The first law of complexity

Alice Bernamonti

UNIVERSITÀ DEGLI STUDI FIRENZE

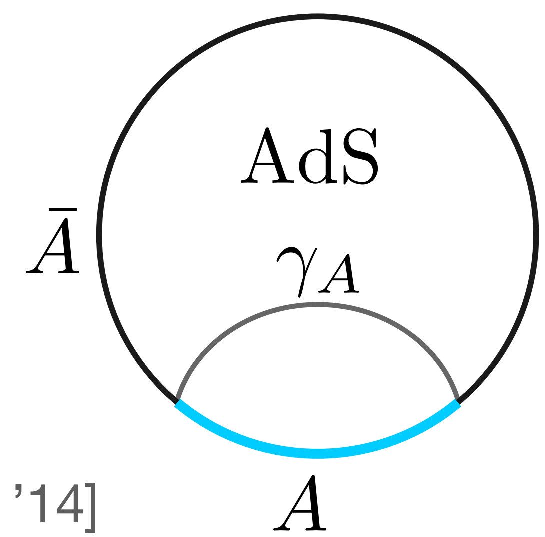
Theories of Fundamental Interactions 2019
INFN Torino, 21-23 Oct 2019

Based on PRL 123 (2019) no.8, 081601 & work in progress
with F. Galli, J. Hernandez, R. Myers, S. Ruan and J. Simón

Quantum information & AdS/CFT

- Cross-over between quantum information and holography led to fruitful bulk-boundary dialogue:
  - ➡ new lessons about QFTs & quantum gravity
- Holographic entanglement entropy [Ryu and Takayanagi, '06]

$$S_A = \mathrm{Min} \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$



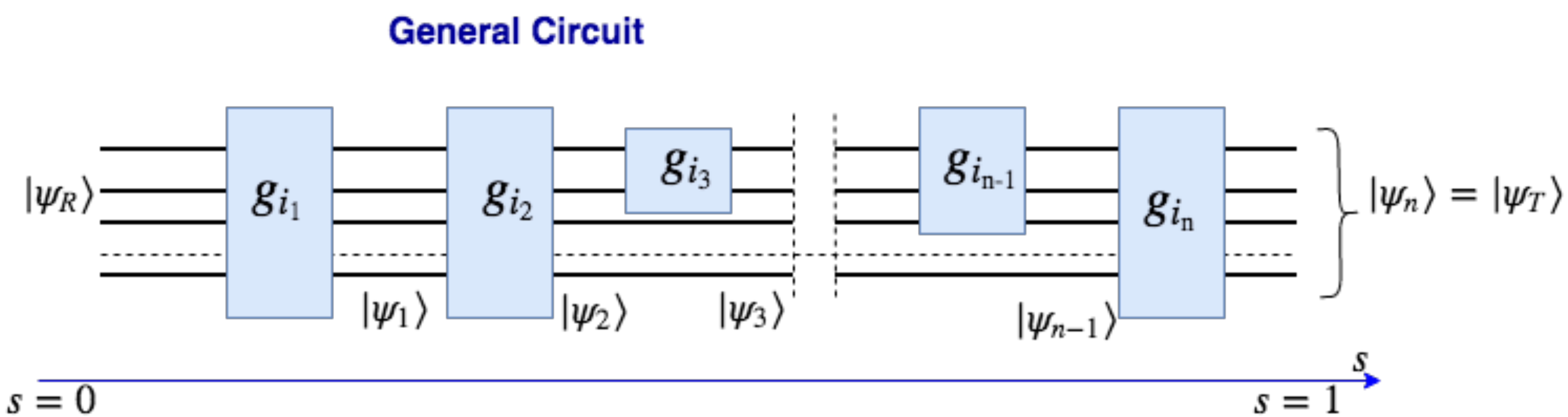
  - ➡ spacetime geometry ~ entanglement
  - ➡ Einstein's eq. from first law of entanglement entropy [Faulkner, Guica, Hartman, Myers, Van Raamsdonk '14]
- Entanglement entropy is not enough (only probes the eigenvalues of the density matrix)
- Operational perspective: generating spacetime, rather than probing it

Quantum circuit complexity

- How difficult is it to implement a task?
  How difficult is it to prepare a particular state?
- Given a reference state $|\Psi_R\rangle$, generate -approximately- a target state

$$|\Psi_T\rangle = U_T|\Psi_R\rangle$$

with a set of generators $\mathcal{O}_I$ of elementary gates



- Complexity quantifies the cost of the optimal circuit generating the unitary U_T, or the state $|\Psi_T\rangle$

Nielsen's geometric approach

[Nielsen et al '06]

