Towards precision heavy flavour physics from lattice QCD

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July 5, 2010

Impact of LQCD on precision heavy flavour physics

Heavy quark sector constrains UT: angles & sides are related to hadronic matrix elements of $\mathcal{H}_{\rm weak}^{\rm (eff)}$, corresponding to mesonic decays/transitions

$$\Delta m_d \propto F_{B_d}^2 \widehat{B}_{B_d} |V_{td} V_{tb}^*|^2 \qquad \frac{\Delta m_s}{\Delta m_d} = \xi^2 \frac{m_{B_s}}{m_{B_d}} \frac{|V_{ts}|^2}{|V_{td}|^2} \qquad \xi = F_{B_s}\sqrt{\widehat{B}_{B_s}} \Big/ F_{B_d}\sqrt{\widehat{B}_{B_d}}$$

- $\exists$ large number of experimental data from heavy flavour-factories (CLEO, BaBar, Belle, LHCb, ...)
- Inputs of theory and predominantly LQCD computations needed to
  - interpret results of experimental measurements
  - determine / pin down heavy quark masses & CKM matrix elements
  - overconstrain unitarity relations $\leftrightarrow$ unveiling New Physics effects

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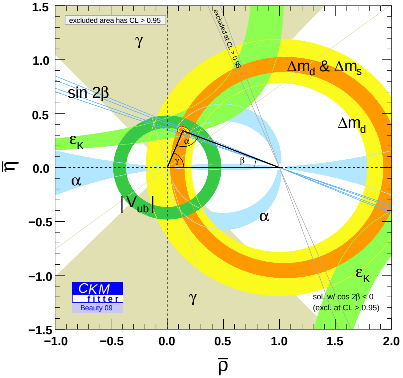
$$\begin{pmatrix} \mathbf{V_{ud}} & \mathbf{V_{us}} & \mathbf{V_{ub}} \\ \pi \to \ell\nu & K \to \ell\nu & B \to \pi\ell\nu \\ & K \to \pi\ell\nu & \\ \mathbf{V_{cd}} & \mathbf{V_{cs}} & \mathbf{V_{cb}} \\ D \to \ell\nu & D_s \to \ell\nu & B \to D\ell\nu \\ D \to \pi\ell\nu & D \to K\ell\nu & B \to D^*\ell\nu \\ \mathbf{V_{td}} & \mathbf{V_{ts}} & \mathbf{V_{tb}} \\ B_d \leftrightarrow \overline{B}_d & B_s \leftrightarrow \overline{B}_s & \end{pmatrix}$$

”Gold-plated” lattice processes

- 1 hadron in the initial state, 0 or 1 hadron in the final state
- stable hadrons (or narrow, far from theshold)
- controlled χ-extrapolation

Impact of LQCD on precision heavy flavour physics



- Constrain apex $(\bar\rho, \bar\eta)$ as precisely as possible by independent processes
- Theory & Exp. sufficiently precise
  $\Rightarrow$ New Physics $=$ inconsistent $(\bar\rho, \bar\eta)$
- LQCD inputs from the heavy sector:
  - B-meson decays & mixing: F_B, B_B
  - $B \longrightarrow D^{(*)}$ decays:
    $F(1), G(1) \hookrightarrow |V_{cb}|$
  - semi-leptonic B-meson decays:
    $f_+(q^2) \hookrightarrow |V_{ub}|$

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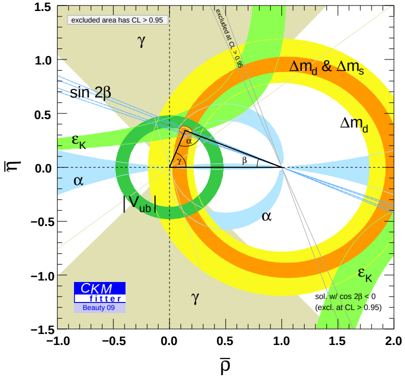

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What is the required precision for key contributions to phenomenology?

- Experiments reach few-% level, even $\leqslant 5\%$ $\Rightarrow$ theory error dominates
  $\Delta m_{d,s}$: $< 1\%$ [PDG,CDF], $\mathcal{B}(D_{(s)} \to \mu\nu)$: $\leqslant 4\%$ [CLEO-c], $\mathcal{B}(B \to D^*\ell\nu)$: 1.5% [HFAG]
- Lattice calculations with an accuracy of $O(5\%)$ or better required
  $\rightarrow$ incl. *all* systematics (unquenching, extrapolations, renormalization, . . .)
- Verification/Agreement of results using different formulations crucial !

Outline

1. Lattice QCD & Heavy flavour physics
   - Basics & Challenges
   - Lattice heavy quark formalisms
2. Non-perturbative HQET in two-flavour QCD
   - Non-perturbative formulation of HQET
   - Mass dependence at leading order in $1/m$
   - Strategy to determine HQET parameters at $O(1/m)$
   - First physical results in the two-flavour theory
3. Conclusions & Outlook

Lattice QCD & Heavy flavour physics

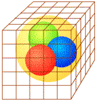
- Basics & Challenges
- Lattice heavy quark formalisms

Lattice QCD — The principle

'Ab initio' approach to determine standard model parameters

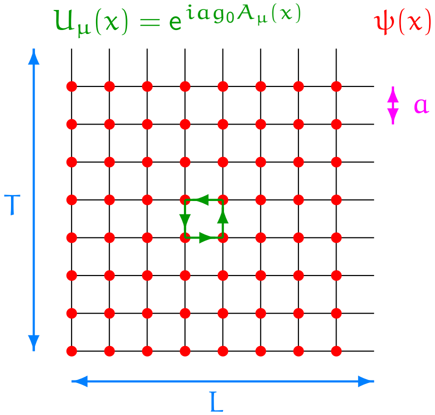
$$\mathcal{L}_{QCD}[g_0, m_f] = -\frac{1}{2g_0^2}\,\mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu}\} + \sum_{f=u,d,s,\ldots} \overline{\psi}_f \{\gamma_\mu(\partial_\mu + g_0 A_\mu) + m_f\}\psi_f$$

$$\underbrace{\begin{bmatrix} F_\pi \\ m_\pi \\ m_K \\ m_D \\ m_B \end{bmatrix}}_{\text{Experiment}} \quad \xrightarrow{\mathcal{L}_{QCD}[g_0, m_f]} \quad \underbrace{\begin{bmatrix} \Lambda_{QCD} \\ M_u, M_d \\ M_s \\ M_c \\ M_b \end{bmatrix}}_{\text{QCD parameters (RGIs)}} \quad + \quad \underbrace{\begin{bmatrix} F_D \\ F_B \\ B_K, B_B \\ \xi \\ \ldots \end{bmatrix}}_{\text{Predictions}}$$



Sources of systematic uncertainties in LQCD computations:

- Part of the vacuum polarization effects is missed, as long as u, d, s (and ideally also c) sea quarks are not incorporated
- Extrapolations to $m_{u,d}$ guided by χPT to connect to the physical world
- Discretization errors, notably from heavy quarks: $O\left[(am_Q)^n\right]$ effects
- Perturbative vs. non-perturbative renormalization



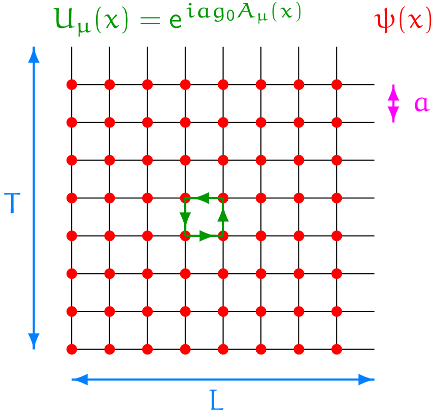
- Lattice cutoff $a^{-1} \sim \Lambda_{\mathrm{UV}}$
- Finite volume $L^3 \times T$
- Lattice action

$$S[U, \overline{\psi}, \psi] = S_G[U] + S_F[U, \overline{\psi}, \psi]$$

$$S_G = \frac{1}{g_0^2} \sum_p \mathrm{Tr}\{1 - U(p)\}$$

$$S_F = a^4 \sum_x \overline{\psi}(x)\, D[U]\, \psi(x)$$

- Physical quantities:
  Expectation values,
  represented as path integrals



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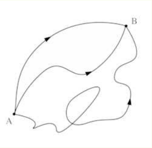
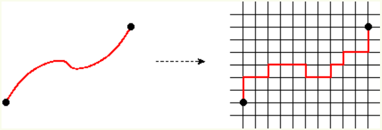
Feynman path (resp. functional) integral

