# Dualities in SUSY Gauge Theories in Various Dimensions

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# Introduction and Motivation

# Quantum Field Theory and Geometry

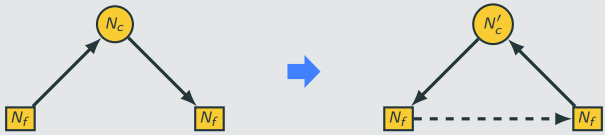
- Progress in our understanding of QFT often comes from finding geometry structures that encapsulate certain salient aspects.
- Engineering QFT's in string theory has been in successful in uncovering these structures especially for QFT's with SUSY.
- One of the remarkable results of this endeavor has been the network of dualities connecting these theories.
- These dualities have been especially powerful for theories with extended supersymmetry.
- But they exist even for minimally supersymmetric theories.

# Seiberg Duality [Seiberg, 1994]

- The prototypical example is Seiberg Duality, which is an infrared equivalence between two $4d\ \mathcal{N}=1$ gauge theories:
  1. An $SU(N_c)$ gauge theory with $N_f$ flavors $Q_i$ and $\tilde{Q}^i$. We assume $N_f > N_c + 1$.
  2. An $SU(N_f - N_c)$ gauge theory with $N_f$ flavors $q_i$ and $\tilde{q}^i$ and mesons $M^i_j$. It has a superpotential coupling flavors and mesons $W = M^j_i \tilde{q}^i q_j$.
- For $N_f < N_c$ the first theory has no SUSY vacua due to non-perturbative ADS superpotential

$$W_{ADS} \propto \left[1/\det\left(Q_i \tilde{Q}^j\right)\right]^{1/(N_c - N_f)}$$

**Seiberg Duality**



# Generalizations and Applications

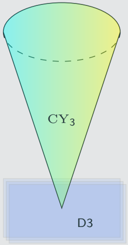
- Since its discovery Seiberg duality has been generalized in many directions.
- On Physics side perhaps the most striking generalizations are
  - A triality for $2d$ $(0,2)$ theories [Gadde, Gukov, Petrov - 2013].
  - A quadrality for $0d$ $\mathcal{N}=1$ matrix models [Franco, Lee, Seong, Vafa - 2016].
- On mathematical side, Seiberg duality plays an important role in cluster algebras under the name of mutations.
- Cluster algebras and mutations crop up in the computation of scattering amplitudes using on-shell diagrams.

# Seiberg Duality and Calabi-Yau Threefolds

# D-branes and Calabi-Yau

- An important realization of Seiberg duality is among worldvolume theories on D-branes probing geometries.
- Supersymmetry requires that the probed geometry is a Calabi-Yau.
- The probed geometry appears as the moduli space of vacua for the corresponding theory.
- This setup can be generalized to realize theories with minimal SUSY in even dimensions starting from 6 to 0.

**D-Branes probing CY**



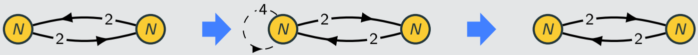
# Geometrization of Seiberg Duality

- While studying these worldvolume theories it was soon realized that many theories correspond to the same geometry.
- All these theories have the same low energy limit and are in fact related by Seiberg duality [Beasley, Plesser; Feng, Hanany, He, Uranga - 2001].
- In this context Seiberg duality can be understood geometrically in complementary ways:
  - A mutation of an exceptional collection in the B-model of topological strings. [Herzog - 2004].
  - From the mirror perspective it is a movement of a vanishing cycle past another. [Feng, He, Kennaway, Vafa - 2008].

# Conifold: An Example [Klebanov, Strassler: 2000]

- Gauge group is $SU(N+M) \times SU(N)$.
- Global symmetry is $SU(2) \times SU(2) \times U(1)_B \times U(1)_R$.
- $W = \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} \operatorname{tr} X_{01}^{\alpha} X_{10}^{\gamma} X_{01}^{\beta} X_{10}^{\delta}$.
- The theory is conformal for $M = 0$.
- At $M = 0$, the theory is also self-dual! Not too exciting!

