TFI 2019 - Theories of Fundamental Interactions 2019

Defects, nested instantons and comet shaped quivers

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mathematical companion paper to appear soon

Outline

1. Introduction/motivation
2. D-branes construction
3. SUSY partition functions
4. Open questions

Introduction/motivation

Why is studying defects interesting?

They characterize physical theories and the related mathematical objects.

We will study **surface defects**: real codimension two submanifolds where a **specific reduction of the gauge connection** takes place.

They have also been studied in many different contexts:

- Study of $S-$duality;
- Correspondences with CFTs (AGT, BPS/CFT,...);
- Quantum integrable systems and their quantization;
- Exact results in non-perturbative QFT;
- Differental invariants (Donaldson, SW, VW,...).

What are defects?

Take for example $\mathcal{N}=4$, $4d$ $SU(N)$ SYM on $\mathbb{R}^4$, and put a surface defect at $(z, w=0) \simeq \mathbb{R}^2 \subset \mathbb{C}^2 = \mathbb{R}^4$ by asking that the gauge field diverges near $w=0$ as

$$A_\mu \sim \operatorname{diag}(\underbrace{\alpha_1, \ldots, \alpha_1}_{n_1}, \ldots, \underbrace{\alpha_M, \ldots, \alpha_M}_{n_M}) \mathrm{i} d\theta.$$

- $G = SU(N)$ broken to $\mathbb{L} = S[U(n_1) \times \cdots \times U(n_M)] \subseteq SU(N)$ along the defect.
- $\mathbb{L}$ has $M-1$ $U(1)$ factors $\Rightarrow$ **fluxes** $\mathfrak{m}^I$, **parameters** η_I and **phase factor** $\exp(\mathrm{i}\mathfrak{m}^I\eta_I)$ in the path integral.
- $\underline{\alpha}$ define a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}$, which is the Lie algebra of some **parabolic subgroup** $\mathcal{P} \subseteq G_{\mathbb{C}}$.
- Surface defect is a $4d$ gauge theory coupled to $2d$ NLSM with target space the **partial flag manifold** $\mathbb{G}_{\mathbb{C}}/\mathcal{P}$.

The defect theory we are interested in is engineered as a low-energy effective theory of a brane system.

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Quiver Gauge Theories

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We are naturally led to the study of **moduli spaces of SUSY vacua**.

In particular we study an $\mathcal{N} = 2^*$ theory (topologically twisted) on the product of two Riemann surfaces.

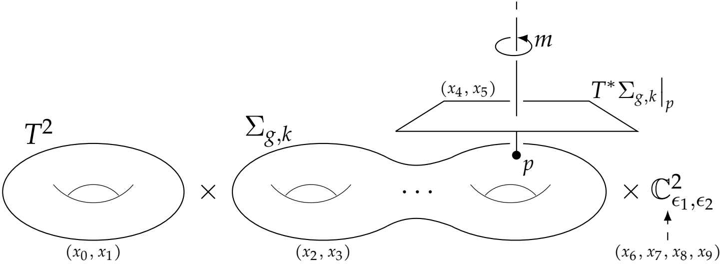
⇩

The defects are located at **punctures** on one of the two surfaces.

⇩

This theory reduces to a NLSM with target space the moduli space parametrizing what we call **nested instantons**.

SUSY background and D-branes



	$\boldsymbol{T^2}$	$\times$ $\boldsymbol{T^*\Sigma_{g,k}}$		$\times$ $\boldsymbol{\mathbb{C}^2_{\epsilon_1,\epsilon_2}}$
	(x_0, x_1)	(x_2, x_3)	(x_4, x_5)	(x_6, x_7, x_8, x_9)
D3	•	•		
D7	•	•		•

Bosonic field content

The D3 effective theory on $T^2 \times \Sigma_g$ in the background of the D7 is a Vafa-Witten topologically twisted $\mathcal{N} = 4$, $D = 4$ theory on $T^2 \times \Sigma_g$.

The bosonic content is comprised of:

- gauge connection $\mathcal{A}$ on $T^2 \times \Sigma_g$;
- 1-form in the adjoint Φ_Σ describing motion within $T^*\Sigma$;
- two complex scalars $B_{1,2}$ describing motion along $\mathbb{C}^2$;
- two complex scalars I, J taking care of the D7 background effect on the D3.