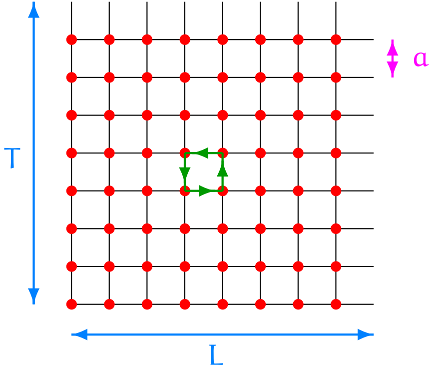


$$\langle B|e^{-H\tau}|A\rangle = \int \mathcal{D}[x(t)]\, e^{-S_E[x]}$$

$$x(0) = A\,,\ x(t) = B$$



$U_\mu(x) = e^{iag_0 A_\mu(x)}$ $\psi(x)$



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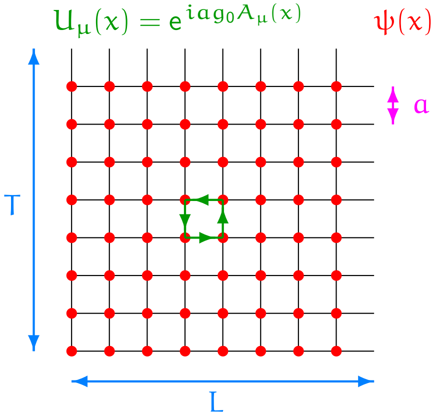
$$\mathcal{Z} = \int \mathcal{D}[U]\mathcal{D}[\overline{\psi}, \psi]\, e^{-S[U,\overline{\psi},\psi]} = \int \mathcal{D}[U] \prod_f \det\left(\not{D} + m_f\right) e^{-S_G[U]}$$

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \prod_{x,\mu} dU_\mu(x)\, O \prod_f \det\left(\not{D} + m_f\right) e^{-S_G[U]} \;\hat{=}\; \text{thermal average}$$



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Stochastic evaluation with *Monte Carlo (MC) methods*

$\rightarrow$ Observables $\langle O \rangle = \frac{1}{N} \sum_{n=1}^{N} O_n \pm \Delta_O$ from numerical simulations

Light sea quark ensembles in use

[in current studies of heavy quark physics]

Quenched approximation ($N_f = 0$)

- No dynamical fermions, not suitable for phenomenology
- Still useful test laboratory, e.g., to understand methodologies etc.

Two-flavour QCD ($N_f = 2$)

- NP'ly $O(a)$ improved Wilson ($=$ clover) fermions — ALPHA, QCDSF
- Twisted mass Wilson fermions — ETMC
- Stout-smeared, chirally improved fermions — BGR

Three-flavour QCD ($N_f = 2 + 1$)

- AsqTad-improved staggered quarks with debated rooting prescription $\left[\det^{(4)}(D_{st} + m)\right]^{\frac{1}{4}} \equiv \det^{(1)}(\gamma_\mu D_\mu + m)$ — MILC & FNAL, HPQCD
- Domain wall fermions — RBC & UKQCD
- NP'ly $O(a)$ improved Wilson fermions — PACS-CS

Four-flavour QCD ($N_f = 2 + 1 + 1$) — in progress, e.g., by ETMC

Light *valence* quarks usually discretized in the same way as the sea

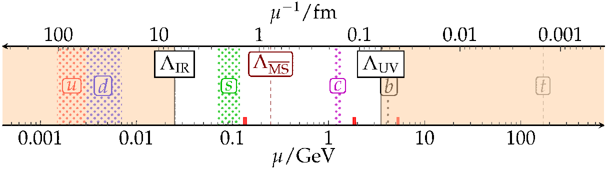
Challenge of LHQP: The multi-scale problem

Predictivity in a quantum field theory relies upon a large scale ratio

interaction range $\ll$ physical length scales

momentum cutoff $\gg$ physical mass scales : $\Lambda_{\mathrm{cut}} \sim a^{-1} \gg E_i, m_j$

This is a challenge in QCD, which has many physical scales:



hierarchy of disparate physical scales to be covered:

$$\Lambda_{\mathrm{IR}} = L^{-1} \ll m_\pi, \ldots, m_D, m_B \ll a^{-1} = \Lambda_{\mathrm{UV}}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left\{ O(e^{-Lm_\pi}) \Rightarrow L \gtrsim \frac{4}{m_\pi} \sim 6\,\mathrm{fm} \right\} \curvearrowright \quad L/a \gtrsim 120 \quad \curvearrowleft \left\{ am_D \lesssim \frac{1}{2} \Rightarrow a \approx 0.05\,\mathrm{fm} \right\}$$

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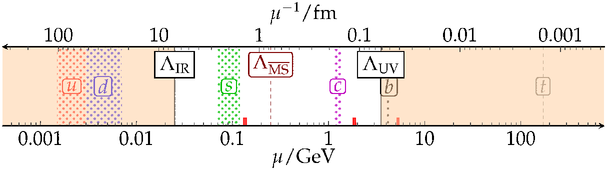


$\Rightarrow$ Difficult to satisfy simultaneously, clever technologies are required

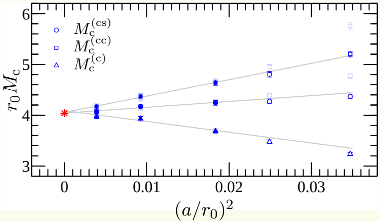
- charm just doable, but lattice artefacts may be substantial
- given the today's computing resources, it seems impossible to work directly with relativistic b-quarks (resolving their propagation) on the currently simulated lattices
- the b-quark scale ($m_b/m_c \sim 4$) has to be separated from the others in a theoretically sound way before simulating the theory

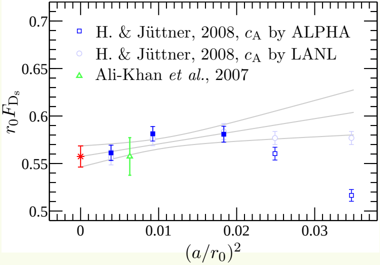
Illustration: Cutoff effects in the charm sector

High-precision computation of the charm quark's mass and F_{D_s} $(N_f = 0)$

- Large volume and small lattice spacings: $a \approx (0.09 - 0.03)$ fm
- $O(a, am_{q,c})$ cutoff effects relevant & removed NP'ly [ALPHA Collaboration]
- Controlling the CL demands scaling study down to very fine lattices

Lattice artefacts may be large for charm physics [H. & Jüttner, 2009]





$\Rightarrow$ *Warning* from F_{D_s}:

Symanzik programme works for charm, but $a < 0.08$ fm mandatory

- Note: small lattice spacings are challenging for $N_f > 0$
- $M_b \simeq 4M_c$ s.th. beauty is not yet accomodated

  $\rightarrow$ for b-quarks: can't control $a \rightarrow 0$ his way, *effective theory* needed

Lattice heavy quark formalisms

Lattice heavy quark physics has to deal with the presence of

strong lattice artefacts : $am_c \lesssim 1 \qquad am_b > 1$

Heavy quarks introduced as valence quarks $=$ "Partially quenched" setting

Relativistic formulations $\longrightarrow$ mainly for D-physics applications

- Wilson-like (clover or twisted mass) quarks
  - $am_c \leqslant 1/2 \ll 1$ desirable — ALPHA, ETMC
- Fermilab approach & its variants $=$ RHQ actions
  - relativistic clover actions with HQET interpretation
  - adopted for charm & beauty — FNAL & MILC, PACS-CS, RBC & UKQCD
- *H*ighly *I*mproved *S*taggered *Q*uarks $=$ HISQ
  - PT'ly improved / smeared glue, reduced taste-changing interactions
  - now also being tried towards the bottom region — HPQCD

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Non-relativistic / effective field theory strategies $\longrightarrow$ B-physics applications

