Duality symmetry and Covariant Actions for (Chiral) $p-$forms

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Standard approach to free theories

The Lorentz-covariant field variable is taken in the same representation as that of the little group carried by the corresponding particle

Some well-known examples

- Trivial representation of the little group corresponds to the spin-zero particle. Lorentz covariant variable – scalar field.
- Vector representation of the little group corresponds to the spin-one particle and is described by a Lorentz vector field (Maxwell potential).
- Symmetric tensor of the little group corresponds to the spin-two particle and is described by the linearised Einstein equations (Fierz-Pauli) and is described by symmetric Lorentz tensor (metric).

Particles and fields

Wigner classification of particles $\leftrightarrow$ field equations (unique?)

For massless spin-zero particle the simplest option is the Klein-Gordon equation

$$\Box\,\phi = 0$$

The scalar here is a single field that carries one degree of freedom: trivial representation of the massless little group. The Lagrangian is

$$\mathcal{L} \sim \frac{1}{2}\phi\,\Box\,\phi$$

Alternative

An alternative formulation of the scalar field is given by so-called Notoph Lagrangian by Ogievetsky and Polubarinov (1966):

$$\mathcal{L} \sim \partial^{\mu} B_{\mu\nu} \partial_{\lambda} B^{\lambda\nu}$$

Interactions depend on the formulation of the free theory

Interacting spin-zero particles

The scalar-field formulation allows for straightforward generalisation to non-linear theory with arbitrary potential:

$$\mathcal{L} \sim \frac{1}{2}\phi\,\Box\,\phi + V(\phi)\,.$$

Instead, the notoph formulation does not allow for any non-derivative self-interactions (those would spoil the gauge symmetry)!

Moral of the story

The choice of the free field formulation plays an important role in deriving possible interacting theories.
Therefore, before addressing the problem of the interacting p-forms, we should find a convenient action for the free fields.

Duality

A $p-$form and its dual

The Lagrangian is given in the form of ("Maxwell Lagrangian")

$$\mathcal{L} \sim F \wedge \star F \,, \quad F = dA \,.$$

Massless $p-$form and a $(d-2-p)$-form fields describe correspondingly particles of $p-$form and a $(d-2-p)$-form representations of the massless little group $ISO(d-2)$, which are dual to each other.

Attention!

Dual formulations do not admit the same interacting deformations!

Duality-symmetric fields

Maxwell action for $p-$forms and $(d-2-p)-$forms describes the same particle content.
When $d = 2p + 2$, the dual variables are of the same type and the Maxwell action itself takes the same form in both variables.

Twisted self-duality equations

The Maxwell equations are equivalent to first-order equations involving both dual potentials:

$$F = \pm \star \tilde{F} \,, \quad F = dA \,, \quad \tilde{F} = d\tilde{A}$$

Duality-symmetric formulations

Zwanziger '70,..., Tseytlin'90, Schwarz-Sen '93, Gibbons-Rasheed '95, Pasti-Sorokin-Tonin '96, Cederwall-Westerberg '97, Ivanov-Zupnik '02, ..., Kuzenko-Theisen '00,...

Chiral $p-$forms in $d = 4k + 2$ Minkowski space

Minkowski vs Euclidean

Since $\star^2 = (-1)^{\sigma+p+1}$ where σ is the number of time directions, only even-forms can be self-dual (chiral) in Minkowski space.

$p = 2k$ forms in $d = 4k + 2$ dimensions

For even $p-$form potentials in special dimensions the corresponding particles are not irreducible but contain two irreps — chiral and anti-chiral halves.

Self-dual (Chiral) fields

There are special representations of the Poincaré algebra which are described by self-dual forms. The covariant equations describing such representations are given as:

$$F = \pm \star F\,, \quad F = dA$$

which implies the regular "Maxwell equations" $d \star F = 0$.

Lagrangian?

Lagrangian formulation of the (free) chiral fields has a long history. Siegel '84, Kavalov-Mkrtchyan '87, Florianini-Jackiw '87, Henneaux-Teitelboim '88, Harada '90, Tseytlin '90, McClain-Yu-Wu '90, Wotzasek '91, ..., Pasti-Sorokin-Tonin '95,...