- Continuum representation of unitary transformations

$$U(\sigma) = \vec{\mathcal{P}} \exp\left[-i \int_0^\sigma ds\, H(s)\right] \quad \text{with} \quad H(s) = \sum Y^I(s) \mathcal{O}_I$$

control functions

- $U(\sigma) \sim$ a path in the space of unitaries. For $\sigma \in [0,1]$:

$$U(\sigma = 0) = \mathbb{I} \qquad \text{and} \qquad U(\sigma = 1) = U_T$$

- Introducing coordinates x^a on the space of unitaries

$$\mathcal{C}(|\Psi_T\rangle) \equiv \text{Min} \int_0^1 F(x^a, \dot{x}^a)$$

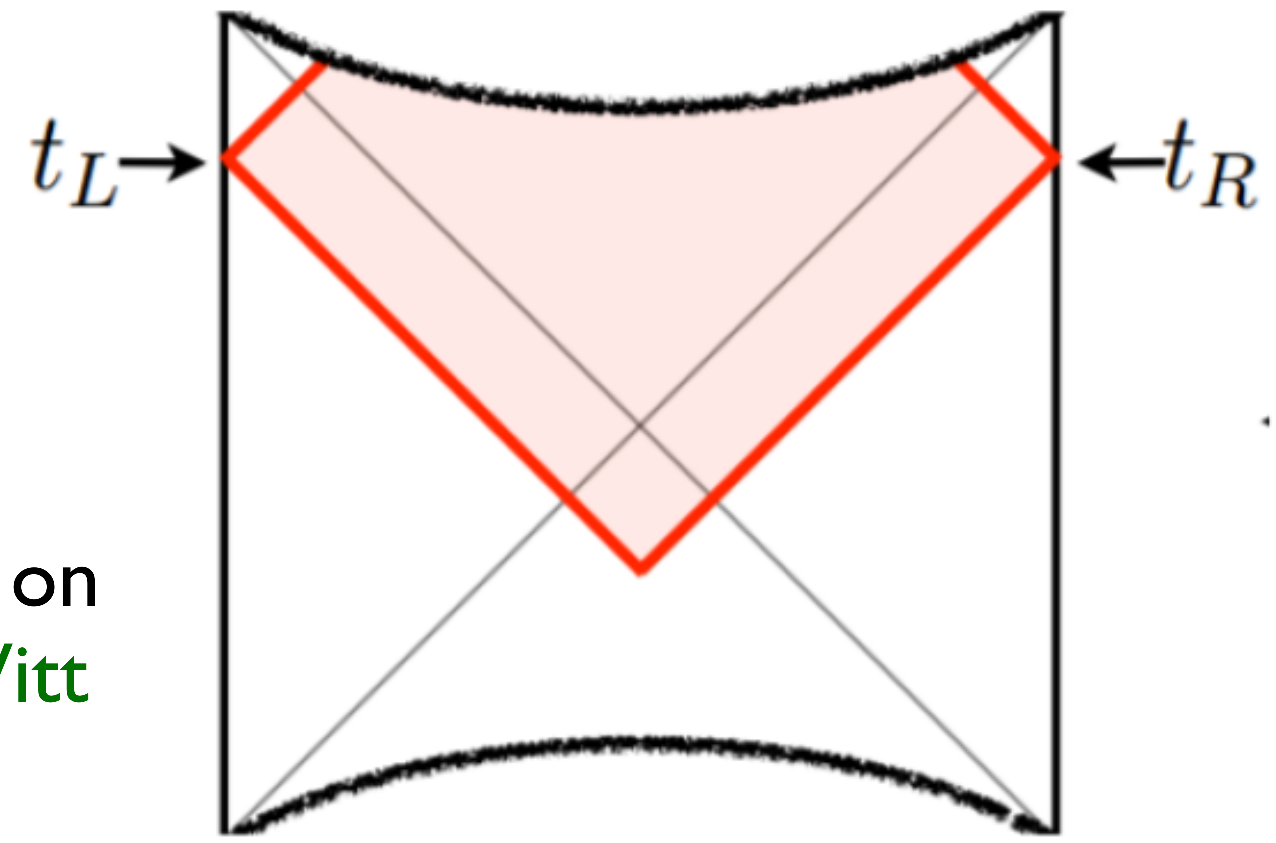
  for a choice of cost function $F(x^a, \dot{x}^a)$

- Optimal circuits generating U_T are mapped to globally minimizing cost trajectories in the space of unitaries.