- NRQCD $=$ Discretized, non-relativistic expansion of continuum $\mathcal{L}_D$
  - $\mathrm{O}\left[\alpha_s^n/(am_Q)\right]$ divergences — HPQCD
- Static approximation $=$ Leading-order HQET
  - HQET-guided extrapolations of fully relativistic simulations in the charm regime, turning into interpolations if the static limit is known
  - also in conjunction with finite-volume / finite-size scaling techniques

INFN-TOV, ALPHA, ETMC

- HQET for the b-quark $=$ Systematic expansion in $\Lambda_{\mathrm{QCD}}/m_b$
  - NP fine-tuning of parameters to $\mathrm{O}(1/m_b)$ & impr. statistical precision
  - connect different volumes iteratively with "step scaling functions"

ALPHA

A glimpse of the status of B-physics parameters

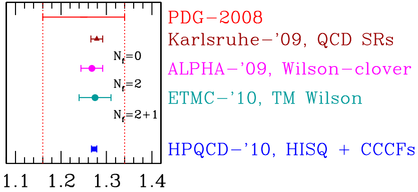
Heavy quark masses

= Inputs to many (B)SM calculations

Hadronic weak matrix elements F_B & F_{B_s}

$\rightarrow$ Extract $|V_{ub}|$ via $\underbrace{\mathcal{B}(B^- \rightarrow \tau^- \bar{\nu}_\tau)}_{\text{experiment}} \propto |V_{ub}|^2 \underbrace{F_B^2}_{\text{lattice}}$

$\overline{m}_c^{\overline{MS}}(\overline{m}_c)$ [GeV]



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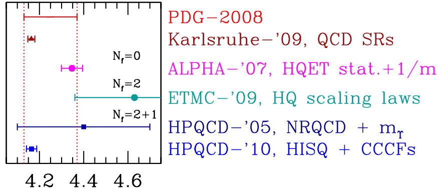
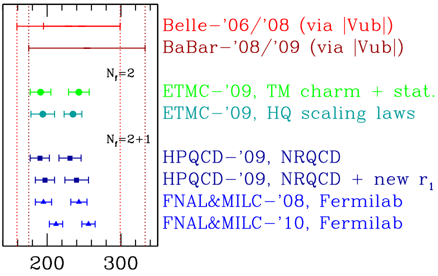


F_B F_{B_s} [MeV]



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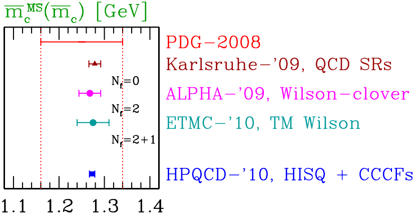

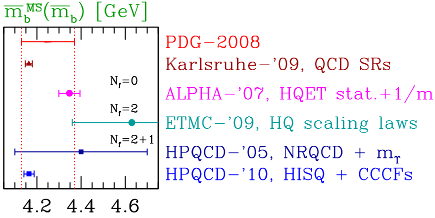

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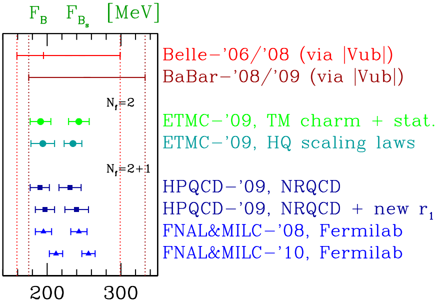

Caveat:

Lattice computations based on NRQCD, Fermilab and HQ scaling laws are standard, however, they all involve *perturbative* renormalization / matching

$\Rightarrow$ Is this accurate enough for precision flavour physics ?

A glimpse of the status of B-physics parameters

Heavy quark masses

= Inputs to many (B)SM calculations

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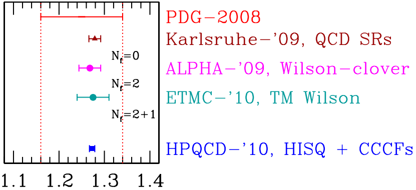
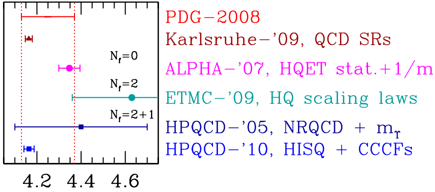


$\overline{m}_b^{\overline{\rm MS}}(\overline{m}_b)$ [GeV]



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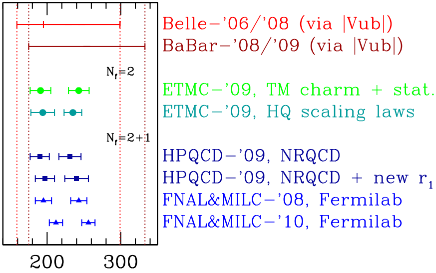

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$\Rightarrow$ Are the claimed small (particularly systematic) errors too optimistic ?

Non-perturbative HQET in two-flavour QCD

ALPHA Collaboration

B. Blossier, J. Bulava, M. Della Morte,
M. Donnellan, P. Fritzsch, N. Garron,
J. H., G.M. von Hippel, N. Tantalo,
H. Simma, R. Sommer

FlaviA net

- Non-perturbative formulation of HQET
- Mass dependence at leading order in $1/m$
- Strategy to determine HQET parameters at $O(1/m)$
- First physical results in the two-flavour theory

Scale, light quark masses from light sector:
F. Knechtli, B. Leder, S. Schaefer, F. Virotta

CLS based

Non-perturbative formulation of HQET

Action: $S_{\mathrm{HQET}}(x) = a^4 \sum_x \mathcal{L}_{\mathrm{HQET}}(x)$ for the b-quark (zero velocity HQET)

[Eichten, 1988; Eichten & Hill, 1990]

$$
\begin{aligned}
\mathcal{L}_{\mathrm{HQET}}(x) &= \mathcal{L}_{\mathrm{stat}}(x) - \omega_{\mathrm{kin}} \mathcal{O}_{\mathrm{kin}}(x) - \omega_{\mathrm{spin}} \mathcal{O}_{\mathrm{spin}}(x) \\
\mathcal{L}_{\mathrm{stat}}(x) &= \overline{\psi}_{\mathrm{h}}(x) \left[D_0 + m_{\mathrm{bare}} \right] \psi_{\mathrm{h}}(x) \qquad \tfrac{1}{2}(1+\gamma_0)\psi_{\mathrm{h}}(x) = \psi_{\mathrm{h}}(x) \\
\mathcal{O}_{\mathrm{kin}}(x) &= \overline{\psi}_{\mathrm{h}}(x)\, \mathbf{D}^2\, \psi_{\mathrm{h}}(x)
\end{aligned}
$$

$\rightarrow$ kinetic energy from heavy quark's residual motion

$$
\mathcal{O}_{\mathrm{spin}}(x) = \overline{\psi}_{\mathrm{h}}(x)\, \boldsymbol{\sigma} \cdot \mathbf{B}\, \psi_{\mathrm{h}}(x)
$$

$\rightarrow$ chromomagnetic interaction with the gluon field

Composite fields: axial current, related to the B-meson decay constant

$F_B \sqrt{m_B} = \langle\, B(\mathbf{p}=0) \,|\, A_0(0) \,|\, 0 \,\rangle$, where $A_0 = \overline{\psi}_{\mathrm{l}} \gamma_0 \gamma_5 \psi_{\mathrm{b}} \;\rightarrow\; A_0^{\mathrm{HQET}}$

$$
\begin{aligned}
A_0^{\mathrm{HQET}}(x) &= Z_A^{\mathrm{HQET}} \left[A_0^{\mathrm{stat}}(x) + c_A^{\mathrm{HQET}} \delta A_0^{\mathrm{stat}}(x) \right] \\
A_0^{\mathrm{stat}}(x) &= \overline{\psi}_{\mathrm{l}}(x) \gamma_0 \gamma_5 \psi_{\mathrm{h}}(x) \\
\delta A_0^{\mathrm{stat}}(x) &= \overline{\psi}_{\mathrm{l}}(x)\, \tfrac{1}{2} \left(\overleftarrow{\nabla}_i + \overleftarrow{\nabla}_i^{*} \right) \gamma_i \gamma_5\, \psi_{\mathrm{h}}(x)
\end{aligned}
$$

EVs $=$ Functional integral representation at the quantum level :