GLSMs and character varieties

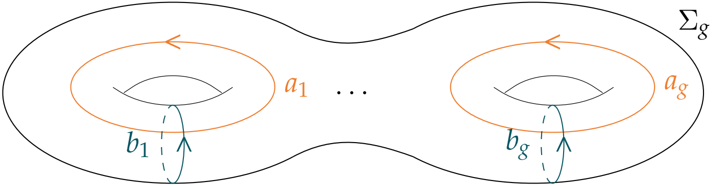
$S = T^2 \times \Sigma_g$: VW theory reduces in the limit of small Σ_g volume to GLSM from T^2 to the Hitchin moduli space on Σ_g.[Bershadsky, et al., 1995]

Hitchin moduli spaces on Riemann surfaces with singularities $\{p_i\}$ are homeomorphic to GL_n-character varieties of Riemann surfaces with marked points.[Simpson, 1990]

We study surface defects describing the parabolic reduction of connections at punctures on a Riemann surface. The quantities we compute have a connection to the **cohomology of character varieties**.[Hausel, et al., 2011],[Chuang, et al., 2013]

Character varieties (I)

They parametrize (flat) connections by their holonomy along non trivial loops.



$$A_i = \mathrm{Hol}_{a_i}(\nabla) = \mathrm{Tr}\left(\mathcal{P}\int_{a_i} A_\mu \mathrm{d}x^\mu\right); \quad B_i = \mathrm{Hol}_{b_i}(\nabla) = \mathrm{Tr}\left(\mathcal{P}\int_{b_i} A_\mu \mathrm{d}x^\mu\right)$$

What if the surface has punctures?

The relevant information is still encoded in monodromy data (and in particular in their eigenvalues multiplicities).

Character varieties (II)

Let now $\Sigma_{g,k}$ be a genus g Riemann surface with punctures at $\{p_1, \ldots, p_k\}$, then fix a $k-$tuple of Young diagrams, $\boldsymbol{\mu} \in \mathcal{P}_n^k$, and a $k-$tuple of semisimple GL_n-conjugacy classes $(C_1, \ldots, C_k)$ of type $\boldsymbol{\mu}$, *i.e.* the eigenvalues of $X_i \in C_i$ have multiplicities given by $\mu_i \in \mathcal{P}_n$.

$\Downarrow$

C_i represents the monodromy datum at the puncture p_i on $\Sigma_{g,k}$.

$\Downarrow$

The character variety in this case is

$$\begin{aligned}\mathcal{M}_{\boldsymbol{\mu}} &= \{\rho \in \mathrm{Hom}(\pi_1(\Sigma_{g,k}), GL_n(\mathbb{C})) : \rho(\gamma_i) \in C_i\} /\!\!/ PGL_n \\ &= \{(A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k = \mathbb{1}, X_i \in C_i\} /\!\!/ PGL_n\end{aligned}$$

HLRV conjecture

Define the genus g Cauchy function $\Omega_g \in \mathbb{Q}(z,w) \otimes_{\mathbb{Z}} \Lambda(\mathbf{x}_1, \dots, \mathbf{x}_k)$, $\mathbf{x}_i = \{x_{i,1}, x_{i,2}, \dots\}$

$$\Omega_g(z,w) = \sum_{\lambda \in \mathcal{P}} \mathcal{H}_\lambda(z,w) \prod_{i=1}^{k} \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2)$$

with

$$\mathcal{H}_\lambda(z,w) = \prod_{s \in \lambda} \frac{(z^{2a(s)+1} - w^{2l(s)+1})^g}{(z^{(2a(s)+2} - w^{2l(s)})(z^{2a(s)} - w^{2l(s)+2})},$$

and $\tilde{H}_\lambda(\mathbf{x}; q, t)$ the modified Macdonald polynomial.