Holographic complexity = Action

[Brown, Roberts, Susskind, Swingle, Zhao '16]

$$\mathcal{C}_A(\Sigma) = \frac{I_{\rm WDW}}{\pi}$$



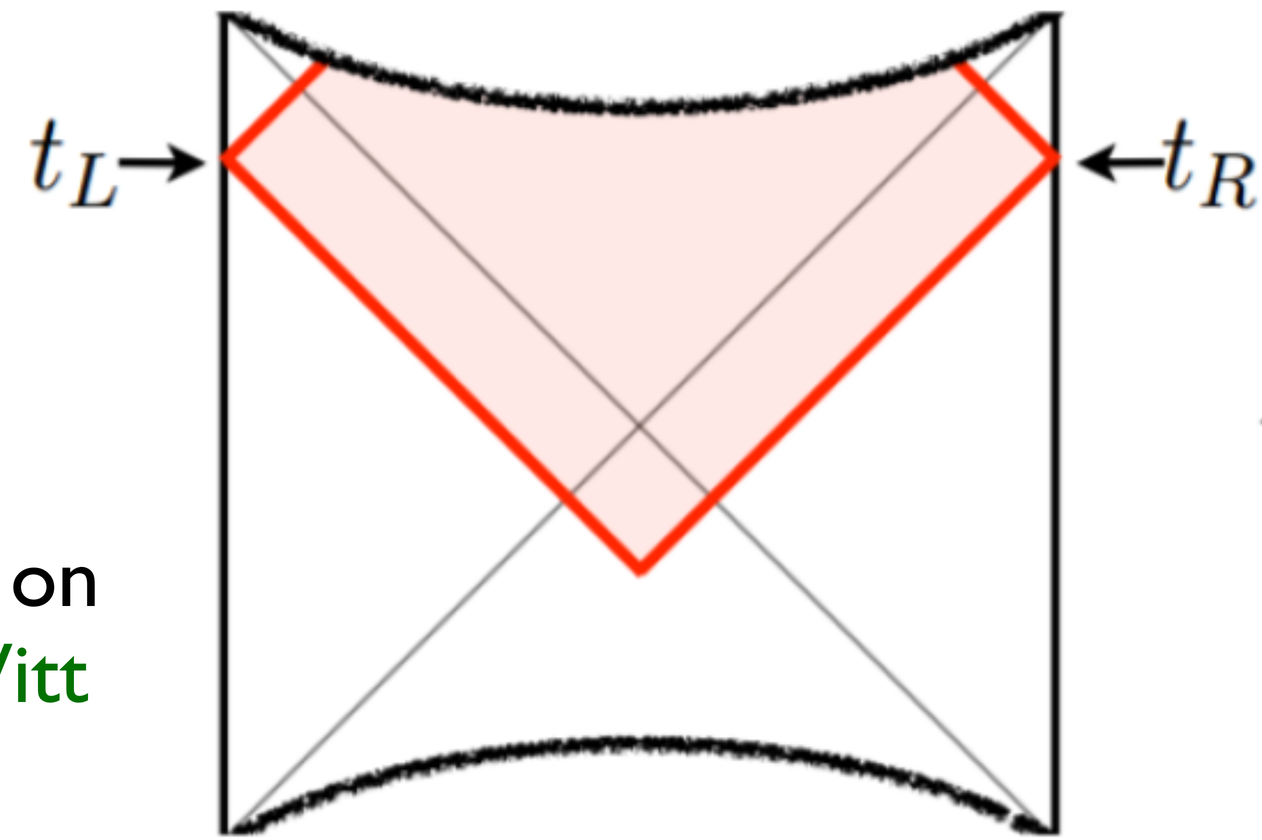
Complexity of $|\Psi_T\rangle$ on boundary Cauchy surface Σ = Gravitational action $I_{\rm WDW}$ on Wheeler-DeWitt patch

- WDW patch: domain of dependence of a bulk spatial slice anchored on Σ

Holographic complexity = Action

[Brown, Roberts, Susskind, Swingle, Zhao '16]

$$\mathcal{C}_A(\Sigma) = \frac{I_{\rm WDW}}{\pi}$$



Complexity of $|\Psi_T\rangle$ on boundary Cauchy surface Σ = Gravitational action $I_{\rm WDW}$ on Wheeler-DeWitt patch

- WDW patch: domain of dependence of a bulk spatial slice anchored on Σ
- This gravitational observable probes the black hole interior
- It reproduces the expected complexity linear growth (at lates times)
- $|\Psi_T\rangle$ on Σ ⟷ classical gravity dual $(g, \{\phi\})$
- $|\Psi_R\rangle$?? Gates ?? Cost function ??

Complexity variations

Study variations of complexity:

$$\delta\mathcal{C} \equiv \mathcal{C}(|\Psi_T + \delta\Psi\rangle) - \mathcal{C}(|\Psi_T\rangle)$$

Why?

- Focus on the dependence on $|\Psi_T\rangle$ and its perturbations, which have a clear geometric interpretation
- Independent of $|\Psi_R\rangle$
- Extract information about implicit choice of cost function $F(x^a, \dot{x}^a)$
- Study properties of new gravitational observable $\mathcal{C}_A$
- Operational perspective: what is the cost of perturbing spacetime?