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi]\, O[\varphi]\, e^{-(S_{\text{rel}}+S_{\text{HQET}})} \qquad \mathcal{Z} = \int \mathcal{D}[\varphi]\, e^{-(S_{\text{rel}}+S_{\text{HQET}})}$$

Instead of including the NLO term in $1/m$ of $\mathcal{L}_{\text{HQET}}$ in the action (as this theory wouldn't be renormalizable), the *FI weight* is expanded in a *power series* in $1/m$

$$\exp\{-S_{\text{HQET}}\} =$$

$$\exp\left\{-a^4 \textstyle\sum_x \mathcal{L}_{\text{stat}}(x)\right\}$$
$$\times \left\{1 - a^4 \textstyle\sum_x \mathcal{L}^{(1)}(x) + \frac{1}{2}\left[a^4 \sum_x \mathcal{L}^{(1)}(x)\right]^2 - a^4 \sum_x \mathcal{L}^{(2)}(x) + \ldots\right\}$$

$$\Rightarrow \ \langle O \rangle \ = \ \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi]\, e^{-S_{\text{rel}} - a^4 \sum_x \mathcal{L}_{\text{stat}}(x)}\, O \, \left\{1 - a^4 \textstyle\sum_x \mathcal{L}^{(1)}(x) \ + \ \ldots\right\}$$

EVs $=$ Functional integral representation at the quantum level:

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi]\, O[\varphi]\, e^{-(S_{\mathrm{rel}}+S_{\mathrm{HQET}})} \qquad \mathcal{Z} = \int \mathcal{D}[\varphi]\, e^{-(S_{\mathrm{rel}}+S_{\mathrm{HQET}})}$$

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Explicitly:

$$\begin{aligned}
\langle O \rangle &= \langle O \rangle_{\mathrm{stat}} + \omega_{\mathrm{kin}} a^4 \sum_x \langle O \mathcal{O}_{\mathrm{kin}}(x) \rangle_{\mathrm{stat}} + \omega_{\mathrm{spin}} a^4 \sum_x \langle O \mathcal{O}_{\mathrm{spin}}(x) \rangle_{\mathrm{stat}} \\
&\equiv \langle O \rangle_{\mathrm{stat}} + \omega_{\mathrm{kin}} \langle O \rangle_{\mathrm{kin}} + \omega_{\mathrm{spin}} \langle O \rangle_{\mathrm{spin}} \\
\langle O \rangle_{\mathrm{stat}} &= \tfrac{1}{\mathcal{Z}} \int_{\mathrm{fields}} O \exp\left\{ -a^4 \sum_x \left[\mathcal{L}_{\mathrm{light}}(x) + \mathcal{L}_{\mathrm{h}}^{\mathrm{stat}}(x) \right] \right\}
\end{aligned}$$

EVs $=$ Functional integral representation at the quantum level:

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi]\, O[\varphi]\, \mathrm{e}^{-(S_{\mathrm{rel}}+S_{\mathrm{HQET}})} \qquad \mathcal{Z} = \int \mathcal{D}[\varphi]\, \mathrm{e}^{-(S_{\mathrm{rel}}+S_{\mathrm{HQET}})}$$

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$$\exp\left\{-a^4 \textstyle\sum_x \mathcal{L}_{\mathrm{stat}}(x)\right\}$$
$$\times \left\{1 - a^4 \textstyle\sum_x \mathcal{L}^{(1)}(x) + \frac{1}{2}\left[a^4 \sum_x \mathcal{L}^{(1)}(x)\right]^2 - a^4 \sum_x \mathcal{L}^{(2)}(x) + \ldots\right\}$$

$$\Rightarrow \quad \langle O \rangle \;=\; \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi]\, \mathrm{e}^{-S_{\mathrm{rel}} - a^4 \sum_x \mathcal{L}_{\mathrm{stat}}(x)}\, O \left\{1 - a^4 \textstyle\sum_x \mathcal{L}^{(1)}(x) \;+\; \ldots\right\}$$

Important implications of this definition of HQET

- $1/m$ – terms appear only as *insertions* of local operators in CFs
  $\Rightarrow$ Power counting: Renormalizability at any given order in $1/m$
- $\Leftrightarrow$ Existence of the continuum limit with universality
- Effective theory $=$ Continuum asymptotic expansion in $1/m$ of QCD

Renormalization & Matching

Renormalization

- The mixing of operators of different dimension in $\mathcal{L}_{\text{HQET}}$ induces power divergences [Maiani, Martinelli & Sachrajda, 1992]

  $\rightarrow$ $\mathcal{L}_{\text{stat}}$: linearly divergent additive mass renormalization δm originates from mixing of $\overline{\psi}_h D_0 \psi_h$ with $\overline{\psi}_h \psi_h$ $\Rightarrow$ $E^{\text{QCD}}_{h,\bar{h}} = E^{\text{stat}}_{h,\bar{h}}\big|_{\delta m = 0} + m_{\text{bare}}$

$$m_{\text{bare}} = \delta m + m \,, \quad \delta m = \frac{c(g_0)}{a} \sim e^{1/(2b_0 g_0^2)} \times \left\{ c_1 g_0^2 + c_2 g_0^4 + \ldots \right\}$$

  $\rightarrow$ PT: uncertainty $=$ truncation error $\sim e^{1/(2b_0 g_0^2)}\, c_{n+1}\, g_0^{2n+2} \xrightarrow{g_0 \to 0} \infty$!

  $\Rightarrow$ Non-perturbative $c(g_0)$ needed, i.e., *NP renormalization of HQET (resp. fixing of its parameters) required for the continuum limit to exist*

- Power-law divergences even worse at the level of $1/m$ – corrections: $a^{-1} \rightarrow a^{-2}$ (e.g., δm picks up a contribution $a^{-2}\omega_{\text{kin}}$)

Matching

- The finite parts of renormalization constants must be fixed s.th. the effective theory describes the underlying theory, QCD
- Proper conditions for these must be imposed from QCD with finite m_b

Mass dependence at leading order in $1/m$

The rôle of perturbative anomalous dimensions

Consider matrix elements of composite fields involving b-quarks as, e.g., obtained from a QCD correlation function of the heavy-light axial current

$$
\begin{aligned}
C_{AA}^{QCD}(x_0) &= Z_A^2 a^3 \sum_{\mathbf{x}} \langle A_0(x)(A_0)^\dagger(0)\rangle_{QCD} \\
\left[\Phi^{QCD}\right]^2 &\equiv F_B^2\, m_B \;=\; \left|\langle B\,|\,Z_A A_0\,|\,0\rangle\right|^2 \\
&= \lim_{x_0\to\infty}\left[2\exp\left\{x_0\, m_B^{eff}(x_0)\right\} C_{AA}^{QCD}(x_0)\right]
\end{aligned}
$$

- B-meson state dominates spectral representation of C_{AA}^{QCD} at large x_0
- $Z_A(g_0)$ fixed by chiral Ward identities, renormalization scale independent

In the static approximation this translates into

$$
\left[\Phi(\mu)\right]^2 = \left|\langle B\,|\,Z_A^{stat} A_0^{stat}\,|\,0\rangle\right|^2 = \lim_{x_0\to\infty}\left[2\exp\left\{x_0\, E_{stat}^{eff}(x_0)\right\} C_{AA}^{stat}(x_0)\right]
$$

- Absence of chiral symmetry in HQET implies a scale dependence
  $\to$ μ – dependence in $Z_A^{stat}(g_0, a\mu) = 1 + g_0^2\left[B_0 - \gamma_0 \ln(a\mu)\right] + O(g_0^4)$
- Better alternative: work with the RGI opertator $(A_{RGI}^{stat})_0$

How does one get from $\Phi_{\rm RGI} = Z^{\rm stat}_{\rm A,RGI} \langle\, B \,|\, A_0^{\rm stat} \,|\, 0 \,\rangle$ to F_B ?