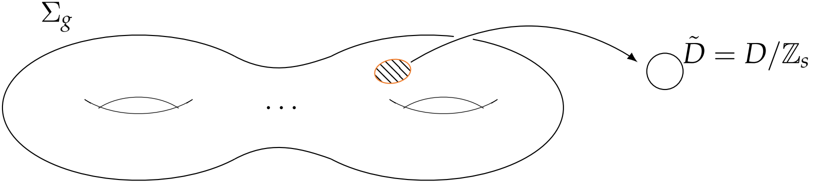
Conjecture (HLRV)

The polynomial $\mathbb{H}_{\boldsymbol{\mu}}(z,w) = (z^2-1)(1-w^2)\langle \mathrm{PL}\,\Omega_g(z,w), h_{\boldsymbol{\mu}}\rangle$ *encodes information about the cohomology of* $\mathcal{M}_{\boldsymbol{\mu}}$.

D-branes construction

Local D3 theory

The local behaviour of D3 branes is obtained by excising a small disk D around each marked point and describing it as an $s-$covering $\tilde{D} = D/\mathbb{Z}_s$.

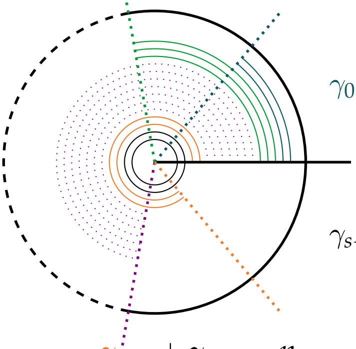


On $\tilde{D}$ there is a natural action of $\mathbb{Z}_s$ inducing a splitting of the Chan-Paton vector spaces into $\mathbb{Z}_s-$irreps.

$$\mathbb{V} = \bigoplus_{\ell=0}^{s-1} \mathbb{V}_\ell \otimes R_\ell; \qquad \mathbb{W} = \bigoplus_{\ell=0}^{s-1} \mathbb{W}_\ell \otimes R_\ell$$

D-branes twisted sectors

$$\gamma_1 + \cdots + \gamma_{s-2} + \gamma_{s-1} = n_1$$



$$\gamma_0 + \gamma_1 + \cdots + \gamma_{s-2} + \gamma_{s-1} = n_0 \quad (= \boldsymbol{n})$$

$$\gamma_{s-1} = n_{s-1}$$

$$\gamma_{s-2} + \gamma_{s-1} = n_{s-2}$$

$$V_j = \bigoplus_{\ell=j}^{s-1} \mathbb{V}_\ell; \qquad W_j = \bigoplus_{\ell=j}^{s-1} \mathbb{W}_\ell$$

Local D3-D7 theory (I)

By gluing back the formal orbifold disk $\tilde{D}$ to Σ_g we get the local quiver description of the D3/D7 system. Open string modes split as:

$$B^j_{1,2} \in \mathrm{End}(V_j); \quad I^j \in \mathrm{Hom}(W_j, V_j); \quad J^j \in \mathrm{Hom}(V_j, W_j);$$
$$G^j \in \mathrm{Hom}(V_j, V_{j+1}); \quad F^j \in \mathrm{Hom}(V_j, V_{j-1})$$

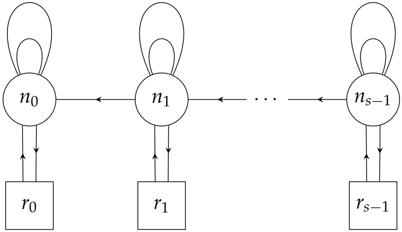
satisfying the relations:

$$[B^j_1, B^j_2] + I^j J^j = 0; \quad B^j_\alpha F^j - F^j B^{j-1}_\alpha; \quad G^j B^j_\alpha - B^{j-1}_\alpha G^j = 0;$$
$$J^j F^j = 0; \quad G^j I^j = 0; \quad F^j G^j = 0.$$

Under suitable stability conditions F^j are **injective** $\Rightarrow G^j = 0$.

