Einstein manifolds with torsion and nonmetricity and some applications in (super)gravity theories

Based on arXiv:1811.11458 [gr-qc] and Phys. Lett. B **793** (2019) 265-270, arXiv:1904.03681 [hep-th], both in collaboration with Silke Klemm

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Overview

- Einstein manifolds with torsion and nonmetricity in d dimensions
- Applications in gravity theories
  1. Einstein-Cartan action $\leftrightarrow$ Scale invariant gravity
  2. Einstein-Hilbert action + 3-form $\leftrightarrow$ Einstein-Cartan gravity

[Silke Klemm, L.R., arXiv:1811.11458 [gr-qc]]

- Applications in SUGRA
  1. Einstein-Weyl spaces and near-horizon geometry (brief review)
     [M. Dunajski, J. Gurowski, W. Sabra, Class. Quant. Grav. **34** (2017), no.4, 045009, arXiv:1610.08953 [hep-th]]
  2. SUSY near-horizion geometry and Einstein-Cartan-Weyl spaces
     [Silke Klemm, L.R., Phys. Lett. B **793** (2019) 265-270, arXiv:1904.03681 [hep-th]]

WHY TORSION AND NONMETRICITY? SOME MOTIVATIONS

- Riemannian geometry $\rightarrow$ Mathematical formulation of General Relativity
- General Relativity is successful and predictive, but there are still some open problems and questions
- Clearer understanding and solutions may need the formulation of a new theoretical framework $\rightarrow$ Generalizations and extensions of Riemannian geometry
- Possible way of generalizing Riemannian geometry: Allowing for non-vanishing torsion and nonmetricity $\rightarrow$ Metric affine gravity
- Several physical and mathematical reasons motivate torsion or nonmetricity (applications in the theory of defects in crystals, explore spacetime microstructure, applications in quantum gravity and cosmology, etc.)

Historical aspects

- Weyl (1918): Attempt of unifying EM with gravity geometrically $\rightarrow$ Generalization of Riemannian geometry (both the direction and the length of vectors are allowed to vary under parallel transport) $\rightarrow$ Connection involving the nonmetricity tensor, whose trace part is called the Weyl vector $\rightsquigarrow$ Observational inconsistencies in Weyl's theory $\Rightarrow$ Weyl's theory of EM fails
- Renewed interest in Weyl geometry, trying to go beyond Weyl's results (scale invariant gravity, higher symmetry approaches to gravity involving conformal invariance, etc.)

NONMETRICITY AND EINSTEIN-WEYL SPACES

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Einstein-Weyl geometry

- *Weyl manifold*: Conformal manifold, torsionless but nonmetric connection preserving the conformal structure
- *Einstein-Weyl manifold*: Weyl manifold for which the symmetric, trace-free part of the Ricci tensor of the connection vanishes and the symmetric part of the Ricci tensor of the Weyl connection is $\propto$ to the metric
- Applications: (Fake) SUSY SUGRA solutions; EW geometry in $d = 3$ has an equivalent formulation in twistor theory $\rightarrow$ Tool for constructing self-dual $d = 4$ geometries (Jones-Tod correspondence); relations with integrable systems; etc.

Historical aspects

- Einstein-Cartan theory: Generalization of Riemannian geometry including torsion, geometrical structure of the manifold modified by allowing for an AS part of the affine connection
- Introduction of torsion widely analyzed in GR and in the setting of teleparallel gravities; torsion tensor also related to the Kalb-Ramond field; relation between torsion and conformal symmetry; etc.

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Einstein-Cartan geometry

- *Einstein-Cartan manifold*: Einstein manifold with a metric connection endowed with non-vanishing torsion
- Applications: Cosmology (torsion as origin for dark energy); Einstein manifolds with skew-symmetric torsion analyzed from the mathematical p.o.v.; condensed matter physics; etc.

EINSTEIN-CARTAN-WEYL GEOMETRY: EINSTEIN MANIFOLDS WITH TORSION AND NONMETRICITY IN d DIMENSIONS

- Einstein manifolds with torsion in d dimensions and relations with Einstein manifolds with nonmetricity in d dimensions (trace part of the torsion $\leftrightarrow$ Weyl vector); Weyl invariance in both cases

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- Conn.: $\Gamma^{\lambda}{}_{\mu\nu} = \underbrace{\tilde{\Gamma}^{\lambda}{}_{\mu\nu}}_{\text{Christoffel}} + \underbrace{N^{\lambda}{}_{\mu\nu}}_{\text{distortion}}$, with $N^{\lambda}{}_{\mu\nu} = \frac{1}{2}\left(T_{\nu\lambda\mu} - T_{\lambda\mu\nu} - T_{\mu\nu\lambda}\right) + \frac{1}{2}\left(Q_{\lambda\mu\nu} + Q_{\lambda\nu\mu} - Q_{\mu\lambda\nu}\right)$
- Torsion: $T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}$; *decomposition*: $T^{\lambda}{}_{\mu\nu} = \breve{T}^{\lambda}{}_{\mu\nu} + \frac{1}{d-1}\left(\delta^{\lambda}{}_{\nu} T_{\mu} - \delta^{\lambda}{}_{\mu} T_{\nu}\right)$, where $\breve{T}^{\nu}{}_{\mu\nu} = 0$ and $T_{\mu} \equiv T^{\nu}{}_{\mu\nu}$
- Nonmetricity: $Q_{\mu\nu\lambda} = -\nabla_{\nu} g_{\lambda\mu}$; *decomposition*: $Q_{\mu\nu\lambda} = -2\Theta_{\nu} g_{\lambda\mu} + \breve{Q}_{\mu\nu\lambda}$, where Θ_{ν} is the Weyl vector