New action for Chiral fields

The new Lagrangian

$$\mathcal{L} = -\frac{1}{2(p+1)}\, F_{\mu_1 \dots \mu_{p+1}}\, F^{\mu_1 \dots \mu_{p+1}} + G^{\mu\nu}\partial_{[\mu}c_{\nu]}$$

$$-\frac{1}{2(p+1)}\,(\mathcal{F}_{\mu_1 \dots \mu_{p+1}} - (p+1)\, c_{[\mu_1}\, R_{\mu_2 \dots \mu_{p+1}]})\times$$

$$\times\,(\mathcal{F}^{\mu_1 \dots \mu_{p+1}} - (p+1)\, c^{[\mu_1}\, R^{\mu_2 \dots \mu_{p+1}]})$$

where

$$F = d\varphi\,, \quad \mathcal{F} = F + \star F\,,$$

or, equivalently,

$$\mathcal{L} \sim -F \wedge \star F + \star G \wedge d\,c - (\mathcal{F} - c \wedge R) \wedge \star(\mathcal{F} - c \wedge R)$$

Equivalence to PST

Integrating out the auxiliary field

We solve the algebraic equation of motion for the field $R_{\mu_1\dots\mu_p}$,

$$\mathcal{F}_{\mu_1\dots\mu_{p+1}}\,c^{\mu_1} + (-1)^{p+1}\,p\,c_{[\mu_2}\,R_{\mu_3\dots\mu_{p+1}]\mu_1}\,c^{\mu_1} - c^2\,R_{\mu_2\dots\mu_{p+1}} = 0$$

as

$$R_{\mu_1\dots\mu_p} = \frac{1}{c^2}\mathcal{F}_{\nu\mu_1\dots\mu_p}\,c^\nu + c_{[\mu_1}\lambda_{\mu_2\dots\mu_p]}$$

and plug back into the action to get:

$$\mathcal{L} = -\frac{1}{2(p+1)}\,F_{\mu_1\dots\mu_{p+1}}\,F^{\mu_1\dots\mu_{p+1}} + G^{\mu\nu}\partial_{[\mu}c_{\nu]} + \frac{1}{2\,c^2}\,\mathcal{F}_{\mu_1\dots\mu_p\nu}\,c^\nu\,\mathcal{F}^{\mu_1\dots\mu_p\rho}\,c_\rho$$

classically equivalent to the celebrated PST action.

Classically equivalent form

Integrating out the field G, we get a classically equivalent Lagrangian

$$\mathcal{L} = -\frac{1}{2(p+1)}\, F_{\mu_1 \ldots \mu_{p+1}}\, F^{\mu_1 \ldots \mu_{p+1}}$$

$$-\ \frac{1}{2(p+1)}\, (\mathcal{F}_{\mu_1 \ldots \mu_{p+1}} - (p+1)\, \partial_{[\mu_1} a\, R_{\mu_2 \ldots \mu_{p+1}]}) \times$$

$$\times\ (\mathcal{F}^{\mu_1 \ldots \mu_{p+1}} - (p+1)\, \partial^{[\mu_1} a\, R^{\mu_2 \ldots \mu_{p+1}]})\,,$$

or

$$\mathcal{L} \sim -F \wedge \star F - (\mathcal{F} - d\,a \wedge\ R) \wedge \star(\mathcal{F} - d\,a \wedge\ R)\,,$$

Manifest double gauge symmetries

Rearranging the Lagrangian

After a field redefinition $\varphi_{\mu_1\dots\mu_p} \to \varphi_{\mu_1\dots\mu_p} + a\, R_{\mu_1\dots\mu_p}$ can be rewritten in the form:

$$\mathcal{L} = -\frac{1}{2(p+1)}\,(F_{\mu_1\dots\mu_{p+1}} + a\, Q_{\mu_1\dots\mu_{p+1}})(F^{\mu_1\dots\mu_{p+1}} + a\, Q^{\mu_1\dots\mu_{p+1}})$$
$$-\frac{1}{(p+1)\,(p+1)!}\epsilon_{\mu_1\dots\mu_{p+1}\nu_1\dots\nu_{p+1}}\, a\, F^{\nu_1\dots\nu_{p+1}}\, Q^{\mu_1\dots\mu_{p+1}}\,,$$