QCD	LO HQET
$Z_A \langle\, B \,|\, A_0(0) \,|\, 0 \,\rangle_{\rm QCD}$	$C_{\rm PS}(M_b/\Lambda)\, Z^{\rm stat}_{\rm A,RGI} \langle\, B \,|\, A_0^{\rm stat}(0) \,|\, 0 \,\rangle_{\rm stat}$
$F_B \sqrt{m_B}$	$F_B \sqrt{m_B} + O(1/m_b)$

- Renormalization problem solved non-perturbatively (via interm. SF scheme)
  $\Rightarrow$ $Z^{\rm stat}_{\rm A,RGI}$: NP'ly known (to $\approx 1\%$ accuracy)
  [$N_f = 0$: H., Kurth & Sommer, 2003; $N_f = 2$: Della Morte, Fritzsch & H., 2007]
- $\langle\, B_{(s)} \,|\, A_0^{\rm stat} \,|\, 0 \,\rangle$: known for $N_f = 0$ and in progress for $N_f = 2$
  [ALPHA Collaboration, Blossier et al., arXiv:1006.5816]

  $\Rightarrow$ $\langle\, B_{(s)} \,|\, A_0^{\rm stat} \,|\, 0 \,\rangle_{\rm RGI} \longrightarrow F_B, F_{B_s}$ by multiplying with $C_{\rm PS}$

How does one get from $\Phi_{\mathrm{RGI}} = Z^{\mathrm{stat}}_{\mathrm{A,RGI}} \langle \mathrm{B} \,|\, A_0^{\mathrm{stat}} \,|\, 0 \rangle$ to F_{B} ?

QCD	LO HQET
$Z_{\mathrm{A}} \langle \mathrm{B} \,\vert\, A_0(0) \,\vert\, 0 \rangle_{\mathrm{QCD}}$	$C_{\mathrm{PS}}(M_{\mathrm{b}}/\Lambda)\, Z^{\mathrm{stat}}_{\mathrm{A,RGI}} \langle \mathrm{B} \,\vert\, A_0^{\mathrm{stat}}(0) \,\vert\, 0 \rangle_{\mathrm{stat}}$
$F_{\mathrm{B}}\sqrt{m_{\mathrm{B}}}$	$F_{\mathrm{B}}\sqrt{m_{\mathrm{B}}} + \mathrm{O}(1/m_{\mathrm{b}})$

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  $\Rightarrow$ $Z^{\mathrm{stat}}_{\mathrm{A,RGI}}$: NP'ly known (to $\approx 1\%$ accuracy)
  [$N_{\mathrm{f}} = 0$: H., Kurth & Sommer, 2003; $N_{\mathrm{f}} = 2$: Della Morte, Fritzsch & H., 2007]
- $\langle \mathrm{B}_{(\mathrm{s})} \,|\, A_0^{\mathrm{stat}} \,|\, 0 \rangle$: known for $N_{\mathrm{f}} = 0$ and in progress for $N_{\mathrm{f}} = 2$
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A closer look at the "conversion function" C_{PS} and γ^{match} :

$$\text{Matching} \quad \Leftrightarrow \quad \begin{aligned} \Phi^{\mathrm{QCD}}(m) &= \widetilde{C}_{\mathrm{match}}(m,\mu) \times \Phi(\mu) + \mathrm{O}(1/m) \\ \widetilde{C}_{\mathrm{match}}(m,\mu) &= 1 + c_1(m/\mu)\,\bar{g}^2(\mu) + \ldots \end{aligned}$$

- $m \leftrightarrow$ heavy (b) quark mass dependence on the QCD side
- $\mu \leftrightarrow$ (arbitrary) renormalization scale dependence in the effective theory
- this fixes the (finite) renormalization $\widetilde{C}_{\mathrm{match}} \longleftrightarrow$ "matching scheme"

QCD observables F_B, F_{B_s}: independent of renormalization scheme & scale

$\Rightarrow$ the $\mu-$dependence is artificial, only the mass dependence is for real

$\Rightarrow$ choose a convenient and common scale:

$$\mu = m_\star = \overline{m}(m_\star) \qquad g_\star = \bar{g}(m_\star)$$

$$\widetilde{C}_{\mathrm{match}}(m_\star, m_\star) = C_{\mathrm{match}}(g_\star) = 1 + c_1(1)\, g_\star^2 + \ldots$$

Eliminate the scheme dependence by passing to the RGI matrix element:

$$\Phi_{\mathrm{RGI}} = \exp\left\{-\int^{\bar{g}(\mu)} dx\, \frac{\gamma(x)}{\beta(x)}\right\} \Phi(\mu)$$

$$\Rightarrow \quad \Phi^{\mathrm{QCD}} = C_{\mathrm{match}}(g_\star)\, \Phi(\mu) = C_{\mathrm{match}}(g_\star) \exp\left\{\int^{g_\star} dx\, \frac{\gamma(x)}{\beta(x)}\right\} \Phi_{\mathrm{RGI}}$$

$$\equiv \exp\left\{\int^{g_\star} dx\, \frac{\gamma^{\mathrm{match}}(x)}{\beta(x)}\right\} \Phi_{\mathrm{RGI}} \qquad \text{defines } \gamma^{\mathrm{match}}$$

- $\gamma^{\mathrm{match}}(g_\star) = \frac{m_\star}{\Phi^{\mathrm{QCD}}} \frac{\partial \Phi^{\mathrm{QCD}}}{\partial m_\star}$ describes the full physical mass dependence . . .
- . . . but there is still a scheme dependence through the choice of $\overline{m}, \bar{g}$

Remove this renormalization scheme dependence by reparametrization in terms of renormalization group invariants Λ, M (= RGI heavy quark mass) :

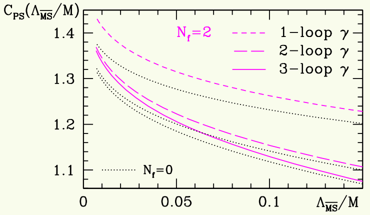
$$\Phi^{\mathrm{QCD}} = C_{\mathrm{PS}}(M/\Lambda) \times \Phi_{\mathrm{RGI}} \;, \quad C_{\mathrm{PS}}(M/\Lambda) = \exp\left\{ \int^{g_\star(\frac{M}{\Lambda})} dx \, \frac{\gamma^{\mathrm{match}}(x)}{\beta(x)} \right\}$$

To evaluate C_{PS}, insert $\gamma^{\mathrm{match}}(g_\star) \overset{g_\star \to 0}{\sim} -\gamma_0 g_\star^2 - \gamma_1^{\mathrm{match}} g_\star^4 - \gamma_2^{\mathrm{match}} g_\star^6 + \ldots$

$\Rightarrow$ leading large-mass behaviour via $\frac{M}{\Phi}\frac{\partial \Phi}{\partial M}\Big|_\Lambda = \frac{M}{C_{\mathrm{PS}}}\frac{\partial C_{\mathrm{PS}}}{\partial M}\Big|_\Lambda = \frac{\gamma^{\mathrm{match}}(g_\star)}{1-\tau(g_\star)}$:

$$C_{\mathrm{PS}} \overset{M \to \infty}{\sim} \left(2b_0 g_\star^2\right)^{-\gamma_0/(2b_0)} \sim \left[\log(M/\Lambda)\right]^{\gamma_0/(2b_0)}$$

C_{PS} perturbatively under control ? [3-loop AD by Chetyrkin & Grozin, 2003]

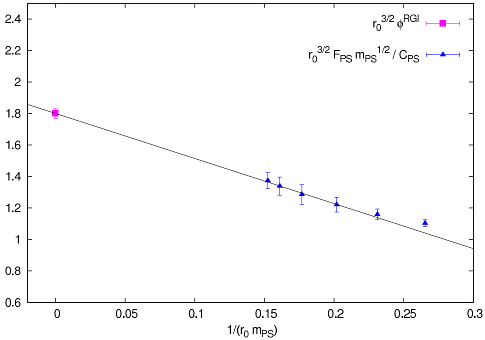


$N_f = 0, 2$

- RGI-*ratio* M/Λ: can be fixed in numerical simulations without perturbative errors
- Full (logarithmic) mass dependence $\in C_{\mathrm{PS}}$
- Fig. seems to indicate that the remaining $\mathrm{O}\left(\bar{g}^6(m_b)\right)$ errors are relatively small
  $\rightarrow$ however: a premature conclusion . . .
- For B-Physics: $\Lambda_{\overline{\mathrm{MS}}}/M_b \approx 0.04$

An application ($N_f = 0$)
Interpolation between the static limit and the charm region

Della Morte, Dürr, Guazzini, H., Jüttner & Sommer, JHEP0802(2008)078



Looks good: under a reasonable smoothness assumption, *interpolate* the mass dependence (linearly) in the inverse PS mass to the physical point

- F_{PS} follows the heavy quark scaling law, no $1/(r_0 m_{PS})^2$ – effects are visible
  $\rightarrow$ $1/m$ – expansion appears to work very well even for charm quarks
  $\leftarrow$ surprising; needs further confirmation, as the perturbative C_{PS} is used
- Question: What is the accuracy of perturbation theory involved in this ?