First law of complexity

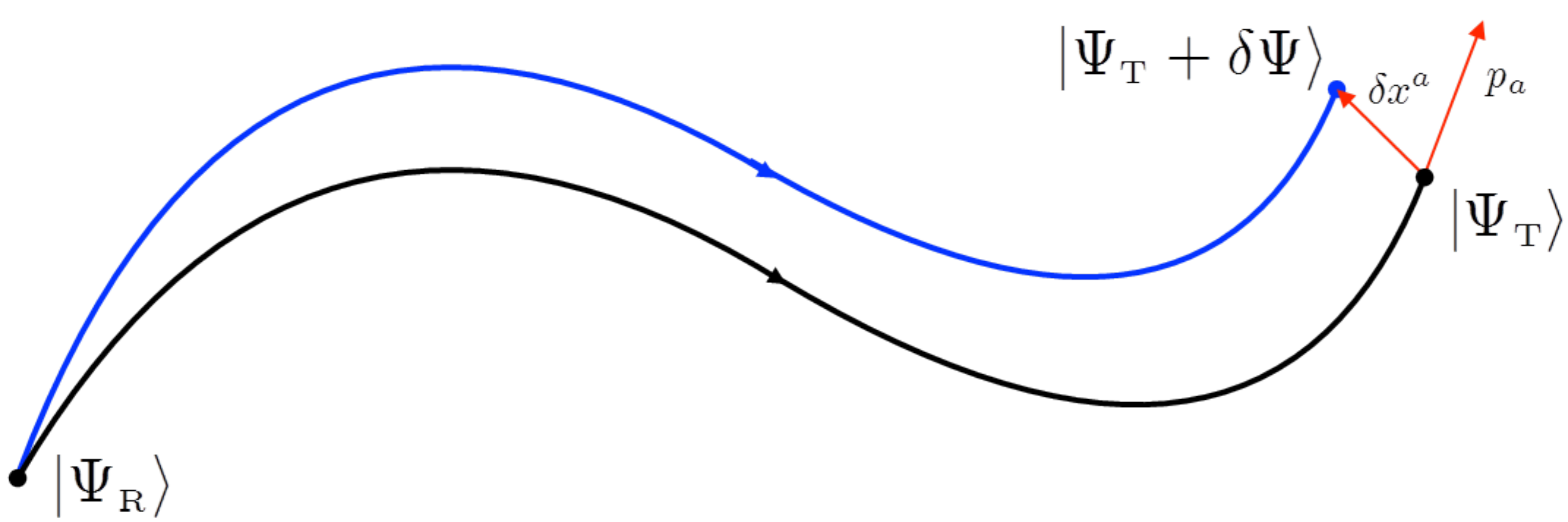
Using the analogy of Nielsen's approach to classical mechanics:

- 1st order variation

$$\delta\mathcal{C} = p_a \delta x^a \big|_{s=1} \quad \text{with} \quad p_a = \frac{\partial F}{\partial \dot{x}^a}$$

- 2nd order variation

$$\delta\mathcal{C} = \frac{1}{2}\delta p_a \delta x^a \Big|_{s=1} \quad \text{with} \quad \delta p_a = \delta x^b \frac{\partial^2 F}{\partial x^b \partial \dot{x}^a} + \delta \dot{x}^b \frac{\partial^2 F}{\partial \dot{x}^b \partial \dot{x}^a}$$



First law of complexity

Using the analogy of Nielsen's approach to classical mechanics:

