adj}(SU(N))$$

- Building block of the SYM action – chiral spinor superfield

$$\mathcal{W}_\alpha(x, \theta) = -i\lambda_\alpha + \theta_\alpha D - \frac{i}{2} F_{mn} \sigma^{mn}_{\alpha \beta} \theta_\beta + \theta^2 \sigma^m_{\alpha \dot{\beta}} \nabla_m \bar{\lambda}^\dot{\beta}, \quad \bar{D}_\alpha \mathcal{W}_\beta = 0.$$  

$$\mathcal{L}_{\text{SYM}} = \frac{1}{4g^2} \int d^2 \theta \text{Tr} \mathcal{W}_\alpha \mathcal{W}_\alpha + \text{c.c.}$$
Review of pure $N=1$ SU(N) SYM

• **Special** chiral scalar superfield

\[
S = \text{Tr} \mathcal{W}^\alpha \mathcal{W}_\alpha = -\text{Tr} \lambda^\alpha \lambda_\alpha + \sqrt{2} \theta^\alpha \chi_\alpha + \theta^2 F
\]

\[
\chi = \sqrt{2} \text{Tr} \left( \frac{1}{2} F_{mn} \sigma^{mn} \lambda - i \lambda D \right)
\]

\[
F = \text{Tr} \left( -2i \lambda \sigma^m \nabla_m \bar{\lambda} - \frac{1}{2} F_{mn} F^{mn} + D^2 - \frac{i}{4} \varepsilon_{mnpl} F^{mn} F^{pl} \right)
\]

• **Important notice**

\[
\text{Im} \, F = * F_4 = - * \text{Tr} \, F_2 \wedge F_2 - \partial_m \text{Tr} \, \lambda \sigma^m \bar{\lambda}
\]

\[
= - * d \text{Tr} \left( A d A + \frac{2i}{3} A^3 + \frac{1}{3!} d x^k d x^n d x^m \epsilon_{mnkl} \text{Tr} \, \lambda \sigma^l \bar{\lambda} \right) \equiv * d C_3
\]

\[
S = - \frac{i}{4} \bar{D}^2 U, \quad U(x, \theta, \bar{\theta}) - \text{real salar superfield} \quad (Gates '81)
\]
Veneziano-Yankielowicz ’82 Lagrangian revisited

• Provides effective description of color-less bound states of the SYM multiplet (glueballs, gluinoballs and their fermionic superpartner), gluino condensate and the N-degeneracy of the SYM vacuum

\[ S = s(x) + \sqrt{2} \theta^\alpha \chi_\alpha(x) + \theta^2 (\hat{D}(x) + i \, *dC_3(x)) \]

\[ *dC_3 = \partial_m C^m, \quad \text{where} \quad C_1 = *C_3 \]

• The form of the VY Lagrangian is (almost) fixed by anomalous superconformal Ward identities of the SU(N) SYM

\[ \mathcal{L}_{VY} = \frac{1}{\rho} \int d^2 \theta d^2 \bar{\theta} (S \bar{S})^{\frac{1}{3}} + \int d^2 \theta W(S) + c.c, \]

\[ W(S) = \frac{N}{16 \pi^2} S \left( \ln \frac{S}{\Lambda^3} - 1 \right) \]
Veneziano-Yankielowicz ’82 Lagrangian revisited

- The superpotential and the Lagrangian are not single valued
  \[ S \rightarrow S e^{2\pi i}, \quad \mathcal{L}_{VY} \rightarrow \mathcal{L}_{VY} + \frac{N}{4\pi} \partial_m C^m \]

- The special form of the chiral superfield \( S = -\frac{i}{4} \bar{D}^2 U \) requires the variation of the VY Lagrangian with respect to independent real superfield \( U \).

- The variation principle is well-defined only with the addition of the boundary (total derivative) term (Bandos, Lanza, D.S. ’19)

\[
\mathcal{L}_{bd} = -\frac{1}{8} \left( \int d^2 \theta \bar{D}^2 - \int d^2 \bar{\theta} D^2 \right) \left[ \left( \frac{1}{12\rho} \bar{D}^2 \frac{\bar{S}^{\frac{1}{3}}}{S^{\frac{2}{3}}} + \frac{1}{16\pi^2} \ln \frac{\Lambda^3 N}{S^N} \right) U \right] + \text{c.c.}
\]
Veneziano-Yankielowicz ’82 Lagrangian revisited

- Bosonic part of the Lagrangian

\[ \mathcal{L}^{\text{bos}}_{\text{VY}} = K_{s\bar{s}} \left( -\partial_m s \partial^m \bar{s} + (\partial_m C^m)^2 + \hat{D}^2 \right) + \left( W_s \left( \hat{D} + i\partial_m C^m \right) + \text{c.c.} \right) + \mathcal{L}^{\text{bos}}_{\text{bd}} \]