Accuracy of perturbation theory in the matching

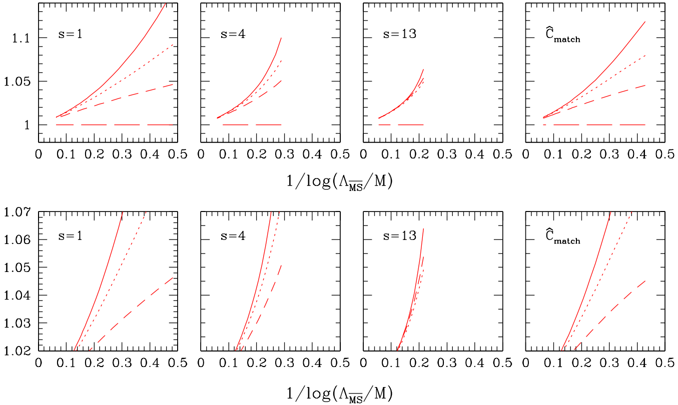
Bekavac, Grozin, Marquard, Piclum, Seidel & Steinhauser, NPB833(2010)46

$C_{\rm match}(g_\star)$ now known to $\rm N^3LO$ for various bilinears $\mathcal{O}_\Gamma = \overline{\psi}_l(x)\,\Gamma\,\psi_h(x)$

$\to$ $\gamma_\Gamma^{\rm match}$: 3-loop, $\gamma_\Gamma^{\rm match} - \gamma_{\Gamma'}^{\rm match}$: 4-loop (unknown 4-lp AD in HQET cancels)

$\Rightarrow$ *Ratios* of conversion functions reflect perturbative 4-loop precision:

$$C_{\Gamma/\Gamma'} = C^{\Gamma}_{\rm match}(m,\mu)/C^{\Gamma'}_{\rm match}(m,\mu)$$



Example

$C_{PS/V} = C_{PS}/C_V$

x-axis $\propto g_\star^2(M/\Lambda)$

For B-physics:

$\Lambda_{\overline{\rm MS}}/M_b \approx 0.04$

$1/\ln(\Lambda_{\overline{\rm MS}}/M_b) \approx 0.3$

PT is badly behaved for beauty and even worse for charm

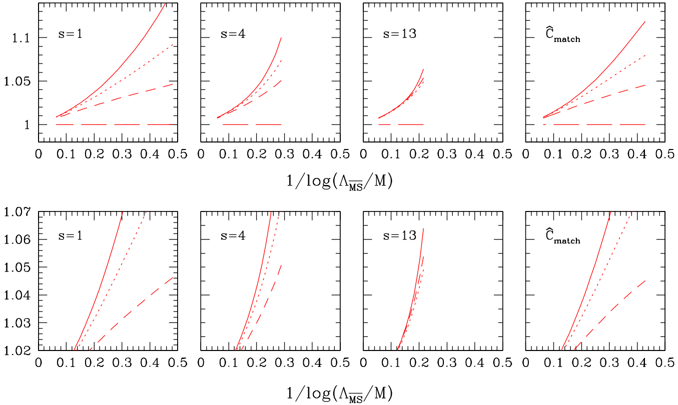
"We find that the perturbative series for f_{B^*}/f_B and $f^T_{B^*}/f_{B^*}$ converge very slowly at best."

[quote from Bekavac at al., 2010]

Freedom to "optimize" the scale

[R. Sommer, private communication]

$$\mu = s^{-1} m_\star = \overline{m}(m_\star)\ ,\ \ \hat{g} = \bar{g}\left(s^{-1} m_\star\right) \qquad C_\Gamma\,(M/\Lambda) = \exp\left\{ \int^{\hat{g}} dx\, \frac{\hat{\gamma}^{\rm match}_\Gamma(x)}{\beta(x)} \right\}$$



Matching below $m_\star$, expect $s > 1$ is better

Decrease of terms in perturbative series improved, once $s \gtrsim 4$

However:

$\alpha(m_b/4)$ is not small, series unreliable again

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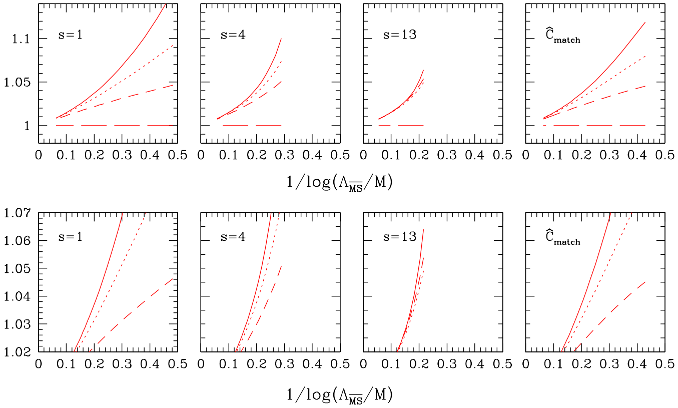

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→ Effective scale is well below $\mu = m_b$; asymptotic convergence of PT only improved far beyond m_b, where it is of limited use for us

⇒ Accuracy of perturbative matching is hard to assess for b- and c-physics

"We find that the perturbative series for f_{B^*}/f_B and $f^T_{B^*}/f_{B^*}$ converge very slowly at best." [quote from Bekavac at al., 2010]

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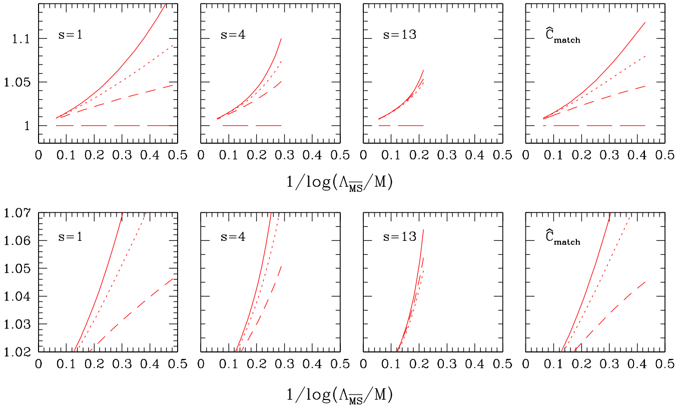

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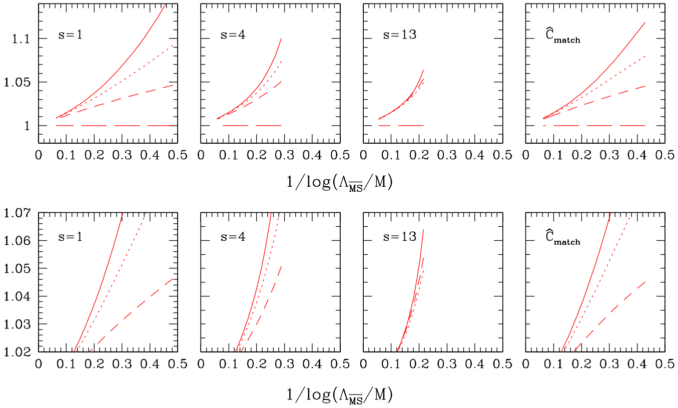
$\Rightarrow$ Error estimates in the literature seem much too optimistic . . .