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- Nonmetricity: $Q_{\mu\nu\lambda} = -\nabla_\nu g_{\lambda\mu}$; *decomposition*: $Q_{\mu\nu\lambda} = -2\Theta_\nu g_{\lambda\mu} + \check{Q}_{\mu\nu\lambda}$, where Θ_ν is the Weyl vector
- Riemann tensor: $R^\lambda{}_{\rho\mu\nu} = \partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\nu \Gamma^\lambda{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho} - \Gamma^\lambda{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho}$
- Ricci tensor of ∇: $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$
- Ricci scalar: $R = g^{\mu\nu} R_{\mu\nu}$

Einstein-Cartan-Weyl spaces and ECW equations

- We define an **ECW space** by

$$R_{(\mu\nu)} = \lambda g_{\mu\nu}$$

- We use

$$R_{(\mu\nu)} = R^{\rho}{}_{(\mu|\rho|\nu)}$$

- We substitute back and take the trace
- We get the form of λ in terms of $\tilde{R}$ (of $\tilde{\nabla}$, Levi-Civita) $+$ torsion and nonmetricity terms
- We substitute back λ
- We get a system of nonlinear PDEs characterizing an ECW manifold (ECW equations)

$$R_{(\mu\nu)} = \frac{1}{d} R g_{\mu\nu}$$

in terms of Riemannian data

(We will see the explicit form of the ECW equations in the $d = 3$ case)

WEYL INVARIANCE OF THE ECW EQUATIONS

- Consider the Weyl rescaling

$$g_{\mu\nu} \mapsto e^{2\omega} g_{\mu\nu}$$

- The Weyl vector and the connection transform as

$$\Theta_\mu \mapsto \Theta_\mu + \xi \partial_\mu \omega \,, \quad \Gamma^\rho{}_{\mu\nu} \mapsto \Gamma^\rho{}_{\mu\nu} + (1-\xi)\delta^\rho{}_\nu \partial_\mu \omega$$

  where ξ denotes an arbitrary parameter
- Then we have

$$T^\lambda{}_{\mu\nu} \mapsto T^\lambda{}_{\mu\nu} + 2(1-\xi)\delta^\lambda{}_{[\nu}\partial_{\mu]}\omega \,, \quad Q^\lambda{}_{\mu\nu} \mapsto Q^\lambda{}_{\mu\nu} - 2\xi \partial_\mu \omega \delta^\lambda{}_\nu$$

- In particular, we find

$$T_\mu \mapsto T_\mu + (1-\xi)(d-1)\partial_\mu \omega \,, \quad \breve{T}^\lambda{}_{\mu\nu} \mapsto \breve{T}^\lambda{}_{\mu\nu} \,, \quad \breve{Q}_{\lambda\mu\nu} \mapsto \breve{Q}_{\lambda\mu\nu}$$

- For the Riemann tensor, the Ricci tensor, and the scalar curvature, we obtain, respectively,

$$R^\sigma{}_{\rho\mu\nu} \mapsto R^\sigma{}_{\rho\mu\nu} \,, \quad R_{\rho\nu} \mapsto R_{\rho\nu} \,, \quad R \mapsto e^{-2\omega} R$$

- $R_{(\mu\nu)} = \lambda g_{\mu\nu}$ implies (trace): $R = \lambda d \Rightarrow R_{(\mu\nu)} = \lambda g_{\mu\nu}$ is equivalent to

$$R_{(\rho\nu)} = \frac{1}{d} R g_{\rho\nu}$$

- The latter is clearly Weyl invariant

TWO PARTICULAR CASES: $\xi = 1$ AND $\xi = 0$

- For $\xi = 1$:

$$T_\mu \mapsto T_\mu \, ,$$
$$\Theta_\mu \mapsto \Theta_\mu + \partial_\mu \omega$$

This corresponds to the Weyl transformation for Θ_μ for manifolds with nonmetricity and vanishing torsion. Moreover, this is the only case in which **the connection is also invariant**, namely

$$\Gamma^\rho{}_{\mu\nu} \mapsto \Gamma^\rho{}_{\mu\nu}$$

Two particular cases: $\xi = 1$ and $\xi = 0$

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- For $\xi = 0$:

$$T_\mu \mapsto T_\mu + (d-1)\partial_\mu \omega \,,$$
$$\Theta_\mu \mapsto \Theta_\mu$$

This corresponds to the Weyl transformation for T_μ for manifolds with torsion and vanishing nonmetricity. In this case, for the connection we get

$$\Gamma^\rho{}_{\mu\nu} \mapsto \Gamma^\rho{}_{\mu\nu} + \delta^\rho{}_\nu \partial_\mu \omega$$