- boundary term

\[ \mathcal{L}^{\text{bos}}_{\text{bd}} = -2\partial_m \left[ C^m \left( K_{s\bar{s}} \partial_n C^m - \text{Im} \ W_s \right) \right], \quad K_{s\bar{s}} \equiv \partial_s \partial_{\bar{s}} K(s, \bar{s}), \quad W_s \equiv \partial_s W(s) \]

- auxiliary field equations of motion

\[ K_{s\bar{s}} \hat{D} + \text{Re} \ W_s = 0 \quad \rightarrow \quad \hat{D} = -\frac{\text{Re} \ W_s}{K_{s\bar{s}}} \]

\[ \partial_m \left( K_{s\bar{s}} \partial_n C^m - \text{Im} \ W_s \right) = 0 \quad \rightarrow \quad \partial_m C^m = \frac{\text{Im} \ W_s - \frac{n}{8\pi}}{K_{s\bar{s}}} \quad \text{<- integration constant} \]

\[ F \equiv \hat{D} + i\partial_m C^m = -\frac{\overline{W_s} + i\frac{n}{8\pi}}{K_{s\bar{s}}} \]
Veneziano-Yankielowicz ’82 Lagrangian revisited

- Scalar field potential (Kovner & Shifman ’97)

\[ V(s, \bar{s}) = \frac{9\rho N}{16\pi^2} |s|^{\frac{4}{3}} \left( \ln^2 \frac{|s|}{\Lambda^3} + (\arg s - \frac{2\pi n}{N})^2 \right), \quad n = 0, 1, 2, \ldots, N - 1 \]

Potential is single-valued, multi-branched, has cusps at \( \arg s = \frac{\pi n}{N} \)

and susy minima at \( \langle s \rangle = \Lambda^3 e^{2\pi i \frac{n}{N}} = SU(N) \text{ SYM vacua} \)
Coupling membrane to SYM

• Supersymmetric and kappa-symmetric membrane action
  \((I. \ Bandos, \ S. \ Lanza, \ D.S. \ '19)\)

\[
S_{M2+SYM} = -\frac{k}{4\pi} \int d^3 \xi \sqrt{-\det h_{ij}} |S| + \frac{k}{4\pi} \int C_3 \quad (k = 0, \pm 1, \pm 2, \ldots)
\]

\[\text{Nambu-Goto} \quad \text{Wess-Zumino}\]

\[
S(x, \theta) = \text{Tr} \, W^\alpha W_\alpha = -\frac{i}{4} \bar{D}^2 U, \quad U(x(\xi), \theta(\xi), \bar{\theta}(\xi)) - \text{real scalar superfield}
\]

\[E^a = dx^a + i \theta \sigma^a \bar{\theta} + c.c.\]

\[
C_3 = i \, E^a \wedge d\theta^\alpha \wedge d\bar{\theta}^{\dot{\alpha}} \sigma_{a\alpha\dot{\alpha}} U
\]
\[
- \frac{1}{4} E^b \wedge E^a \wedge d\theta^\alpha \sigma_{ab} \alpha^{\beta} D_\beta U - \frac{1}{4} E^b \wedge E^a \wedge d\bar{\theta}^{\dot{\alpha}} \bar{\sigma}_{ab} \dot{\beta} \dot{\alpha} \bar{D}_{\dot{\beta}} U
\]
\[
- \frac{1}{48} E^c \wedge E^b \wedge E^a \epsilon_{abcd} \bar{\sigma}^{d\alpha\dot{\alpha}} [D_\alpha, \bar{D}_{\dot{\alpha}}] U.
\]

\[C_3|_{\theta=0} = C_3 = \text{Tr} \left( AdA + \frac{2i}{3} A^3 \right)\]
Kappa-symmetry

• Counterpart of local worldvolume supersymmetry

\[ \delta \theta^\alpha = \kappa^\alpha (\xi), \quad \delta x^m = i \kappa \sigma^m \bar{\theta} + c.c. \]

\[ \kappa_\alpha = \Gamma_\alpha \dot{\bar{\kappa}}^\dot{\alpha}, \quad \Gamma^2 = 1 \]

Gauges away 2 of 4 fermionic modes \( \theta^\alpha (\xi), \bar{\theta}^{\dot{\alpha}} (\xi) \) of the membrane

Worldvolume reparametrization gauges away 3 of 4 bosonic modes \( x^m (\xi) \)