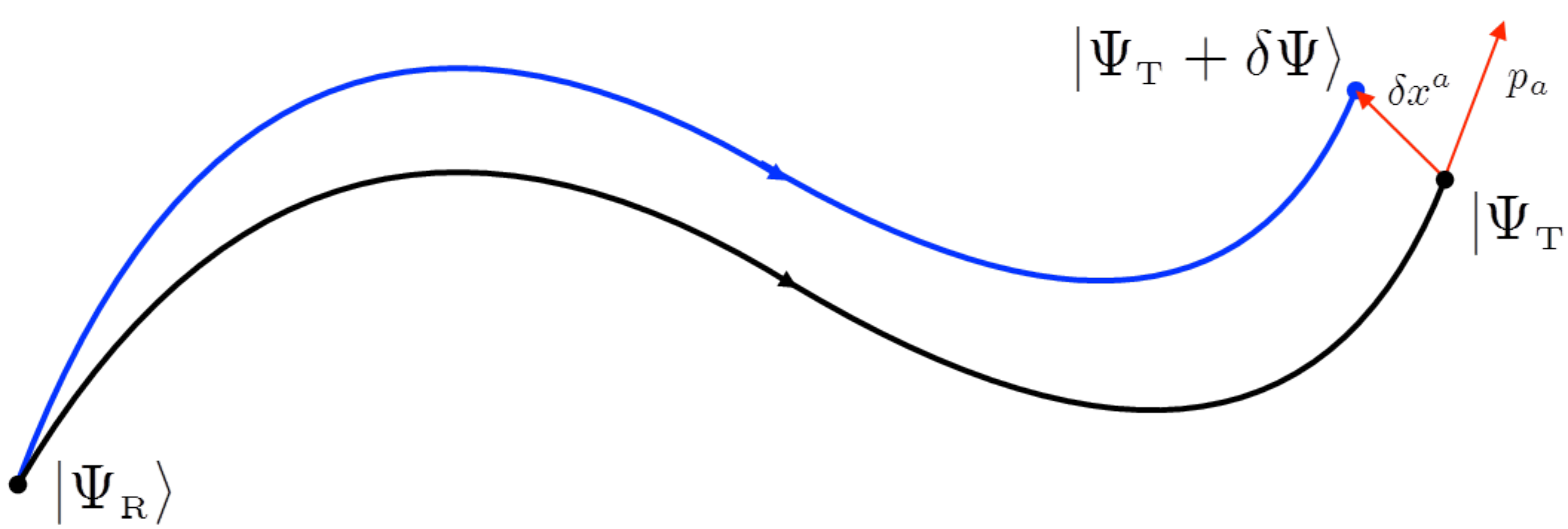
- 1st order variation

$$\delta\mathcal{C} = p_a \delta x^a \big|_{s=1} \quad \text{with} \quad p_a = \frac{\partial F}{\partial \dot{x}^a}$$

- 2nd order variation

$$\delta\mathcal{C} = \frac{1}{2}\delta p_a \delta x^a \Big|_{s=1} \quad \text{with} \quad \delta p_a = \delta x^b \frac{\partial^2 F}{\partial x^b \partial \dot{x}^a} + \delta \dot{x}^b \frac{\partial^2 F}{\partial \dot{x}^b \partial \dot{x}^a}$$

only contributions from the endpoint



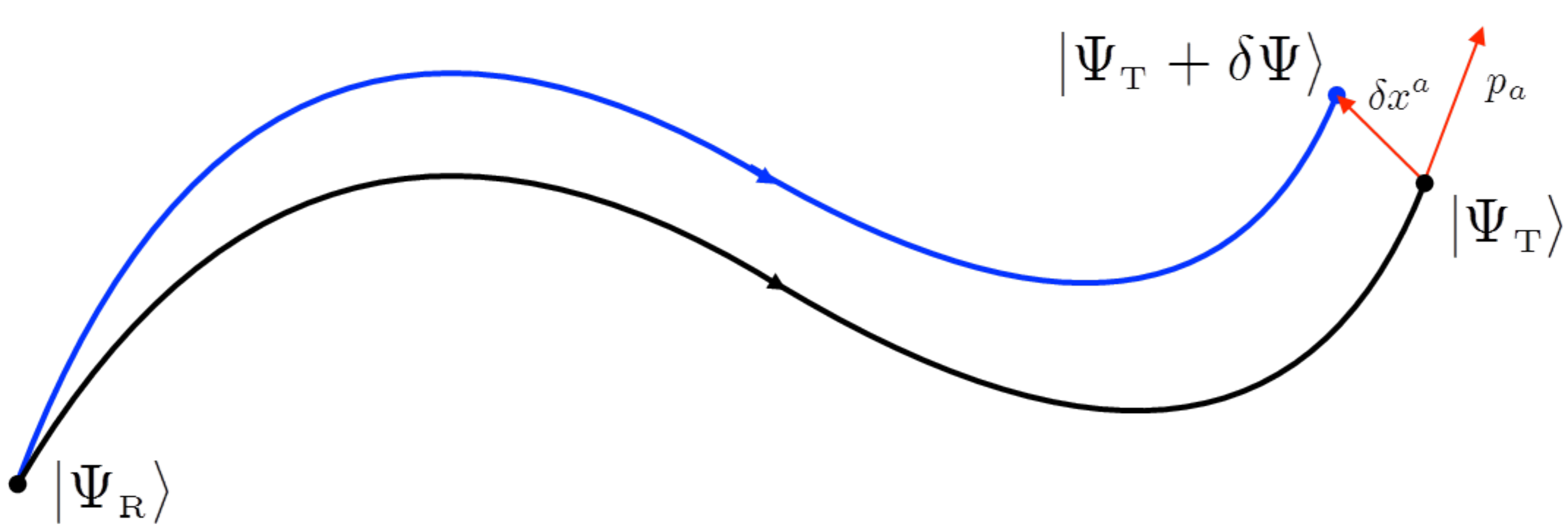
Caveat

- $\mathcal{C}$ ~ minimal cost, *i.e.* global minimum over all possible circuits

- Assume: the circuit globally minimizing the cost function stays close to the original optimal circuit, *i.e.* the family of globally minimizing circuits is continuous in the amplitude of the perturbation.

- It does not hold in general, but we expect it to hold in the example we consider (*cf.* free QFT complexity calculations).