Obs.: We can reabsorb the Weyl vector into the trace part of the torsion by defining $\hat{T}_\mu = T_\mu + (d-1)\Theta_\mu$ (and vice versa)

Einstein-Cartan-Weyl geometry in $d = 3$

- Consider a $d = 3$ Einstein manifold endowed with a metric γ; Conn.: $\Gamma^l{}_{ij} = \tilde{\Gamma}^l{}_{ij} + N^l{}_{ij}$, with torsion and nonmetricity
- For simplicity: Traceless part of the distortion $\breve{N}_{lij} \equiv \frac{1}{2}\left(\breve{T}_{jli} - \breve{T}_{lji} - \breve{T}_{ijl}\right) + \frac{1}{2}\left(\breve{Q}_{lij} + \breve{Q}_{lji} - \breve{Q}_{ilj}\right)$
- **ECW space**: $R_{(ij)} = \frac{1}{3} R \gamma_{ij}$
- In terms of Riemannian data, the latter becomes (ECW equations)

$$\tilde{R}_{ij} + \tilde{\nabla}_{(i}\Theta_{j)} + \Theta_i\Theta_j + \frac{1}{2}\tilde{\nabla}_{(i}T_{j)} + \frac{1}{4}T_iT_j + \Theta_{(i}T_{j)} - \breve{N}^{lm}{}_{(i}\breve{N}_{j)lm} + \Theta^l\breve{N}_{(ij)l} + \frac{1}{2}T^l\breve{N}_{(ij)l} - \tilde{\nabla}_l\breve{N}^l_{(i\,j)}$$
$$= \frac{1}{3}\gamma_{ij}\left(\tilde{R} + \tilde{\nabla}^k\Theta_k + \Theta^k\Theta_k + \frac{1}{2}\tilde{\nabla}^kT_k + \frac{1}{4}T^kT_k + \Theta^kT_k - \breve{N}^{lmn}\breve{N}_{mnl}\right)$$

- Ricci scalar for a $d = 3$ ECW manifold:

$$R = \tilde{R} + 4\tilde{\nabla}^k\Theta_k - 2\Theta^k\Theta_k + 2\tilde{\nabla}^kT_k - \frac{1}{2}T^kT_k - 2\Theta^kT_k - \breve{N}^{lmn}\breve{N}_{mnl}$$

- Obs.: We can define $\breve{\Theta}_i \equiv \Theta_i + \frac{1}{2}T_i$ such that

$$\tilde{R}_{ij} + \tilde{\nabla}_{(i}\breve{\Theta}_{j)} + \breve{\Theta}_i\breve{\Theta}_j - \breve{N}^{lm}{}_{(i}\breve{N}_{j)lm} + \breve{\Theta}^l\breve{N}_{(ij)l} - \tilde{\nabla}_l\breve{N}^l_{(i\,j)} = \frac{1}{3}\gamma_{ij}\left(\tilde{R} + \tilde{\nabla}^k\breve{\Theta}_k + \breve{\Theta}^k\breve{\Theta}_k - \breve{N}^{lmn}\breve{N}_{mnl}\right),$$
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Applications in the context of gravity theories

- Einstein-Cartan action $\leftrightarrow$ Scale invariant gravity
- Einstein-Hilbert action + 3-form $\leftrightarrow$ Einstein-Cartan gravity

Einstein-Cartan action ↔ Scale invariant gravity

- Consider the Einstein-Cartan action:

$$
\begin{aligned}
S_1 &= \int d^d x \sqrt{-g}\phi^2 \left(R - \kappa \phi^{\frac{4}{d-2}} \right) \\
&= \int d^d x \sqrt{-g}\phi^2 \left(\tilde{R} - \frac{d-2}{d-1} T_\mu T^\mu + 2\tilde{\nabla}_\mu T^\mu + \frac{1}{4} \breve{T}_{\mu\nu\rho} \breve{T}^{\mu\nu\rho} - \frac{1}{2} \breve{T}_{\nu\rho\mu} \breve{T}^{\mu\nu\rho} - \kappa \phi^{\frac{4}{d-2}} \right)
\end{aligned}
$$

R: Ricci scalar of a torsionful but metric connection, ϕ: scalar field, κ: constant

- The action is invariant under: $g_{\mu\nu} \mapsto e^{2\omega} g_{\mu\nu}$, $\phi \mapsto e^{\frac{2-d}{2}\omega}\phi$, $T_\mu \mapsto T_\mu + (d-1)\partial_\mu \omega$, $\breve{T}^\lambda{}_{\mu\nu} \mapsto \breve{T}^\lambda{}_{\mu\nu}$

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- Variation w.r.t. T_μ and traceless part of the contorsion: $T_\mu=-\frac{2(d-1)}{d-2}\frac{\tilde{\nabla}_\mu\phi}{\phi}$, $\breve{T}_{\mu\nu\rho}=0$
- Variation w.r.t $g_{\mu\nu}$:

$$\phi^2\left(\tilde{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}\right)+\frac{2d}{d-2}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi-2\phi\tilde{\nabla}_\nu\tilde{\nabla}_\mu\phi+2g_{\mu\nu}\phi\tilde{\nabla}_\rho\tilde{\nabla}^\rho\phi-\frac{2}{d-2}g_{\mu\nu}\tilde{\nabla}_\rho\phi\tilde{\nabla}^\rho\phi+\frac{1}{2}g_{\mu\nu}\kappa\phi^{\frac{2d}{d-2}}=0$$