"We find that the perturbative series for f_{B^*}/f_B and $f^T_{B^*}/f_{B^*}$ converge very slowly at best." [quote from Bekavac at al., 2010]

Freedom to "optimize" the scale [R. Sommer, private communication]

$$\mu = s^{-1} m_\star = \overline{m}(m_\star)\ ,\ \hat{g} = \bar{g}\left(s^{-1} m_\star\right) \qquad C_\Gamma\left(M/\Lambda\right) = \exp\left\{\int^{\hat{g}} dx\, \frac{\hat{\gamma}_\Gamma^{\text{match}}(x)}{\beta(x)}\right\}$$



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However:

$\alpha(m_b/4)$ is not small, series unreliable again

$\Rightarrow\ \bar{g}^{2l}(m_b) \propto \left[2b_0 \ln\left(m_b/\Lambda_{QCD}\right)\right]^{-l} \overset{m_b\to\infty}{\gg} \Lambda_{QCD}/m_b$: Pert. matching theor. consistent only at LO in $1/m_b$, a few–% error budget requires NP matching

Mass dependence in finite-volume QCD ($N_f = 2$)

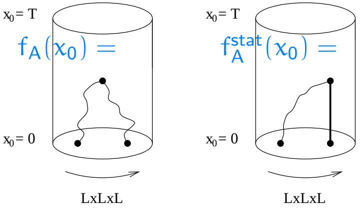
Della Morte, Fritzsch, H. & Sommer, PoS LATTICE2008(2008)226
Fritzsch & H., in progress

Non-perturbative computation of the *heavy quark mass dependence* of heavy-light meson observables in the continuum limit of finite-volume QCD

$\longrightarrow$ Explicit pure theory tests that HQET is an *effective* theory of QCD

$\longrightarrow$ Constraining the large-mass behaviour of QCD by the static limit

- QCD with Schrödinger Functional boundary conditions (T, L, θ):



- Renormalization [ALPHA Collaboration, 2005-2008]
  - Fix $\bar{g}^2(L_1) = 4.484$ s.th. $L_1 \approx 0.5\,\text{fm}$, $L_1/a = 20, 24, 32, 40$, $L_2 = 2L_1$
  - Fix RGI (heavy) quark masses via its NP relation to bare parameters:

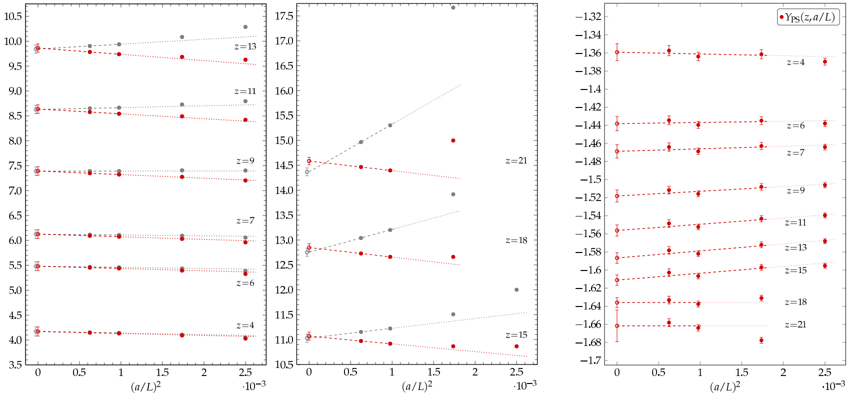
$$z \equiv L_1 M = Z_m \frac{M}{\overline{m}(\mu_0)} (1 + b_m a m_q) \times L_1 m_q \qquad Z_m = \frac{Z(g_0)\, Z_A(g_0)}{Z_P(g_0, a\mu_0)}$$

[Fritzsch, H. & Tantalo, arXiv:1004.3978]

Mass dependence in finite-volume QCD ($N_f = 2$)

The B-system in finite-volume QCD ($L = L_1$)

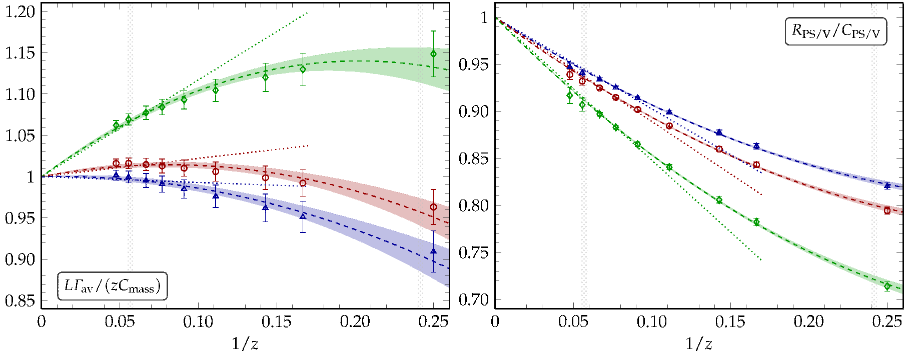
- $L_1 = 0.5\,\mathrm{fm}$, z–values covering the b-quark down to the charm quark region
- Removal of all $O\left(\left(\frac{a}{L}\right)^n\right)$ effects at tree-level: $O \rightarrow O_{\mathrm{impr}}(a/L) = \frac{O(a/L)}{1+\delta(a/L)}$
- Examples of continuum extrapolations (B-meson mass & decay constant):



Mass dependence in finite-volume QCD ($N_f = 2$)

The B-system in finite-volume QCD ($L = L_1$)

- Tests of HQET: validating and demonstrating the applicability of HQET
- Verification of the approach to the spin-symmetric limit:
  (B-meson mass & ratio of PS to V decay constants)



$\Rightarrow$ Large-mass asymptotics ($1/z \to 0$) confirms HQET predictions

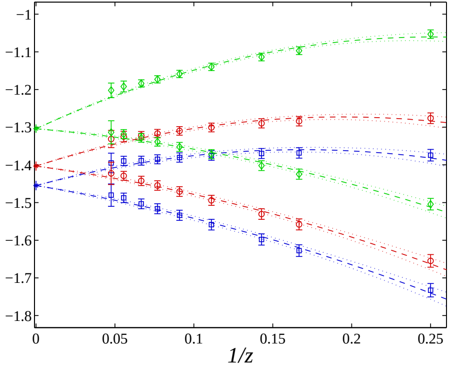
Mass dependence in finite-volume QCD ($N_f = 2$)

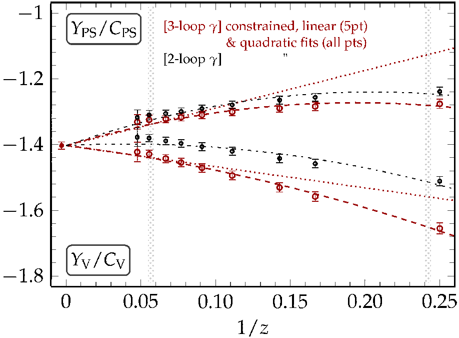
The B-system in finite-volume QCD ($L = L_1$)

- But: some numerical evidence for the previous doubts in the reliability of PT in the b-quark region is found with Y_{PS}, Y_V and its effective theory predictions

$$Y_{PS}(L,z)/C_{PS}(M/\Lambda) = X_{RGI}(L) + O(1/z)$$

$$Y_{PS}(L,z;\theta) \propto Z_A \frac{f_A(L/2,\theta)}{\sqrt{f_1(\theta)}} \qquad X_{RGI}(L;\theta) \propto Z^{stat}_{A,RGI} \underbrace{\frac{f_A^{stat}(L/2,\theta)}{\sqrt{f_1^{stat}(\theta)}}}_{=X^{stat}(\theta)}$$





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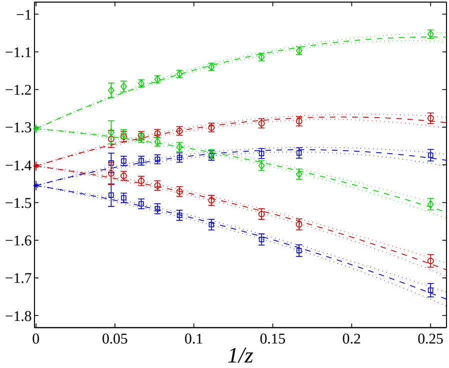
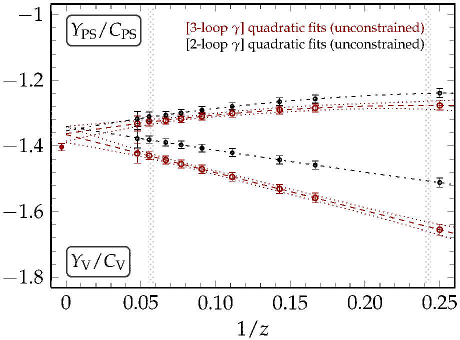
$$Y_{PS}(L,z;\theta) \propto Z_A \frac{f_A(L/2,\theta)}{\sqrt{f_1(\theta)}} \qquad X_{RGI}(L;\theta) \propto Z^{stat}_{A,RGI} \underbrace{\frac{f^{stat}_A(L/2,\theta)}{\sqrt{f^{stat}_1(\theta)}}}_{=X^{stat}(\theta)}$$