Local D3-D7 theory (II)

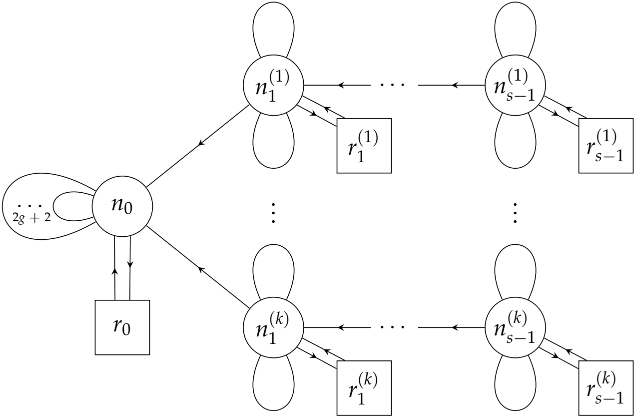
The previous data realize a representation of the following quiver, whose stable representations describe nested instantons.



The theory is **anomalous**: the coupling to the D7 branes **breaks half of the chiral supersymmetry** $\Rightarrow$ add $\overline{\text{D7}}$ branes.

Global D3/D7 theory

Gluing back all formal orbifold disks cut out at each marked point of Σ_g gives the global theory describing the D3/D7 branes.



Vacua of the effective field theory

NLSM of maps $\varphi : T^2 \to \mathcal{V}_{g,r,\underline{\lambda},n,\underline{\mu}}$

$\mathcal{V}_{g,r,\underline{\lambda},n,\underline{\mu}}$ describes the effective dynamics of D3 branes wrapping $T^2 \times \Sigma_{g,k}$ in the bg of the D7;

$\underline{\lambda}$ is a partition describing the behaviour of the r D7 branes at the punctures on $\Sigma_{g,k}$;

$\underline{\mu}$ is a partition describing the behaviour of the n D3 branes at the punctures on $\Sigma_{g,k}$;

$\pi_{\mathcal{V}} : \mathcal{V}_{g,r,\underline{\lambda},n,\underline{\mu}} \to \mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}}$ is actually a vector bundle over the moduli space describing **nested instantons**.

The bundle $\mathcal{V}_{g,r,\underline{\lambda},n,\underline{\mu}}$

Pullback of the natural projection $\pi : \mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}} \longrightarrow \mathcal{M}_{r,n}$

Coupling to the background connection of $\mathrm{Det}(\not{D})$

$$\mathcal{V}_{g,r,\underline{\lambda},n,\underline{\mu}} = \pi^* \left[(T^\vee \mathcal{M}_{r,n})^{\oplus g} \otimes \mathrm{Det}(\mathcal{T})^{\otimes(1-g)} \right]$$

Cotangent to instanton moduli space, describing g adjoint hypermultiplets

Tautological bundle over $\mathcal{M}_{r,n}$

What we have so far...

- The dynamics of the surface defect is described by a $2d$ $\mathcal{N} = (2,2)$ **comet-shaped** quiver gauge theory on T^2.
- The dynamics in the background of D7-branes is governed by a $2d$ $\mathcal{N} = (0,2)$ **framed** comet-shaped quiver gauge theory.
- The IR LEET of the D3-D7 system is a NLSM of maps from T^2 to the moduli space of BPS vacua, which is a vector bundle $\mathcal{V}_{g,r,\underline{\lambda},n,\underline{\mu}} \xrightarrow{\pi} \mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}}$.

Partition functions are computable via SUSY localization and give conjectural explicit formulae for (virtual) invariants of $\boldsymbol{\mathcal{V}_{g,r,\underline{\lambda},n,\underline{\mu}} \xrightarrow{\pi} \mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}}}$.

Moduli spaces

Geometrical interpretation:

When $r = 1$ and $k = 1$ nested pointlike instantons on $\mathbb{C}^2$ have a nice geometrical counterpart:

$$\mathcal{N}_{1,[1^1],n,\mu(n)} \simeq \mathrm{Hilb}^{(n_0, n_0 - n_{s-1}, \dots, n_0 - n_1)}(\mathbb{C}^2).$$

In general moduli spaces of nested instantons arise naturally in the context of computation of VW invariants. These have increasingly gained attention lately, especially among the mathematical community, and are now being widely studied.

In this context, the objects we work with provide the local model for the computation of monopole contributions to VW invariants.