where $Q_{\mu_1\dots\mu_{p+1}} = (p+1)\partial_{[\mu_1} R_{\mu_2\dots\mu_{p+1}]}.$, or

$$\mathcal{L} \sim -(F + a\, Q) \wedge \star(F + a\, Q) - a\, F \wedge Q$$

where $F = d\,\varphi\,, \quad Q = d\, R\,.$

Yet another form of the action

Finally, one can rewrite the action in the form:

$$\mathcal{L} = -\frac{1}{2(p+1)}\, F_{\mu_1\dots\mu_{p+1}}\, F^{\mu_1\dots\mu_{p+1}} \\ - \frac{1}{2(p+1)}\, (\mathcal{F}_{\mu_1\dots\mu_{p+1}} + a\, Q_{\mu_1\dots\mu_{p+1}})^2$$

We will use the notation:

$$Q^{\pm}_{\mu_1\dots\mu_{p+1}} = Q_{\mu_1\dots\mu_{p+1}} \pm \frac{1}{(p+1)!}\epsilon_{\mu_1\dots\mu_{p+1}\nu_1\dots\nu_{p+1}} Q^{\nu_1\dots\nu_{p+1}}$$

for (anti)self-dual part of the $(p+1)-$form $Q_{\mu_1\dots\mu_{p+1}}$.

Equations and PST symmetry

Combining the equations of motion E^{φ} , E^{R} for the fields $\varphi_{\mu_1\dots\mu_p}$ and $R_{\mu_1\dots\mu_p}$ one gets

$$E^{R}_{\mu_2\dots\mu_{p+1}} + a\, E^{\varphi}_{\mu_2\dots\mu_{p+1}} = \partial^{\mu_1} a\, P_{\mu_1\dots\mu_{p+1}} = 0\,,$$
$$P_{\mu_1\dots\mu_{p+1}} \equiv \mathcal{F}_{\mu_1\dots\mu_{p+1}} + a\, Q^{+}_{\mu_1\dots\mu_{p+1}}\,,$$

which implies

$$P_{\mu_1\dots\mu_{p+1}} = 0\,,$$

automatically satisfying the equation of motion E^{a} for the a field,

$$E^{a} = Q_{\mu_1\dots\mu_{p+1}}\, P^{\mu_1\dots\mu_{p+1}} = 0\,.$$

This indicates the existence of a PST like symmetry (shift for a).

A Generalisation

An interesting generalisation of the Lagrangian is:

$$\mathcal{L} = -\frac{1}{2}\, f(a)\, (\sqrt{a}\, F + \frac{1}{\sqrt{a}}\, Q)^2 + \, f(a)\, F \wedge Q\,.$$

For $f(a) \sim 1/a$, this Lagrangian is equivalent to the chiral one written earlier and describes a single chiral p-form carried in field φ.

For $f(a) \sim a$, it describes an anti-chiral $p-$form field carried by R. The exchange $\varphi \leftrightarrow R\,, a \to -\frac{1}{a}\,, f(a) \to -f(a)$ is a symmetry of the Lagrangian.

$d = 2$

One can integrate out the c_μ field in the original action:

$$c_\mu = \frac{1}{R}\mathcal{F}_\mu + \frac{1}{R^2}\epsilon_{\mu\nu}\partial^\nu \tilde{r}\,,$$

plugging back into action (renaming $\frac{1}{R} \to r$) we get:

$$S = \int \Big(-\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\,r^2\,\partial_\mu\tilde{r}\,\partial^\mu\tilde{r} - r\,\mathcal{F}^\mu\,\partial_\mu\tilde{r}\Big)\,d^2x\,.$$

Another parametrisation gives:

$$\mathcal{L}_\pm = -\frac{1}{8}\,[(r+1)\partial_\mu\varphi \pm (r-1)\partial_\mu\tilde{\varphi}]^2 + \frac{1}{4}\,\epsilon^{\mu\nu}\,r\,\partial_\mu\varphi\,\partial_\nu\tilde{\varphi}\,,.$$

where different signs correspond to different chiralities. Here $\varphi = \varphi_+ + \varphi_-$ and $\tilde{\varphi} = \varphi_+ - \varphi_-$. The two actions transform into each other under $\varphi \leftrightarrow \tilde{\varphi}\,,\ r \to -r$.