[Guo, Hernandez, Myers, Ruan '18]



Holographic framework

Bulk:

$$I_{\rm bulk} = \frac{1}{16\pi G_N}\int d^4y\sqrt{-g}\left[R + \frac{6}{L^2} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right]$$

$|\Psi_T\rangle$: empty AdS_4 of radius L

$|\Psi_T + \delta\Psi\rangle$: small amplitude coherent state of the bulk scalar

- Given $m_\phi^2 = 0$ scalar: $\phi(y^\mu) = \sum_n \left(u_n(y^\mu)a_n + u_n^*(y^\mu)a_n^\dagger\right)$

  we consider an excited state

  $$|\varepsilon\alpha_j\rangle = \mathrm{e}^{\varepsilon\sum D(\alpha_j)}|0\rangle \text{ with } D(\alpha_j) = \alpha_j a_j^\dagger - \alpha_j^* a_j$$

  where a few modes $\{j\}$ are given classical expectation values

  $$\langle\varepsilon\alpha_j|\phi|\varepsilon\alpha_j\rangle = \varepsilon\sum\left(\alpha_j u_j + \alpha_j^* u_j^*\right) \equiv \varepsilon\phi_{\rm cl}$$

  and work perturbatively in $\varepsilon \ll 1$

Holographic framework

<u>Boundary</u>:

- In AdS/CFT, bulk and boundary theories provide equivalent descriptions of the same quantum states.
- $|\varepsilon\alpha_j\rangle$ are also coherent states in the boundary CFT corresponding to excitations of the vacuum by the dual generalized free field operator $\mathcal{O}_{\Delta=3}$ and its descendants $\Box^j\mathcal{O}_{\Delta=3}$

Consequences:

- Quantum circuit technology in QFT [Jefferson, Myers '17] applied to coherent states [Guo, Hernandez, Myers, Ruan '18] can be equivalently applied in the bulk.
- Classical gravity duals $(g, \varepsilon\phi_{\rm cl})$ are suitable to compute holographic complexity.

Complexity = Action

Variational principle for Dirichlet BCs on $\partial\mathrm{WDW}$

[Lehner, Myers, Poisson, Sorkin '16]

$$I \supset I_{\rm bulk} + I_{\rm null} + I_{\rm counterterm}$$

$$= \frac{1}{16\pi G_N}\int d^4y\sqrt{-g}\left[R + \frac{6}{L^2} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right]$$

$$+\frac{1}{8\pi G_N}\int_{\partial\mathrm{WDW}} ds\, d^2\Omega\,\sqrt{\gamma}\,\kappa + \frac{1}{8\pi G_N}\int_{\partial\mathrm{WDW}} ds\, d^2\Omega\,\sqrt{\gamma}\,\Theta\,\log(\ell_{\rm ct}\Theta)$$

- κ measures how much non-affine the parametrization s of $\partial\mathrm{WDW}$ is
- $\Theta = \partial_s \log\sqrt{\gamma}$ expansion scalar of null generators
- $\ell_{\rm ct}$ arbitrary scale

Variation of holographic complexity

$$\delta I \equiv I[g_0 + \delta g, \delta\phi] - I[g_0, 0]$$

for a spherically symmetric perturbation $(\delta g, \delta\phi)$ in a small amplitude expansion $\delta\phi = \varepsilon\phi_{\rm cl}$ around global AdS_4 (g_0)

Structure at $\mathcal{O}(\varepsilon^2)$:

$$\delta\mathcal{C}_A(\Sigma) = \frac{\delta I}{\pi} = \frac{1}{\pi}(\delta I_{\rm WDW} + I_{\delta\rm WDW})$$

Variation of holographic complexity

$$\delta I \equiv I[g_0 + \delta g, \delta\phi] - I[g_0, 0]$$

for a spherically symmetric perturbation $(\delta g, \delta\phi)$ in a small amplitude expansion $\delta\phi = \varepsilon\phi_{\mathrm{cl}}$ around global AdS_4 (g_0)

Structure at $\mathcal{O}(\varepsilon^2)$:

$$\delta\mathcal{C}_A(\Sigma) = \frac{\delta I}{\pi} = \frac{1}{\pi}(\delta I_{\mathrm{WDW}} + I_{\delta\mathrm{WDW}})$$

captures $(\delta g, \delta\phi)$ on undeformed WDW patch

Variation of holographic complexity

$$\delta I \equiv I[g_0 + \delta g, \delta\phi] - I[g_0, 0]$$

for a spherically symmetric perturbation $(\delta g, \delta\phi)$ in a small amplitude expansion $\delta\phi = \varepsilon\phi_{\text{cl}}$ around global AdS_4 (g_0)

Structure at $\mathcal{O}(\varepsilon^2)$:

$$\delta\mathcal{C}_A(\Sigma) = \frac{\delta I}{\pi} = \frac{1}{\pi}(\delta I_{\text{WDW}} + I_{\delta\text{WDW}})$$

captures $(\delta g, \delta\phi)$ on undeformed WDW patch

captures g_0 on deformed WDW patch

Variation of holographic complexity

$$\delta \mathcal{C}_A(\Sigma) = \frac{\delta I_{\rm matter}}{\pi} = -\frac{\varepsilon^2}{64\pi^2 G_N} \int_{\partial \rm WDW} ds\, d^2\Omega\, \sqrt{\gamma}\, \partial_s(\phi_{\rm cl}^2)$$