- Variation w.r.t. ϕ: $\quad \phi\tilde{R}-\frac{4(d-1)}{d-2}\tilde{\nabla}_\rho\tilde{\nabla}^\rho\phi-\frac{d}{d-2}\kappa\phi^{\frac{d+2}{d-2}}=0$
- Obs.: In the last two we have already used the expressions for the torsion; the trace of the 1st implies the 2nd (being ϕ pure gauge)

- Now consider the scale invariant gravity action:

$$S_2 = \int d^d x \sqrt{-g} \left[\phi^2 \tilde{R} + \frac{4(d-1)}{d-2} \tilde{\nabla}_\mu \phi \tilde{\nabla}^\mu \phi - \kappa \phi^{\frac{2d}{d-2}} \right]$$

- The e.o.m. following from the scale invariant gravity action are precisely the ones obtained from the Einstein-Cartan action by varying the latter w.r.t. $g_{\mu\nu}$ and ϕ, after having used the expressions for the torsion
- Obs.: Plugging the expression for T_μ in terms of ϕ and $\check{T}_{\mu\nu\rho} = 0$ into the Einstein-Cartan action, one gets, up to a surface term, the scale invariant gravity action
- Obs.: Weyl invariance allows to rescale $\phi \mapsto e^{\frac{2-d}{d}\omega} \phi \Rightarrow$ One can use this freedom to gauge fix $\phi = 1/(4\sqrt{\pi G}) \rightarrow$ The scale invariant gravity action becomes EH $+$ cosmological constant

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- Obs.: Weyl invariance allows to rescale $\phi \mapsto e^{\frac{2-d}{d}\omega}\phi \Rightarrow$ One can use this freedom to gauge fix $\phi = 1/(4\sqrt{\pi G}) \rightarrow$ The scale invariant gravity action becomes EH + cosmological constant
- Moreover, **the Einstein-Cartan action implies that the spacetime is Einstein with torsion**: The e.o.m. obtained from the Einstein-Cartan action by varying the latter w.r.t. $g_{\mu\nu}$ (using also the one obtained from the variation w.r.t. ϕ), can be recast into the form

$$\tilde{R}_{\mu\nu} + \frac{2d}{d-2} \frac{\tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi}{\phi^2} - 2 \frac{\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi}{\phi} = \frac{1}{d} g_{\mu\nu} \left(\tilde{R} - 2 \frac{\tilde{\nabla}_\rho \tilde{\nabla}^\rho \phi}{\phi} + \frac{2d}{d-2} \frac{\tilde{\nabla}_\rho \phi \tilde{\nabla}^\rho \phi}{\phi^2} \right)$$

  On the other hand, considering the system of PDEs characterizing an Einstein-Cartan manifold and using the expressions for the torsion, the system of PDEs boils down to the equation above

Einstein-Hilbert action + 3-form ↔ Einstein-Cartan gravity

- Consider the Einstein-Hilbert action coupled to a 3-form field-strength:

$$S_3 = \int d^d x \sqrt{-g} \left(\tilde{R} - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) ,$$

  where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, with $B_{\mu\nu} = -B_{\nu\mu}$
- Variation w.r.t. $B_{\mu\nu}$: $\tilde{\nabla}^\mu H_{\mu\nu\rho} = 0$
- Variation w.r.t. $g^{\rho\nu}$: $\tilde{R}_{\rho\nu} - \frac{1}{2} g_{\rho\nu} \tilde{R} + \frac{1}{24} g_{\rho\nu} H_{\mu\tau\sigma} H^{\mu\tau\sigma} - \frac{1}{4} H_\rho{}^{\mu\sigma} H_{\nu\mu\sigma} = 0$

EINSTEIN-HILBERT ACTION + 3-FORM $\leftrightarrow$ EINSTEIN-CARTAN GRAVITY

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- Consider the system of PDEs satisfied by an Einstein manifold with torsion, assuming $T_\mu = 0$ and $\breve{T}_{\mu\nu\rho}$ completely AS: $\quad \tilde{R}_{\rho\nu} - \frac{1}{4} \breve{T}_{\mu\sigma\nu} \breve{T}_\rho{}^{\mu\sigma} = \frac{1}{d} g_{\rho\nu} \left(\tilde{R} - \frac{1}{4} \breve{T}^{\mu\tau\sigma} \breve{T}_{\mu\tau\sigma} \right)$
- **Compare the system of PDEs with the e.o.m.** obtained from the variation of S_3 w.r.t. $g^{\rho\nu}$

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  where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, with $B_{\mu\nu} = -B_{\nu\mu}$
- Variation w.r.t. $B_{\mu\nu}$: $\tilde{\nabla}^\mu H_{\mu\nu\rho} = 0$
- Variation w.r.t. $g^{\rho\nu}$: $\tilde{R}_{\rho\nu} - \frac{1}{2} g_{\rho\nu} \tilde{R} + \frac{1}{24} g_{\rho\nu} H_{\mu\tau\sigma} H^{\mu\tau\sigma} - \frac{1}{4} H_\rho{}^{\mu\sigma} H_{\nu\mu\sigma} = 0$