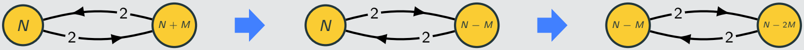
## Conformal Conifold



## Conifold Continued: The RG Flow

- Non-zero $M$ triggers an RG flow. Eventually one of the gauge couplings diverges.
- Seiberg duality allows us to continue past this divergence.
- The resulting theory is similar to our starting point except the gauge group is now $SU(N) \times SU(N-M)$ i.e $N \rightarrow N - M$
- This phenomenon called duality cascade continues until the smaller gauge groups becomes less than $M$.
- At this point supersymmetry is dynamically broken by the ADS superpotential and the cascade stops.

### The Conifold Cascade

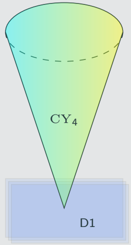


# Down the Dimensions: Triality and Quadrality

# D1 Branes Probing Calabi-Yau Fourfolds

- The setup of D-branes probing Calabi-Yau can be generalized to other dimensions.
- In the case of D1-branes probing CY-fourfolds case the worldvolume theory has at least $2d\,(0,2)$ SUSY.
- In addition to vector and chiral multiplets there are Fermi multiplets $\Lambda_a$. They have no on-shell bosonic degree of freedom.
- Interactions are given by $J_a(X_i)$ and $E_a(X_i)$.
- Gauge invariant "superpotential" $W = \sum_a (\operatorname{tr} \Lambda_a J_a + \operatorname{tr} \bar{\Lambda}_a E_a)$.
- Supersymmetry requires: $\sum \operatorname{tr} E_a J_a = 0$.

**D-Branes probing CY**



# D$(-1)$ Branes Probing Calabi-Yau Fivefolds

- Going down further gives us $0d\,\mathcal{N}=1$ matrix models.
- The different supermultiplets needed to build such a theory are vector, chiral ($X_i$) and Fermi ($\Lambda_a$).
- Interactions are encoded in $J_a(X_i)$ and $H_{ab}(X_i)$ terms:
- Using them we can write a gauge invariant superpotential

$$W = \mathrm{tr}\,\Lambda_a J_a + \mathrm{tr}\,\bar{\Lambda}_a \bar{\Lambda}_b H_{ab}$$

- Supersymmetry requires that for each Fermi $\Lambda_a$

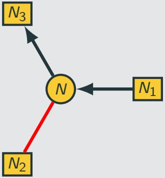
$$\sum_b H_{ab} J_b = 0$$

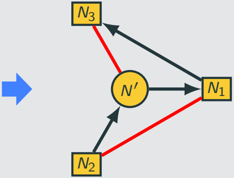
.

# Triality and Quadrality

As in $4d$ case, the map between QFT and geometry is many to one for worldvolume $2d\,(0,2)$ theories and $0d\,\mathcal{N}=1$ Matrix models.

**$2d\,(0,2)$ Triality** [Gadde, Gukov, Petrov - 2013]




- Gauge anomaly cancellation: $2N - N_1 + N_2 - N_3 = 0$
- $N' = N_1 - N$

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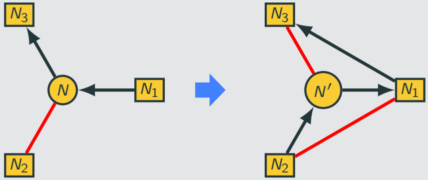
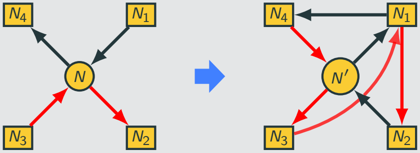
## $2d\,(0,2)$ **Triality** [Gadde, Gukov, Petrov - 2013]




- Gauge anomaly cancellation: $2N - N_1 + N_2 - N_3 = 0$
- $N' = N_1 - N$

## $0d\,\mathcal{N}=1$ **Quadrality** [Franco, Lee, Seong, Vafa - 2016]




- Gauge anomaly cancellation: $N_1 - N_2 + N_3 - N_4 = 0$
- $N' = N_1 - N$

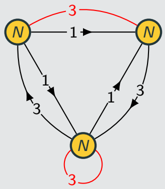
# A Generalization of Conifold: $Y^{1,0}(\mathbb{P}^m)$ [Closset, Franco, Guo, AH - 2018]