A compact form

The Lagrangian in a simpler form

$$\mathcal{L} \sim -\mathcal{M}_{IJ}\, F^I \wedge \star F^J - \mathcal{K}_{IJ}\, F^I \wedge F^J\,,$$

with

$$\mathcal{M}_{IJ} = \begin{bmatrix} 1 & a \\ a & a^2 \end{bmatrix}, \quad \mathcal{K}_{IJ} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, \quad F^I = \begin{bmatrix} F \\ Q \end{bmatrix},$$

where F^I is a two-vector with $p+1$-form components. $\mathcal{M}$ is of rank one. The "background matrix" $\mathcal{E} = \mathcal{M} + \mathcal{K}$ is invertible.

An observation

The same action with the inverted background matrix $\mathcal{E}^{-1}$ describes the same degrees of freedom, exchanging the roles of φ and R.

Duality-symmetric Electromagnetism

The Lagrangian for a single massless spin-one field

$$\mathcal{L} = -\frac{1}{8}\, F^a_{\mu\nu}\, F^{a\,\mu\nu} + G^{\mu\nu}\,\partial_{[\mu}c_{\nu]}$$
$$-\,\frac{1}{8}\,(\mathcal{F}^a_{\mu\nu} - 2\,c_{[\mu}\,R^a_{\nu]})(\mathcal{F}^{a\,\mu\nu} - 2\,c^{[\mu}\,R^{a\,\nu]})\,,$$

where $a, b = 1, 2$, and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu\,, \quad \mathcal{F}^a_{\mu\nu} = F^a_{\mu\nu} - \frac{1}{2}\epsilon_{ab}\,\varepsilon_{\mu\nu\lambda\rho}F^{b\,\lambda\rho}\,,$$
$$\epsilon_{ab} = -\epsilon_{ba}\,, \quad \epsilon_{12} = 1 = \epsilon^{12}\,, \quad \varepsilon_{0123} = 1 = -\varepsilon^{0123}\,.$$

The following identities hold (Einstein summation rule is assumed for both types of indices):

$$\mathcal{F}^a_{\mu\nu}\,\mathcal{F}^{a\,\mu\nu} = 0\,, \quad \varepsilon_{\mu\nu\lambda\rho}\mathcal{F}^{a\,\lambda\rho} = 2\,\epsilon^{ab}\,\mathcal{F}^b_{\mu\nu}\,,$$

Other form of the action

$$\mathcal{L} = -\frac{1}{8}\, F^a_{\mu\nu}\, F^{a\,\mu\nu} - \frac{1}{8}\,(\mathcal{F}^a_{\mu\nu} + a\, Q^a_{\mu\nu})\,(\mathcal{F}^{a\,\mu\nu} + a\, Q^{a\,\mu\nu})$$

where $Q^a_{\mu\nu} = \partial_\mu\, R^a_\nu - \partial_\nu\, R^a_\mu$. This Lagrangian describes a single Maxwell field, using four vectors and a scalar. It can be written as:

$$\mathcal{L} = -\frac{1}{8}\, \mathcal{M}_{IJ}\, F^I_{\mu\nu}\, F^{J\,\mu\nu} - \frac{1}{16}\, \mathcal{K}_{IJ}\epsilon^{\mu\nu\alpha\beta}\, F^I_{\mu\nu}\, F^J_{\alpha\beta}\,,$$

where

$$\mathcal{M}_{IJ} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ a & 0 & a^2 & 0 \\ 0 & a & 0 & a^2 \end{bmatrix}, \quad \mathcal{K}_{IJ} = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix}, \quad F^I = \begin{bmatrix} F^1 \\ F^2 \\ Q^1 \\ Q^2 \end{bmatrix}$$

Problems

A list of related long-standing problems

- Duality-symmetric formulation for non-abelian gauge theory.
- Interacting theory of non-abelian (chiral) $p-$forms.
- Extensions to Gravity and beyond.

Thank you for your attention!