- Consider the system of PDEs satisfied by an Einstein manifold with torsion, assuming $T_\mu = 0$ and $\breve{T}_{\mu\nu\rho}$ completely AS: $\tilde{R}_{\rho\nu} - \frac{1}{4} \breve{T}_{\mu\sigma\nu} \breve{T}_\rho{}^{\mu\sigma} = \frac{1}{d} g_{\rho\nu} \left(\tilde{R} - \frac{1}{4} \breve{T}^{\mu\tau\sigma} \breve{T}_{\mu\tau\sigma} \right)$
- **Compare the system of PDEs with the e.o.m.** obtained from the variation of S_3 w.r.t. $g^{\rho\nu}$
- Take the trace of the e.o.m.: $\tilde{R} = \frac{d-6}{12(d-2)} H^2, \quad H^2 \equiv H_{\mu\tau\sigma} H^{\mu\tau\sigma}$
- Subtract its trace part: $\tilde{R}_{\rho\nu} - \frac{1}{d} g_{\rho\nu} \tilde{R} - \frac{1}{4} H_\rho{}^{\mu\sigma} H_{\nu\mu\sigma} + \frac{1}{4d} g_{\rho\nu} H^2 = 0$
- **The latter coincides with the system of PDEs** under the **identification**: $H_{\mu\nu\rho} = \breve{T}_{\mu\nu\rho}$
- The e.o.m. following from S_3 can thus be interpreted as implying that the spacetime is Einstein with skew-symmetric torsion $H_{\mu\nu\rho}$ (however, the e.o.m. are more restrictive than the system of PDEs)

EINSTEIN-HILBERT ACTION + 3-FORM $\leftrightarrow$ EINSTEIN-CARTAN GRAVITY

- Obs.: The e.o.m. for S_3 can also be retrieved from the following (constrained) action:

$$
\begin{aligned}
S_4 &= \int d^dx\sqrt{-g}\left[R + \lambda^{\mu\nu\rho}\left(\breve{T}_{\mu\nu\rho} - \frac{1}{\sqrt{3}}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})\right)\right] \\
&= \int d^dx\sqrt{-g}\left[\tilde{R} - \frac{d-2}{d-1}T_\mu T^\mu + 2\tilde{\nabla}_\mu T^\mu + \frac{1}{4}\breve{T}_{\mu\nu\rho}\breve{T}^{\mu\nu\rho} - \frac{1}{2}\breve{T}_{\nu\rho\mu}\breve{T}^{\mu\nu\rho}\right. \\
&\qquad \left. + \lambda^{\mu\nu\rho}\left(\breve{T}_{\mu\nu\rho} - \frac{1}{\sqrt{3}}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})\right)\right],
\end{aligned}
$$

  where R denotes the scalar curvature of a torsionful but metric connection, $\lambda^{\mu\nu\rho}$ is a Lagrange multiplier
- Variation w.r.t. T_μ: $\quad T_\mu = 0$
- Variation w.r.t. $B_{\mu\nu}$: $\quad \tilde{\nabla}_\mu \lambda^{[\mu\nu\rho]} = 0$
- Variation w.r.t. $\lambda^{\mu\nu\rho}$: $\quad \breve{T}_{\mu\nu\rho} = \frac{1}{\sqrt{3}}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}) \quad \Rightarrow \quad$ Completely AS
- Variation w.r.t. $\breve{T}_{\mu\nu\rho}$: $\quad \lambda^{\mu\nu\rho} = \frac{1}{2}\left(\breve{T}^{\nu\rho\mu} + \breve{T}^{\rho\mu\nu} - \breve{T}^{\mu\nu\rho}\right) \quad \Rightarrow \quad \lambda^{\mu\nu\rho} = \frac{1}{2}\breve{T}^{\mu\nu\rho}$
- Variation w.r.t. $g^{\mu\nu}$: $\quad \tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R} + \frac{1}{8}g_{\mu\nu}\breve{T}_{\tau\rho\sigma}\breve{T}^{\tau\rho\sigma} - \frac{3}{4}\breve{T}_\mu{}^{\tau\rho}\breve{T}_{\nu\tau\rho} = 0$
- Substituting $\lambda^{\mu\nu\rho}$ in terms of $\breve{T}^{\mu\nu\rho}$ in eq. from variation w.r.t. $B_{\mu\nu}$: $\quad \tilde{\nabla}_\mu \breve{T}^{\mu\nu\rho} = 0$
- Putting all together we get the e.o.m. obtained for S_3

Applications in the context of SUGRA:

- Einstein-Weyl spaces and near-horizon geometry (brief review)
- SUSY near-horizion geometry and Einstein-Cartan-Weyl spaces

Einstein-Weyl spaces and near-horizon geometry

In the last decade there has been significant progress in **classifying near-horizon geometries** $\rightarrow$ Useful in the problem of reconstructing all SUSY solutions from a given n.h. geometry

- For minimal gauged $d = 5$ SUGRA n.h. geometries are at least half-supersymmetric; if they preserve a larger fraction of SUSY, then they are locally isometric to AdS_5 with vanishing 2-form field-strength

