Dualities in SUSY Gauge Theories in Various Dimensions

arXiv:1801.00799 (JHEP05(2018)082)
arXiv:1904.07954

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Introduction and Motivation
• Progress in our understanding of QFT often comes from finding geometry structures that encapsulate certain salient aspects.

• Engineering QFT’s in string theory has been in successful in uncovering these structures especially for QFT’s with SUSY.

• One of the remarkable results of this endeavor has been the network of dualities connecting these theories.

• These dualities have been especially powerful for theories with extended supersymmetry.

• But they exist even for minimally supersymmetric theories.
The prototypical example is Seiberg Duality, which is an infrared equivalence between two 4d $\mathcal{N} = 1$ gauge theories:

1. An $SU(N_c)$ gauge theory with $N_f$ flavors $Q_i$ and $\tilde{Q}^i$. We assume $N_f > N_c + 1$.

2. An $SU(N_f - N_c)$ gauge theory with $N_f$ flavors $q_i$ and $\tilde{q}^i$ and mesons $M^i_j$. It has a superpotential coupling flavors and mesons $W = M^i_j \tilde{q}^i q_j$.

For $N_f < N_c$ the first theory has no SUSY vacua due to non-perturbative ADS superpotential

$$W_{ADS} \propto \left[ \frac{1}{\det (Q_i \tilde{Q}^j)} \right]^{1/(N_c - N_f)}$$
Generalizations and Applications

- Since its discovery Seiberg duality has been generalized in many directions.
- On Physics side perhaps the most striking generalizations are
  - A triality for $2d \ (0, 2)$ theories \([\text{Gadde, Gukov, Petrov - 2013}]\).
  - A quadrality for $0d \mathcal{N} = 1$ matrix models \([\text{Franco, Lee, Seong, Vafa - 2016}]\).
- On mathematical side, Seiberg duality plays an important role in cluster algebras under the name of mutations.
- Cluster algebras and mutations crop up in the computation of scattering amplitudes using on-shell diagrams.
Seiberg Duality and Calabi-Yau Threefolds
D-branes and Calabi-Yau

- An important realization of Seiberg duality is among worldvolume theories on D-branes probing geometries.
- Supersymmetry requires that the probed geometry is a Calabi-Yau.
- The probed geometry appears as the moduli space of vacua for the corresponding theory.
- This setup can be generalized to realize theories with minimal SUSY in even dimensions starting from 6 to 0.
• While studying these worldvolume theories it was soon realized that many theories correspond to the same geometry.
• All these theories have the same low energy limit and are in fact related by Seiberg duality [Beasley, Plesser; Feng, Hanany, He, Uranga - 2001].
• In this context Seiberg duality can be understood geometrically in complementary ways:
  • A mutation of an exceptional collection in the B-model of topological strings. [Herzog - 2004].
  • From the mirror perspective it is a movement of a vanishing cycle past another. [Feng, He, Kennaway, Vafa - 2008].
● Gauge group is $SU(N + M) \times SU(N)$.
● Global symmetry is $SU(2) \times SU(2) \times U(1)_B \times U(1)_R$.
● $W = \frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} \text{tr} X_{\alpha}^{\gamma} X_{\beta}^{\gamma} X_{\delta}^{\gamma}$.
● The theory is conformal for $M = 0$.
● At $M = 0$, the theory is also self-dual! Not too exciting!
Non-zero $M$ triggers an RG flow. Eventually one of the gauge couplings diverges.

Seiberg duality allows us to continue past this divergence.

The resulting theory is similar to our starting point except the gauge group is now $SU(N) \times SU(N - M)$ i.e $N \rightarrow N - M$

This phenomenon called duality cascade continues until the smaller gauge groups becomes less than $M$.

At this point supersymmetry is dynamically broken by the ADS superpotential and the cascade stops.

The Conifold Cascade
Down the Dimensions: Triality and Quadrality
The setup of D-branes probing Calabi-Yau can be generalized to other dimensions.

In the case of D1-branes probing CY-fourfolds case the worldvolume theory has at least $2d (0, 2)$ SUSY.