SUSY partition functions

T^2−partition functions (I)

The D3-D7 T^2 partition function gives a **generating function for the (virtual equivariant) elliptic genera** of $\mathcal{V}_g$ over moduli spaces of nested instantons:

$$\mathcal{Z}_k^{\text{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \ldots, q_{s-1}^i\}) = \sum_n \sum_{\underline{\mu} \in \mathcal{P}(n)^k} (\mathbf{q}^{\underline{\mu}})^{\mathbf{r}} \, \mathrm{Ell}^{\text{vir}} \left(\mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}}, \mathcal{V}_g \right),$$

where

$$\mathcal{V}_g = \pi^* \left[(T^\vee \mathcal{M}_{r,n})^{\oplus g} \otimes \det\left(\mathcal{T}\right)^{\otimes(1-g)} \right]$$

Computation made possible by SUSY localization

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SUSY localization

Infinite dimensional version of ABBV localisation formula.

Let $\mathcal{X}$ be a supermanifold acted upon by a certain supergroup. Then, following an argument by Witten, one can show that in

$$\int_{\mathcal{X}} f = \int_{\mathcal{F}} \mathrm{d}\theta \int_{\mathcal{X}/\mathcal{F}} f$$

only $\mathcal{Q}-$fixed points contribute to the integral of invariant superfunctions f.

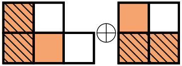
In this case SUSY action closes on $T-$action: **SUSY fixed points are $\boldsymbol{T}-$fixed points**.

$T-$fixed points

On $\mathcal{N}_{r,\lambda,n,\mu}$ there is a natural action of $T = (\mathbb{C}^*)^r \times (\mathbb{C}^*)^2$, where the $(\mathbb{C}^*)^2-$action is the lift of the natural equivariant action on $\mathbb{C}^2_{\epsilon_1,\epsilon_2}$, while $(\mathbb{C}^*)^r$, from the point of view of the D3-branes, is the effect of the complete higgsing of the $U(r)$ flavor group.

$T-$fixed points are isolated and finite, and can be described as $r-$tuples of **nested colored partitions** $\boldsymbol{\mu}_1 \subseteq \cdots \subseteq \boldsymbol{\mu}_{s-1} \subseteq \boldsymbol{\mu}_0$, with $|\boldsymbol{\mu}_0| = n_0$ and $|\boldsymbol{\mu}_0 \setminus \boldsymbol{\mu}_{i>0}| = n_{i>0}$.

As an example take $r = 2,\ n_0 = 9,\ n_1 = 5,\ n_2 = 3$: in the $T-$fixed locus $\boldsymbol{\mu}_1 \subseteq \boldsymbol{\mu}_2 \subseteq \boldsymbol{\mu}_0$ we have, *e.g.*

$((1,1) \subset (2,1) \subset (3,2)) \oplus ((2) \subset (2,1) \subset (2,2)) \longleftrightarrow$ 

T^2-partition functions (II)

The partition function localizes to a sum over $T-$fixed points. In the case $r = 1$:

$$\mathcal{Z}_k^{\mathrm{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}) = \sum_{\mu_0} q_0^{|\mu_0|} \sum_{\{\mu_1^i \subseteq \cdots \subseteq \mu_{s-1}^i} \prod_{j=1}^{k} \left(q_1^{|\mu_0/\mu_1^j|} \cdots \right.$$
$$\left. \cdots q_{s-1}^{|\mu_0/\mu_{s-1}^j|} \right) Z^{\mathrm{ell}}_{(\mu_0;\{\mu_1^i,\dots,\mu_{s-1}^i\})}$$

The term $Z^{\mathrm{ell}}_{(\mu_0;\{\mu_1^i,\dots,\mu_{s-1}^i\})}$ represents the contribution of each tail of the comet shaped quiver at fixed instanton profile:

$$Z^{\mathrm{ell}}_{(\mu_0;\{\mu_1^i,\dots,\mu_{s-1}^i\})} = \mathcal{L}^{\mathrm{ell}}_{\mu_0} \mathcal{N}^{\mathrm{ell}}_{\mu_0} \overline{\mathcal{N}}^{\mathrm{ell}}_{\mu_0} \mathcal{E}^{\mathrm{ell}}_{g,\mu_0} \overline{\mathcal{E}}^{\mathrm{ell}}_{g,\mu_0} \prod_{i=1}^{k} \mathcal{T}^{\mathrm{ell}}_{\mu_0,\mu_1^i} \overline{\mathcal{T}}^{\mathrm{ell}}_{\mu_0,\mu_1^i} \mathcal{W}^{\mathrm{ell}}_{\mu_0,\mu_1^i,\dots,\mu_{s-1}^i}.$$