- Pure $\mathcal{O}(\varepsilon^2)$ matter contribution
- Localized on boundary of undeformed WDW patch
- Independent of arbitrary counterterm scale $\ell_{\rm ct}$

Variation of holographic complexity

$$\delta\mathcal{C}_A(\Sigma) = \frac{\delta I_{\rm matter}}{\pi} = -\frac{\varepsilon^2}{64\pi^2 G_N}\int_{\partial\rm WDW} ds\, d^2\Omega\,\sqrt{\gamma}\,\partial_s(\phi_{\rm cl}^2)$$

- Pure $\mathcal{O}(\varepsilon^2)$ matter contribution
- Localized on boundary of undeformed WDW patch
- Independent of arbitrary counterterm scale $\ell_{\rm ct}$

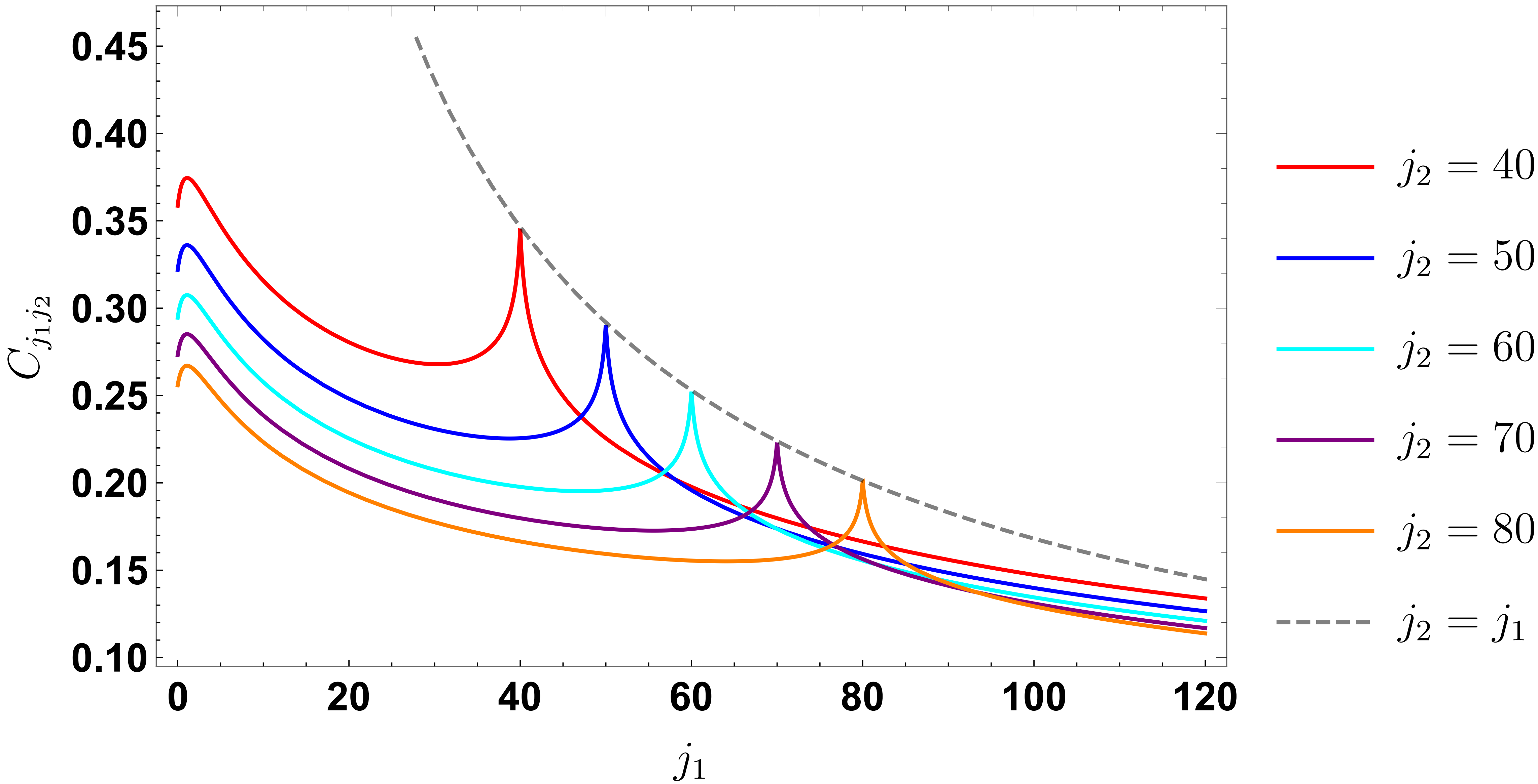
Explicitly at $t=0$

$$\delta\mathcal{C}_A(\Sigma) = \frac{\varepsilon^2}{\pi^2}\sum_{j_1,j_2}\alpha_{j_1}\alpha_{j_2}C_{j_1 j_2}$$