- Resolved conifold can be realized as the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{P}^1$.
- A interesting generalization of Conifold dubbed $Y^{1,0}(\mathbb{P}^m)$ is the bundle

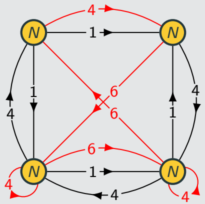
$$\mathrm{Tot}(\mathcal{O}(-m) \oplus \mathcal{O}(-1) \to \mathbb{P}^m)$$

- $Y^{1,0}(\mathbb{P}^m)$ is a Calabi-Yau for any $m$ and the corresponding worldvolume theories have been constructed.

$Y^{1,0}(\mathbb{P}^2)$

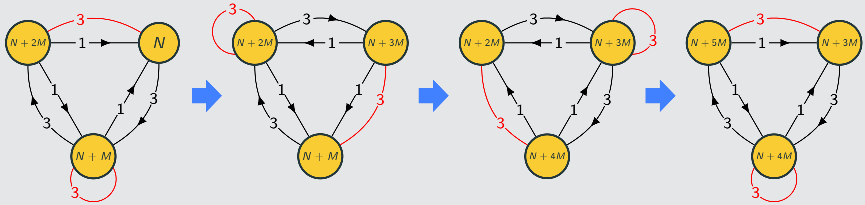



$Y^{1,0}(\mathbb{P}^3)$



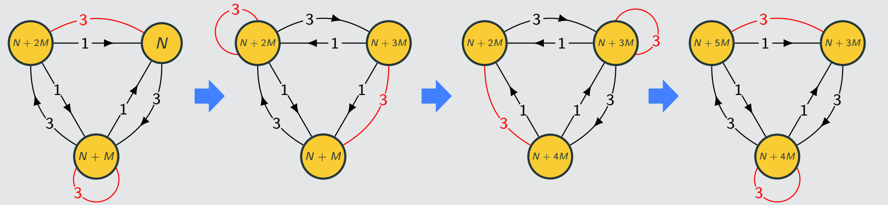
# Cascade for Generalized Conifolds

## $Y^{1,0}(\mathbb{P}^2)$

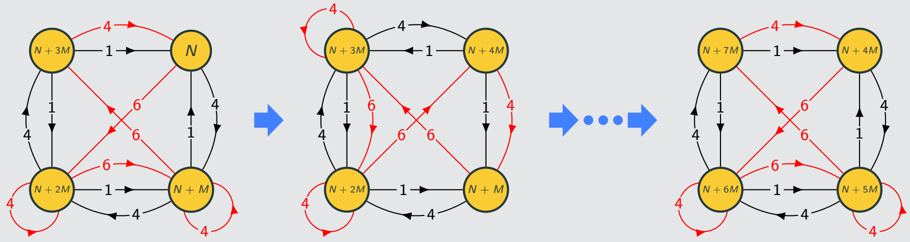


# Cascade for Generalized Conifolds

## $Y^{1,0}(\mathbb{P}^2)$




## $Y^{1,0}(\mathbb{P}^3)$



# Outlook and Conclusions

# Outlook

- The formal similarities between the cascade of $Y^{1,0}(\mathbb{P}^m)$ and conifold are striking!
- They are also an invitation to study the RG flow and dynamical SUSY breaking in more detail for them.
- There is also a need for exploring in more detail the geometric structure of these dualities:
  - The mirror symmetry interpretation of triality and quadrality is known.
  - Herzog's argument relating Seiberg duality to exceptional mutations from algebraic geometry can be generalized. But it should be possible to expand it and include superpotential in the discussion.

# Conclusions

- Seiberg duality and its generalizations not only allow us connect different manifestations of the same theory, they are needed for a complete description of it.
- Seiberg duality has been extensively studied in various contexts, but its lower dimensional generalizations are more recent and a fertile ground for further research.
- Past adventures in utilizing and extending Seiberg duality give us a template for some of these studies.