T^2-partition functions (III)

We defined:

$$\mathcal{N}^{\text{ell}}_{\mu_0} = \prod_{s\in Y_{\mu_0}} \frac{1}{\theta_1(\tau|E(s))\theta_1(\tau|E(s)-\epsilon)},$$

$$\mathcal{E}^{\text{ell}}_{g,\mu_0} = \prod_{s\in Y_{\mu_0}} \theta_1^g(\tau|E(s)-m)\theta_1^g(\tau|E(s)-\epsilon+m),$$

$$\mathcal{T}^{\text{ell}}_{\mu_0,\mu_1} = \prod_{i=1}^{M_0}\prod_{j=1}^{\mu_{0,i}-\mu_{1,i}} \theta_1(\tau|\epsilon_1 i+\epsilon_2(j+\mu'_{1,i}))$$

$$\mathcal{W}_{\mu_0,\dots,\mu_{s-1}} = \prod_{k=0}^{s-2}\left[\prod_{i=1}^{M_0}\prod_{j=1}^{N_0}\frac{\theta_1(\tau|\epsilon_1(i+\mu_{k,j})+\epsilon_2(\mu'_{k+1,i}-j+1))}{\theta_1(\tau|\epsilon_1(i-\mu_{k+1,j})+\epsilon_2(\mu'_{k+1,i}-j+1))}\right.$$
$$\left.\prod_{i=1}^{M_0}\prod_{j=1}^{N_0}\frac{\theta_1(\tau|\epsilon_1(i-\mu_{k+1,j})+\epsilon_2(\mu'_{0,i}-j+1))}{\theta_1(\tau|\epsilon_1(i-\mu_{k,j})+\epsilon_2(\mu'_{0,i}-j+1))}\right],$$

with $E(s) = -\epsilon_1 l(s)+\epsilon_2(a(s)+1)$.

T^2−partition functions (IV)

If we sum over the "tail partitions" $\{\mu_1^i, \dots, \mu_{s-1}^i\}$ first, we can rewrite the partition function as

$$\mathcal{Z}_k^{\text{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}) = \sum_{\mu_0} \mathcal{Y}_{g,\mu_0}^{\text{ell}} \prod_{i=1}^{k} P_{i,\mu_0}^{\text{ell}},$$

with

$$\begin{aligned}
\mathcal{Y}_{g,\mu_0}^{\text{ell}} &= \mathcal{L}_{\mu_0}^{\text{ell}} \mathcal{N}_{\mu_0}^{\text{ell}} \overline{\mathcal{N}}_{\mu_0}^{\text{ell}} \mathcal{E}_{g,\mu_0}^{\text{ell}} \overline{\mathcal{E}}_{g,\mu_0}^{\text{ell}}, \\
P_{i,\mu_0}^{\text{ell}} &= \sum_{\mu_1^i \subseteq \cdots \subseteq \mu_{s-1}^i} \mathcal{T}_{\mu_0,\mu_1^i}^{\text{ell}} \overline{\mathcal{T}}_{\mu_0,\mu_1^i}^{\text{ell}} \mathcal{W}_{\mu_0,\mu_1^i,\dots,\mu_{s-1}^i}^{\text{ell}}.
\end{aligned}$$

An interesting property is that:

$$\mathcal{Z}_k^{\text{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}) \xrightarrow[\substack{j=1,\dots,s-1 \\ i=k'+1,\dots,k}]{q_j^i \to 1} \mathcal{Z}_{k'<k}^{\text{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}).$$

Trigonometric limit

When $\tau \to \mathrm{i}\infty$ (degeneration limit of the torus) $\mathcal{Z}_k^{\text{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots\})$ recovers the $5d$ partition function:

$$\mathcal{Z}_k^{\text{ell}}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}) \xrightarrow{\tau \to \mathrm{i}\infty} \mathcal{Z}_k^{\mathbb{R}^4 \times S^1}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}).$$