$$C_{j_1j_2} = \sqrt{\frac{(j_1+\frac{3}{2})(j_2+\frac{3}{2})}{(j_1+1)(j_1+2)(j_2+1)(j_2+2)}}$$
$$\times\left(H_{j_1+\frac{1}{2}}+H_{j_1+\frac{3}{2}}+H_{j_2+\frac{1}{2}}+H_{j_2+\frac{3}{2}}-H_{j_1+j_2+\frac{5}{2}}-H_{j_1-j_2-\frac{1}{2}}-2+4\log 2\right)$$

with $H_\beta = \partial_\beta \log\Gamma(\beta+1)+\gamma$ harmonic numbers

Main features



- Two peaks at $j_1 = 1$ and $j_1 = j_2$, with values decaying as j_2 grows
- Near the diagonal peak: $$\lim_{j\to\infty} C_{j,j+\delta j} = 3\frac{\log 2j}{j} + \mathcal{O}\left(\frac{1}{j}\right)$$

Remarks

Holographic

- $\delta\mathcal{C}_A$ is scale independent: UV finite and independent of $\ell_{\rm ct}/L$
- $I_{\rm counterterm}$ is crucial for gravitational action cancellation
- $\delta\mathcal{C}_A$ is an integral over boundary of undeformed WDW patch

Quantum circuit

- $\delta\mathcal{C}_A \sim \varepsilon^2\alpha^2 \Rightarrow p_a\delta x^a|_{s=1} = 0$

  coherent state directions are orthogonal to the direction along the circuit preparing the CFT vacuum

- $\delta\mathcal{C}$ only depends on data at the end of the circuit

➡ does the quantum circuit end on ∂WDW?

- Specific choices of cost function F lead to relation with $C_{j_1j_2}$

Comparison with $\kappa = 2$ measure

$$F_{\kappa=2} = \sum_I |Y^I|^2$$

[Guo, Hernandez, Myers, Ruan '18]

$$\delta\mathcal{C}_{\kappa=2}(\Sigma) = \frac{\varepsilon^2}{\pi^2} \sum_{j_1, j_2} \alpha_{j_1} \alpha_{j_2} C^{\kappa=2}_{j_1 j_2}$$

$$C^{\kappa=2}_{j_1 j_2} \to \delta_{j_1 j_2} \frac{\mu R}{(\mu x_0)^2} \frac{\pi^2}{j_1} \log \frac{2 j_1}{\mu R}$$

Comparison with $\kappa = 2$ measure

$$F_{\kappa=2} = \sum_I |Y^I|^2$$

[Guo, Hernandez, Myers, Ruan '18]

$$\delta\mathcal{C}_{\kappa=2}(\Sigma) = \frac{\varepsilon^2}{\pi^2} \sum_{j_1, j_2} \alpha_{j_1} \alpha_{j_2} C^{\kappa=2}_{j_1 j_2}$$

$$C^{\kappa=2}_{j_1 j_2} \to \delta_{j_1 j_2} \frac{\mu R}{(\mu x_0)^2} \frac{\pi^2}{j_1} \log \frac{2 j_1}{\mu R}$$

frequency $|\Psi_R\rangle$

scale of coherent state gates

length scale in the metric
to produce a dimensionful time

Comparison with $\kappa = 2$ measure

$$F_{\kappa=2} = \sum_I |Y^I|^2$$

[Guo, Hernandez, Myers, Ruan '18]

$$\delta\mathcal{C}_{\kappa=2}(\Sigma) = \frac{\varepsilon^2}{\pi^2} \sum_{j_1, j_2} \alpha_{j_1} \alpha_{j_2} C^{\kappa=2}_{j_1 j_2}$$

$$C^{\kappa=2}_{j_1 j_2} \to \delta_{j_1 j_2} \frac{\mu R}{(\mu x_0)^2} \frac{\pi^2}{j_1} \log \frac{2 j_1}{\mu R}$$

- $\mathcal{O}(\varepsilon^2)$ ✓
- large radial quantum number limit ✓
- all coherent states are mutually orthogonal ✗
- absence of scales requires $\mu x_0 \sim 1 \sim R\mu$

Conclusions

- Exploring holographic complexity and developing the concept of circuit complexity for QFTs are two parallel lines of inquiry.
- The first law of complexity provides a new approach to build a bridge between holographic and circuit complexity.
- It allows to investigate the implicit choice of cost function in $\mathcal{C}_A$
- Extensions:
    - other fields and excited states
    - higher spacetime dimensions
    - complexity = volume
    - path integral optimization, Fubini-Study approach, …
- How generic is the cancellation in the gravitational sector?

Thank you!