\( (x^3 (\xi), \psi^\alpha (\xi)) \) Goldstone supermultiplet

associated with \( \frac{1}{2} \) broken supersymmetry in the 4D bulk, while another \( \frac{1}{2} \) of susy remains unbroken allowing for BPS configurations
BPS domain wall solutions sourced by the membrane

• Consider a static membrane in the Veneziano-Yankielowicz model

\[ \theta^\alpha(\xi) = \bar{\theta}^{\dot{\alpha}}(\xi) = 0, \quad \xi^i = x^i \ (i = 0, 1, 2), \quad x^3 = 0 \]

• The presence of the membrane modifies the bulk field equations by source terms, in particular the gauge 3-form eq.

\[ \partial_m (K_{s\bar{s}} \partial_n C^m - \text{Im} \ W_s) = -\frac{k}{8\pi} \delta^3_m \delta(x^3) \]

\[ \text{Im} \ F = \partial_m C^m = \frac{8\pi \text{Im} \ W_s - (n + k \Theta(x^3))}{8\pi K_{s\bar{s}}} \]

\[ \langle s \rangle = \Lambda^3 e^{2\pi i \frac{n}{N}} \quad \text{or} \quad \langle s \rangle = \Lambda^3 e^{2\pi i \frac{n+k}{N}} \]
BPS domain wall solutions sourced by the membrane

- BPS domain-wall equation dictated by $\frac{1}{2}$ susy conservation $\delta \chi_\alpha = 0$

$$
\frac{\partial s(x^3)}{\partial x^3} \equiv \dot{s} = ie^{i\alpha} F = -ie^{i\alpha} \frac{\overline{W}_s + \frac{i}{8\pi} (n + k\Theta(x^3))}{K_{s\overline{s}}} s, \quad e^{i\alpha} = \frac{|s(0)|}{s(0)} \quad \text{(on M2)}
$$

Substituting the form of $W$ and $K$ of the Veneziano-Yankielowiz (VY) model, we get

$$
\dot{s} = 9i \rho N |s|^\frac{4}{3} e^{i\alpha} \left( \ln \frac{\Lambda^3}{|s|} + i \arg s - \frac{2\pi i}{N} (n + k\Theta(x^3)) \right)
$$

BPS value of the on-shell action for the VY model + membrane

$$
S_{\text{BPS}} = S_{\text{VY}} + S_{\text{membr}} = -2 \int d^3 \xi |W_{+\infty} - W_{-\infty}|
$$

$$
T_{\text{DW}} = T_s + T_{\text{membr}} = 2 |W_{+\infty} - W_{-\infty}| = \frac{N}{8\pi^2} \Lambda^3 \left| e^{2\pi i \frac{n + k}{N}} - e^{2\pi i \frac{n}{N}} \right|
$$
Shape of BPS domain walls

• $s(x)$-continuous domain wall solutions of the BPS equations exist for the membrane charge having the following values

$$|k| \leq \frac{N}{3}, \quad |k| = N$$

½ BPS domain wall between the same vacuum

Examples $k = 1, \quad N = 3, 4, 5, 6, 7, \ldots$

Form of the superpotential

$$\text{Im} \left( W \frac{|s(0)|}{\bar{s}(0)} \right)$$

$$\text{Re} \left( W \frac{|s(0)|}{\bar{s}(0)} \right)$$

membrane sits here
Shape $s(x)$ of BPS domain walls with $|k| \leq \frac{N}{3}$

$|s| / \Lambda^3$

"trench"

$N=3$

$N=4$

$N=5$

$x^3$

arg $s$ smooth

$N=3$

$N=4$

$N=5$

$|k|=N$ domain wall

$|s| / \Lambda^3$

membrane sits here

membrane sits here

$\text{arg } s$

smooth
Conclusions

• We have constructed the supersymmetric and kappa-invariant action describing the coupling of a membrane to $N=1$, $D=4$ SYM and its Veneziano-Yankielowicz effective sigma-model.

• The membrane of charge $k$ separates two SYM vacua with different phases of the gluino condensate:

\[
\langle s \rangle = \Lambda^3 e^{2\pi i \frac{n}{N}} \quad \text{and} \quad \langle \bar{s} \rangle = \Lambda^3 e^{2\pi i \frac{n+k}{N}}
\]

• and creates BPS domain walls interpolating between these vacua:

\[
T_{DW} = \frac{N \Lambda^3}{8\pi^2} \left| e^{2\pi i \frac{n+k}{N}} - e^{2\pi i \frac{n}{N}} \right|
\]

• Explicit domain wall configurations have been found for $|k| \leq \frac{N}{3}$ and $|k| = N$.

• $|k| = N$ BPS domain wall interpolates between the same SYM vacuum and has zero total tension.