Geometric interpretation for the $5d$ partition function:

$$\mathcal{Z}_k^{\mathbb{R}^4 \times S^1}(\mathcal{C}_g; q_0, \{q_1^i, \dots, q_{s-1}^i\}) = \sum_n \sum_{\underline{\mu} \in \mathcal{P}(n)^k} (\mathbf{q}^{\underline{\mu}})^{\mathbf{r}} \chi_{-y}^{\text{vir}} \left(\mathcal{N}_{r, \underline{\lambda}, n, \underline{\mu}}, \mathcal{V}_g \right).$$

Analogously to the T^2 case it's expressed in terms of fixed instanton profile contributions. After decoupling the $\overline{\mathrm{D7}}-$branes:

$$Z_{(\mu_0, \{\mu_1^i, \dots, \mu_{s-1}^i\})} = \mathcal{L}_{\mu_0} \mathcal{N}_{\mu_0} \mathcal{E}_{g, \mu_0} \prod_{i=1}^k \mathcal{T}_{\mu_0, \mu_1^i} \mathcal{W}_{\mu_0, \mu_1^i, \dots, \mu_{s-1}^i}.$$

Polynomial contributions

Conjecture

Resumming over the monodromies we get a simplified structure. In particular

$$P_{\mu_0} = \sum_{\mu_1 \subseteq \cdots \subseteq \mu_{s-1}} \mathcal{T}_{\mu_0,\mu_1} \mathcal{W}_{\mu_0,\ldots,\mu_{s-1}}$$

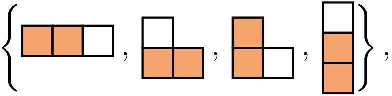
are polynomials in the characters of $(\mathbb{C}^*)^2 \curvearrowright \mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}}$.

- P_{μ_0} always contains modified Macdonald polynomials $\tilde{H}_{\mu_0}$.
- $\mathcal{Z}_1(\mathcal{C}_g; q_0, \ldots, q_{s-1})$ provides a **virtual refinement** of LHRV formulae, while $\mathcal{Z}_1^{\mathrm{ell}}(\mathcal{C}_g; q_0, \ldots, q_{s-1})$ is their virtual elliptic refinement.

Some examples (I)

Consider $\mathcal{N}(1,3,1)$: it is **smooth**.[Cheah, 1998] Its fixed point locus is given by:

$$\{\mu_1 \subseteq \mu_0\} =$$



whence

$$\begin{cases} P^{\text{ell}}_{\ydiagram{3}}(\mathbf{x};\epsilon_1,\epsilon_2)|_{x_0^2 x_1} = \dfrac{\theta_1(\tau|3\epsilon_1)}{\theta_1(\tau|\epsilon_1)} \\ P^{\text{ell}}_{\ydiagram{2,1}}(\mathbf{x};\epsilon_1,\epsilon_2)|_{x_0^2 x_1} = \left(\dfrac{\theta_1(\tau|2\epsilon_1-\epsilon_2)}{\theta_1(\tau|\epsilon_1-\epsilon_2)} + \dfrac{\theta_1(\tau|2\epsilon_2-\epsilon_1)}{\theta_1(\tau|\epsilon_2-\epsilon_1)}\right) \\ P^{\text{ell}}_{\ydiagram{1,1,1}}(\mathbf{x};\epsilon_1,\epsilon_2)|_{x_0^2 x_1} = \dfrac{\theta_1(\tau|3\epsilon_2)}{\theta_1(\tau|\epsilon_2)} \end{cases}$$

Some examples (I)

Consider $\mathcal{N}(1,3,1)$: it is **smooth**.[Cheah, 1998] Its fixed point locus is given by:

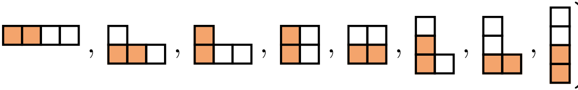
$$\{\mu_1 \subseteq \mu_0\} = \left\{ \ydiagram[*(orange)]{2}*{3}\, , \ \ydiagram[*(orange)]{0,2}*{1,2}\, , \ \ydiagram[*(orange)]{1,1}*{1,2}\, , \ \ydiagram[*(orange)]{0,1,1}*{1,1,1} \right\},$$

whence

$$\begin{cases} P_{\ydiagram{3}}(\mathbf{x};q,t)|_{x_0^2x_1} = 1+q+q^2 = \tilde{H}_{\ydiagram{3}}(\mathbf{x};q,t)|_{x_0^2x_1} \\ P_{\ydiagram{1,2}}(\mathbf{x};q,t)|_{x_0^2x_1} = 1+q+t = \tilde{H}_{\ydiagram{1,2}}(\mathbf{x};q,t)|_{x_0^2x_1} \\ P_{\ydiagram{1,1,1}}(\mathbf{x};q,t)|_{x_0^2x_1} = 1+t+t^2 = \tilde{H}_{\ydiagram{1,1,1}}(\mathbf{x};q,t)|_{x_0^2x_1} \end{cases}$$

Some examples (II)

Consider $\mathcal{N}(1,4,2)$: it is **singular**.[Cheah, 1998] Its fixed point locus is given by:

$$\{\mu_1 \subseteq \mu_0\} = \Big\{$$  $$\Big\},$$

whence

$$P(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+q+2q^2+q^3+q^4-q^2t-q^3t-2q^4t-q^5t-q^6t,$$

$$P(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+q+2q^2+t+qt-q^2t-q^3t-q^4t-qt^2-q^2t^2-q^3t^2,$$

$$P(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+q+q^2+t+qt+t^2-q^2t-qt^2-2q^2t^2-q^3t^2-q^2t^3,$$

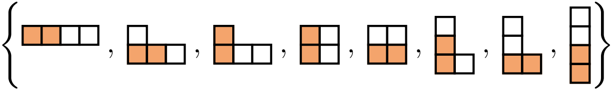
$$P(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+q+t+qt+2t^2-q^2t-qt^2-q^2t^2-qt^3-q^2t^3-qt^4,$$

$$P(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+t+2t^2+t^3+t^4-qt^2-qt^3-2qt^4-qt^5-qt^6.$$

Some examples (II)

Consider $\mathcal{N}(1,4,2)$: it is **singular**.[Cheah, 1998] Its fixed point locus is given by:

$$\{\mu_1 \subseteq \mu_0\} =$$



while

$$\tilde{H}_{\ydiagram{4}}(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+q+2q^2+q^3+q^4,$$

$$\tilde{H}_{\ydiagram{3,1}}(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+q+2q^2+t+qt,$$

$$\tilde{H}_{\ydiagram{2,2}}(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+q+q^2+t+qt+t^2,$$

$$\tilde{H}_{\ydiagram{2,1,1}}(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+q+t+qt+2t^2,$$

$$\tilde{H}_{\ydiagram{1,1,1,1}}(\mathbf{x},q,t)|_{x_0^2x_1^2} = 1+t+2t^2+t^3+t^4.$$

Summary

- We studied **surface operators** in $4d$ gauge theories on a local surface $T^2 \times \Sigma_{g,k}$ describing the parabolic reduction of the gauge connection at punctures on $\Sigma_{g,k}$.
- The moduli space of vacua of the SUSY gauge theory, $\mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}}$, describes **nested instantons**.
- The T^2 partition function provides conjectural explicit formulae for virtual equivariant elliptic genera of vector bundles over $\mathcal{N}_{r,\underline{\lambda},n,\underline{\mu}}$.
- Nested instanton moduli spaces also have a nice geometrical interpretation which seems to be relevant in VW invariants computations.
- The QM limit of the T^2 partition function has a connection to the cohomology of GL_n-character varieties on $\Sigma_{g,k}$ through a virtual refinement of the LHRV conjecture.

Open questions

- Construction of the moduli space from disk amplitudes in brane theories.
- Characterization of the P_{μ_0} polynomials (integrable systems, VOA).
- Modular properties of the partition functions are still to be studied.
- It would be interesting to study the case of S a general elliptic fibration over Σ_g.
- Uplift to $F-$theory.
- It would be interesting to have a geometrical interpretation of the more general nested instantons quiver $\mathcal{N}_{r,\lambda,n,\underline{\mu}}$.

Thank You
for Your Attention