[J. Grover, J. B. Gutowski, G. Papadopoulos and W. A. Sabra, JHEP **1406** (2014) 020, arXiv:1303.0853 [hep-th]]

- Ungauged case: SUSY horizon geometries are given by $d = 3$ EW structures of hyper-CR type (Gauduchon-Tod spaces) $\rightarrow$ A class of solutions of minimal $d = 5$ SUGRA is given by lifts of $d = 3$ EW structures of hyper-CR type; this class was characterized as the most general n.h. limit of SUSY solutions to the $d = 5$ theory
- **Classification**: A compact spatial section of a horizon can only be a Berger sphere, a product metric on $S^1 \times S^2$, or a flat three-torus

[M. Dunajski, J. Gurowski, W. Sabra, Class. Quant. Grav. **34** (2017), no.4, 045009, arXiv:1610.08953 [hep-th]]

- We extended the analysis of horizon geometry of SUSY black holes to the case to minimal gauged $d = 5$ SUGRA $\rightarrow$ **SUSY n.h. geometry and Einstein-Cartan-Weyl spaces**

[Silke Klemm, L.R., Phys. Lett. B **793** (2019) 265-270, arXiv:1904.03681 [hep-th]]

- Bosonic action of minimal $N=2$, $d=5$ gauged SUGRA:

$$S=\frac{1}{4\pi G}\int\left[\frac{1}{4}\left(\tilde{R}+\frac{12}{\ell^2}\right)\star_5 1-\frac{1}{2}F\wedge\star_5 F-\frac{2}{3\sqrt{3}}F\wedge F\wedge A\right],$$

where $F=dA$, ℓ related to the cosmological constant by $\Lambda=-6/\ell^2$, $\star_5$: Hodge endomorphism in $d=5$

- E.o.m.:

$$\tilde{R}_{\alpha\beta}-2F_{\alpha\gamma}F_{\beta}{}^{\gamma}+\frac{1}{3}g_{\alpha\beta}\left(F^2+\frac{12}{\ell^2}\right)=0\,,\quad d\star_5 F+\frac{2}{\sqrt{3}}F\wedge F=0\,,$$

with $F^2\equiv F_{\alpha\beta}F^{\alpha\beta}$

$\mathcal{N} = 2$, $d = 5$ GAUGED SUGRA AND THE NEAR-HORIZON LIMIT OF BPS BLACK HOLES

- Bosonic action of minimal $N = 2$, $d = 5$ gauged SUGRA:

$$S = \frac{1}{4\pi G} \int \left[\frac{1}{4} \left(\tilde{R} + \frac{12}{\ell^2} \right) \star_5 1 - \frac{1}{2} F \wedge \star_5 F - \frac{2}{3\sqrt{3}} F \wedge F \wedge A \right] ,$$

where $F = dA$, ℓ related to the cosmological constant by $\Lambda = -6/\ell^2$, $\star_5$: Hodge endomorphism in $d = 5$

- E.o.m.:

$$\tilde{R}_{\alpha\beta} - 2F_{\alpha\gamma}F_{\beta}{}^{\gamma} + \frac{1}{3} g_{\alpha\beta} \left(F^2 + \frac{12}{\ell^2} \right) = 0 , \quad d \star_5 F + \frac{2}{\sqrt{3}} F \wedge F = 0 ,$$

with $F^2 \equiv F_{\alpha\beta}F^{\alpha\beta}$

- **Gaussian null coordinates** (u, r, y^i) defined in a neighborhood of a Killing horizon; horizon at $r = 0$; y^i are local coordinates on a $d = 3$ Riemannian manifold Σ with metric γ (spatial cross section of the horizon)
- Metric, 2-form field-strength F, and 1-form gauge potential associated to F:

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + \gamma_{ij} dy^i dy^j , \quad F = -\frac{\sqrt{3}}{2} \Phi \mathbf{e}^+ \wedge \mathbf{e}^- - \frac{\sqrt{3}}{2} r \mathbf{e}^+ \wedge (d\Phi - h\Phi) + dB , \quad A = \frac{\sqrt{3}}{2} r\Phi du + B ,$$

with $\mathbf{e}^+ = du$, $\mathbf{e}^- = dr + rh - \frac{1}{2} r^2 \Delta du$, where the scalars Δ, Φ, the 1-forms h, B, and the Riemannian metric γ depend only on y^i $(i, j = 1, 2, 3)$

$\mathcal{N}=2$, $d=5$ GAUGED SUGRA AND THE NEAR-HORIZON LIMIT OF BPS BLACK HOLES

- In the n.h. limit the bosonic field equations boil down to a set of equations on the $d=3$ manifold Σ
- From the gauge field equations: $d\star_3 dB + \frac{\sqrt{3}}{2}\star_3 (d\Phi - \Phi h) - h \wedge \star_3 dB - 2\Phi dB = 0$
- The non-trivial components of the Einstein equations, namely (ur) and (ij), become, respectively,

$$\frac{1}{2}\tilde{\nabla}^i h_i - \frac{1}{2}h^2 + \frac{1}{3}dB_{mn}dB^{mn} + \Phi^2 - \Delta + \frac{4}{\ell^2} = 0\,,$$

$$\tilde{R}_{ij} + \tilde{\nabla}_{(i}h_{j)} - \frac{1}{2}h_i h_j - 2dB_{ik}dB_j{}^k + \gamma_{ij}\left(\frac{1}{3}dB_{kl}dB^{kl} - \frac{1}{2}\Phi^2 + \frac{4}{\ell^2}\right) = 0$$