In addition to vector and chiral multiplets there are Fermi multiplets $\Lambda_a$. They have no on-shell bosonic degree of freedom.

Interactions are given by $J_a(X_i)$ and $E_a(X_i)$.

Gauge invariant “superpotential”

$$W = \sum_a (\text{tr} \Lambda_a J_a + \text{tr} \bar{\Lambda}_a E_a).$$

Supersymmetry requires: $\sum \text{tr} E_a J_a = 0$. 

D-Branes probing CY
\[ D(-1) \text{ Branes Probing Calabi-Yau Fivefolds} \]

- Going down further gives us $0d \mathcal{N} = 1$ matrix models.
- The different supermultiplets needed to build such a theory are vector, chiral ($X_i$) and Fermi ($\Lambda_a$).
- Interactions are encoded in $J_a(X_i)$ and $H_{ab}(X_i)$ terms:
- Using them we can write a gauge invariant superpotential
  \[
  W = \text{tr} \, \Lambda_a J_a + \text{tr} \, \bar{\Lambda}_a \bar{\Lambda}_b H_{ab}
  \]
- Supersymmetry requires that for each Fermi $\Lambda_a$
  \[
  \sum_b H_{ab} J_b = 0
  \]
As in 4d case, the map between QFT and geometry is many to one for worldvolume 2d (0, 2) theories and 0d $\mathcal{N} = 1$ Matrix models.

### 2d (0, 2) Triality

- **Gauge anomaly cancellation:**
  \[2N - N_1 + N_2 - N_3 = 0\]
- \[N' = N_1 - N\]
Triality and Quadrality

As in $4d$ case, the map between QFT and geometry is many to one for worldvolume $2d (0, 2)$ theories and $0d \mathcal{N} = 1$ Matrix models.

### $2d (0, 2)$ Triality [Gadde, Gukov, Petrov - 2013]

- Gauge anomaly cancellation:
  \[ 2N - N_1 + N_2 - N_3 = 0 \]
- $N' = N_1 - N$

### $0d \mathcal{N} = 1$ Quadrality [Franco, Lee, Seong, Vafa - 2016]

- Gauge anomaly cancellation:
  \[ N_1 - N_2 + N_3 - N_4 = 0 \]
- $N' = N_1 - N$
A Generalization of Conifold: $Y^{1,0}(\mathbb{P}^m)$ [Closset, Franco, Guo, AH - 2018]

- Resolved conifold can be realized as the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{P}^1$.
- A interesting generalization of Conifold dubbed $Y^{1,0}(\mathbb{P}^m)$ is the bundle

$$\text{Tot}(\mathcal{O}(-m) \oplus \mathcal{O}(-1) \to \mathbb{P}^m)$$

- $Y^{1,0}(\mathbb{P}^m)$ is a Calabi-Yau for any $m$ and the corresponding worldvolume theories have been constructed.
Cascade for Generalized Conifolds

\[ Y_{1,0}(\mathbb{P}^2) \]

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Cascade for Generalized Conifolds

$\mathcal{Y}^{1,0}(\mathbb{P}^2)$

$\mathcal{Y}^{1,0}(\mathbb{P}^3)$
Outlook and Conclusions
Outlook

- The formal similarities between the cascade of $Y^{1,0}(\mathbb{P}^m)$ and conifold are striking!
- They are also an invitation to study the RG flow and dynamical SUSY breaking in more detail for them.
- There is also a need for exploring in more detail the geometric structure of these dualities:
  - The mirror symmetry interpretation of triality and quadrality is known.
  - Herzog’s argument relating Seiberg duality to exceptional mutations from algebraic geometry can be generalized. But it should be possible to expand it and include superpotential in the discussion.
Conclusions

- Seiberg duality and its generalizations not only allow us connect different manifestations of the same theory, they are needed for a complete description of it.
- Seiberg duality has been extensively studied in various contexts, but its lower dimensional generalizations are more recent and a fertile ground for further research.
- Past adventures in utilizing and extending Seiberg duality give us a template for some of these studies.