$\mathcal{N}=2$, $d=5$ GAUGED SUGRA AND THE NEAR-HORIZON LIMIT OF BPS BLACK HOLES

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- **Necess. and suff. conditions** for a n.h. geometry to be a **SUSY** solution of minimal $d=5$ gauged SUGRA:

$$\Delta = \Phi^2\,, \quad \left(\frac{1}{2}h + \frac{1}{\sqrt{3}}\star_3 dB\right)^2 = \frac{1}{\ell^2} \quad \rightsquigarrow \quad \frac{1}{2}h + \frac{1}{\sqrt{3}}\star_3 dB = \frac{1}{\ell}Z\,, \quad Z^2 \equiv Z^i Z_i = 1$$

  Furthermore one must have

$$\tilde{\nabla}_i Z_j = \left(-\frac{3}{\ell} + h^m Z_m\right)\gamma_{ij} + \frac{3}{\ell}Z_i Z_j - Z_i h_j - \frac{1}{2}\Phi(\star_3 Z)_{ij}\,, \quad \text{with} \quad (\star_3 Z)_{ij} = \epsilon_{ijk}Z^k$$

- Then, we find

$$\star_3 dh = d\Phi - 2\Phi h - 2\sqrt{3}\Phi\star_3 dB$$

  (can be rewritten as a generalized monopole equation)

Horizon geom. for SUSY b.h. sols. of minimal $d=5$ gauged SUGRA is that of a particular $d=3$ ECW structure

- **Consider the field equations previously seen and assume that the SUSY constraints hold**
- We have:

$$dB_{jk} = -\frac{\sqrt{3}}{2}\epsilon_{ijk}h_i + \frac{\sqrt{3}}{\ell}\epsilon_{ijk}Z_i\,, \quad dB_{im}dB_j{}^m = \frac{3}{4}(\gamma_{ij}h^2 - h_ih_j) + \frac{3}{\ell^2}(\gamma_{ij}Z^2 - Z_iZ_j) - \frac{3}{\ell}(\gamma_{ij}h^mZ_m - h_{(i}Z_{j)})\,,$$

$$dB_{im}dB^{im} = \frac{3}{2}h^2 + \frac{6}{\ell^2}Z^2 - \frac{6}{\ell}h^iZ_i\,, \quad \star_3\left[d\Phi + \left(h - \frac{6}{\ell}Z\right)\Phi\right] = dh$$

- The (ur) component of the Einstein equations becomes: $\tilde{\nabla}^i h_i = -\frac{12}{\ell^2}Z^iZ_i + \frac{4}{\ell}h^iZ_i$
- We also have

$$\tilde{\nabla}_{(i}Z_{j)} = \left(-\frac{3}{\ell}Z^mZ_m + h^mZ_m\right)\delta_{ij} + \frac{3}{\ell}Z_iZ_j - Z_{(i}h_{j)}\,, \quad \tilde{\nabla}^iZ_i = 2h^iZ_i - \frac{6}{\ell}Z^iZ_i\,, \quad \tilde{\nabla}^ih_i = \frac{2}{\ell}\tilde{\nabla}^iZ_i$$

- The (ij)-components of the Einstein equations yield

$$\tilde{R}_{ij} + \tilde{\nabla}_{(i}h_{j)} + h_ih_j + \frac{6}{\ell^2}Z_iZ_j - \frac{6}{\ell}h_{(i}Z_{j)} = \left(\frac{1}{2}\Phi^2 + h^kh_k - \frac{4}{\ell}h^kZ_k\right)\gamma_{ij}\,,$$

$$\tilde{R} = \frac{1}{2}\left(3\Phi^2 + 4h^ih_i + \frac{12}{\ell^2}Z^iZ_i - \frac{20}{\ell}h^iZ_i\right)$$

$d = 3$ Einstein-Cartan-Weyl structures and $\mathcal{N} = 2$, $d = 5$ gauged SUGRA

To show that the horizon geom. for BPS b.h. in minimal $d = 5$ gauged SUGRA is that of a particular $d = 3$ ECW, **consider a $d = 3$ ECW space for which the following conditions hold**:

1. $\exists$ a scalar Φ of conformal weight -1 that, together with the nonmetricity and torsion traces Θ and T, satisfies the following generalized monopole equation: $\quad \star_3 \left(d\Phi + \check{\Theta}\Phi \right) = d\Theta \,, \quad \check{\Theta}_i \equiv \Theta_i + \frac{1}{2} T_i$
2. The trace part of the torsion satisfies $T^2 \equiv T^i T_i = c^2$, where c is a *constant*, and
$$\tilde{\nabla}_i T_j = \left(\frac{1}{4} T^k T_k + \Theta^k T_k \right) \gamma_{ij} - T_i \Theta_j - \frac{1}{4} T_i T_j - \frac{1}{2} \Phi \epsilon_{ijk} T^k \quad \rightarrow \quad \tilde{\nabla}^i T_i = 2\Theta^i T_i + \frac{1}{2} T^i T_i \,,$$
$$\tilde{\nabla}_{(i} T_{j)} = \left(\frac{1}{4} T^k T_k + \Theta^k T_k \right) \gamma_{ij} - T_{(i} \Theta_{j)} - \frac{1}{4} T_i T_j$$
3. The Weyl vector obeys: $\quad \tilde{\nabla}^i \Theta_i = -\frac{1}{3} \Theta^i T_i - \frac{1}{12} T^i T_i \quad \Rightarrow \quad \tilde{\nabla}^i \Theta_i = -\frac{1}{6} \tilde{\nabla}^i T_i$
4. The traceless part of the torsion and the traceless part of the nonmetricity read, respectively,
$$\check{T}_{lmn} = \Phi \epsilon_{lmn} \,, \quad \check{Q}_{mln} = \frac{2c}{\sqrt{3}} \epsilon_{lk(m} T^k T_{n)} \quad \rightarrow \quad \check{N}_{lmn} = \frac{c}{\sqrt{3}} \left(\epsilon_{lmk} T^k T_n + \epsilon_{lnk} T^k T_m \right) + \frac{1}{2} \Phi \epsilon_{lmn}$$
5. The Ricci scalar of the affine connection is $R = -\frac{3c}{2} \Theta^i T_i + \frac{9}{2} c^2$

$d=3$ Einstein-Cartan-Weyl structures and $\mathcal{N}=2$, $d=5$ gauged SUGRA

- We now identify

$$\Theta = h\,, \quad T = -\frac{12}{\ell}Z\,, \quad c = \frac{12}{\ell}$$

such that

$$\breve{Q}_{mln} = \frac{4\sqrt{3}}{\ell}\epsilon_{lk(m}Z^k Z_{n)}\,,$$

$$\breve{N}_{lmn} = \frac{2\sqrt{3}}{\ell}\left(\epsilon_{lmk}Z^k Z_n + \epsilon_{lnk}Z^k Z_m\right) + \frac{1}{2}\Phi\epsilon_{lmn}\,,$$

$$R = -\frac{18}{\ell}h^i Z_i + \frac{54}{\ell^2}$$

$\Rightarrow$ **The equations we get coincide with the ones obtained on the SUGRA sol. side**

- In our case, the ECW equations read

$$\tilde{R}_{ij} + \nabla_{(i}h_{j)} + h_i h_j + \frac{6}{\ell^2}Z_i Z_j - \frac{6}{\ell}h_{(i}Z_{j)} = \frac{1}{3}\gamma_{ij}\left(\tilde{R} + h^k h_k - \frac{6}{\ell^2}Z^k Z_k - \frac{2}{\ell}h^k Z_k\right)$$

- **We conclude that the horizon geom. for SUSY b.h. sols. of minimal $d=5$ gauged SUGRA is that of a particular ECW structure in $d=3$ fulfilling the conditions** 1. **to** 5.
- Obs.: The conditions 2. and 5. break conformal invariance (expected, since the SUGRA theory we started with is not conformally invariant)

OBSERVATIONS ON THE LIMIT $\ell \to \infty$

- In the limit $\ell \to \infty$ (cosmological constant goes to zero) we find:

$$\tilde{R}_{ij} + \tilde{\nabla}_{(i}h_{j)} + h_i h_j = \left(\frac{1}{2}\Phi^2 + h^k h_k\right)\gamma_{ij}\,,$$

$$\tilde{R} = \frac{1}{2}\left(3\Phi^2 + 4h^k h_k\right)\,,$$

$$\star_3\left(d\Phi + h\Phi\right) = dh$$

- Results of Dunajski *et al* exactly reproduced
- For $\ell \to \infty$ the **conditions on the ECW geometry** boil down to

$$T_i = 0\,,\quad \tilde{\nabla}^i h_i = 0 \quad \text{(Gauduchon gauge)}\,,\quad \breve{N}_{lmn} = \frac{1}{2}\Phi\epsilon_{lmn}\,,\quad R = 0$$

- The horizon geometry for SUSY black holes in $d = 5$ ungauged SUGRA not only corresponds to a $d = 3$ hyper-CR EW structure in the Gauduchon gauge of Dunajski *et al*, but also to an **ECW structure in the Gauduchon gauge and subject to the constraints above** (this ambiguity comes from the fact that the sets of nonlinear PDEs characterizing the hyper-CR EW structure of Dunajski *et al* and the ECW structure we have defined coincide)

Final remarks

Conclusions

- Connections with torsion and nonmetricity are interesting both from the physical and the mathematical p.o.v.
- We have generalized some results that appeared previously in the literature and presented some new applications in (super)gravity

Open directions

- Possible generalizations of the Jones-Tod correspondence between self-dual conformal four-manifolds with a conformal vector field and abelian monopoles on EW spaces in $d = 3$ (one could ask whether **ECW manifolds** can arise in a similar way by symmetry reduction from higher dimensions)
- **ECW manifolds** may have applications in the classification of (fake) SUSY SUGRA solutions in the same way as EW manifolds provide the base space for fake SUSY solutions in dS SUGRA
- Possible extensions to higher dimensions and to the matter-coupled case

Thank you!