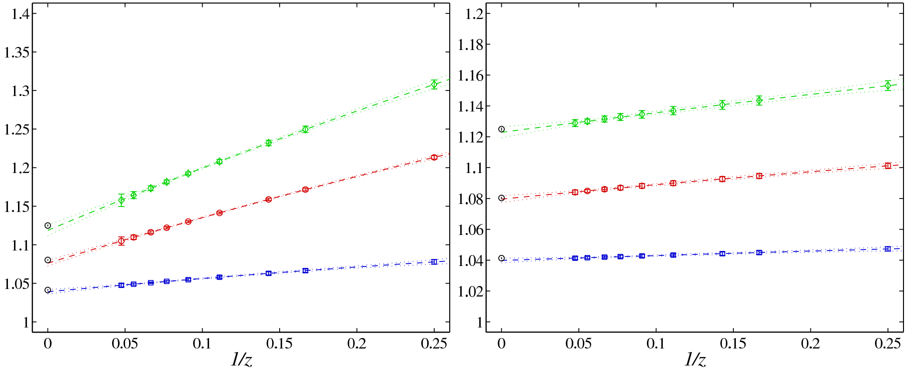


Mass dependence in finite-volume QCD ($N_f = 2$)

The B-system in finite-volume QCD ($L = L_1$)

- Consider *ratios* instead, where C_{PS} cancels completely:

$$\frac{Y_{PS}(z;\theta_1)}{Y_{PS}(z;\theta_2)} = \frac{X^{stat}(\theta_1)}{X^{stat}(\theta_2)} + O(1/z)$$



$\Rightarrow$ These turn smoothly & unconstrained into effective theory predictions

Determination of HQET parameters at $\mathrm{O}(1/m)$

Blossier, Della Morte, Garron & Sommer, arXiv:1001.4783

Vector of the $N_{\mathrm{HQET}} = 5$ parameters in $S_{\mathrm{HQET}}, A_0^{\mathrm{HQET}}$ up to $\mathrm{O}(1/m_b)$:

$$\omega = \begin{pmatrix} \omega^{\mathrm{stat}} \\ \omega^{(1/m)} \end{pmatrix}$$

$$\omega^{\mathrm{stat}} = \left(m_{\mathrm{bare}} \, , \, \ln(Z_A^{\mathrm{HQET}}) \right)^t$$

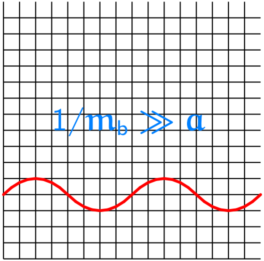
$$\omega^{(1/m)} = \left(c_A^{\mathrm{HQET}} \, , \, \omega_{\mathrm{kin}} \, , \, \omega_{\mathrm{spin}} \right)^t$$

ω_i	classical value	static value
m_{bare}	m_b	$m_{\mathrm{bare}}^{\mathrm{stat}}$
$\ln(Z_A^{\mathrm{HQET}})$	0	$\ln(Z_{A,RGI}^{\mathrm{stat}} C_{PS})$
c_A^{HQET}	$-1/(2m_b)$	$a c_A^{\mathrm{stat}}$
ω_{kin}	$1/(2m_b)$	0
ω_{spin}	$1/(2m_b)$	0

$\Rightarrow$ Trick: *non-perturbative* matching of HQET to QCD in a *finite* volume

[H. & Sommer, JHEP0402(2004)022]

QCD

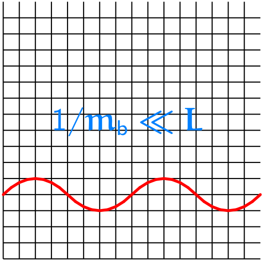


Matching conditions

$$\boxed{\Phi_i^{\mathrm{QCD}} = \Phi_i^{\mathrm{HQET}}}$$

for observables Φ_i

(renormal. quantities, computable for $a \to 0$)



HQET

NP matching in $L = L_1$

Suitable observables in the Schrödinger functional, $L = T = L_1 \approx 0.5\,\mathrm{fm}$

$$\Phi_i(L_1, M, a) \qquad i = 1, \ldots, N_{\mathrm{HQET}}$$

Matching conditions for $i = 1, \ldots, N_{\mathrm{HQET}}$ (note: $a \leftrightarrow g_0$)

$$\lim_{a\to 0} \Phi_i^{\mathrm{QCD}}(L_1, M, a) = \Phi_i^{\mathrm{QCD}}(L_1, M, 0) = \Phi_i^{\mathrm{HQET}}(L_1, M, a)$$

Conveniently, one chooses observables linear in ω_i, e.g.

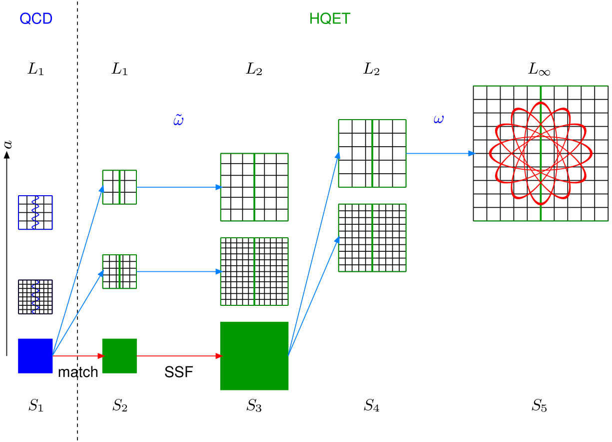
$$\begin{aligned}
\Phi(L, M, a) &= \quad \eta(L, a) + \phi(L, a)\,\omega(M, a) \\
\Phi_1 &= \quad L\,\langle\, B(L)\,|\,\mathbb{H}\,|\,B(L)\,\rangle \;\overset{L\to\infty}{\sim}\; L m_B \\
\Phi_2 &= \quad \ln\left(L^{3/2}\,\langle\, \Omega(L)\,|\,A_0\,|\,B(L)\,\rangle\right) \;\overset{L\to\infty}{\sim}\; \ln\left(L^{3/2} F_B \sqrt{m_B/2}\right) \\
&\ldots
\end{aligned}$$

$$\eta = \begin{pmatrix} \Gamma^{\mathrm{stat}} = \langle\, B(L)\,|\,\mathbb{H}\,|\,B(L)\,\rangle_{\mathrm{stat}} \\ \zeta_A = \ln\left(L^{3/2}\,\langle\, \Omega(L)\,|\,A_0\,|\,B(L)\,\rangle_{\mathrm{stat}}\right) \\ \ldots \end{pmatrix} \qquad \phi = \begin{pmatrix} L & 0 & \cdots \\ 0 & 1 & \cdots \\ \ldots & & \end{pmatrix}$$

Step scaling to $\mathrm{L} = \mathrm{L}_2$

Matching volume $\mathrm{L}_1 \approx 0.5\,\mathrm{fm}$ has very small a, but larger a are needed

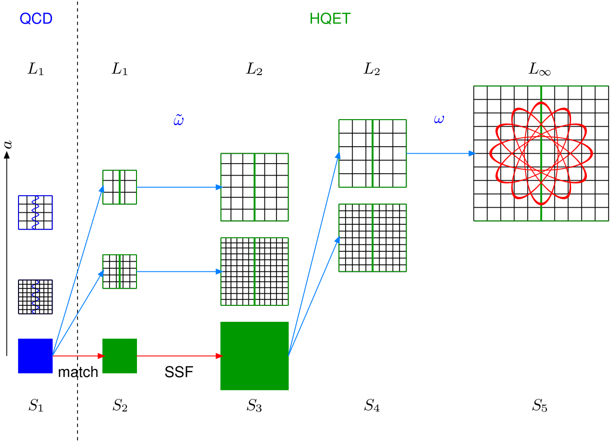
$\Rightarrow$ Gap to large volume & practicable lattice spacings, where physical quantities (m_B, F_B) are extracted, bridged by finite-size scaling steps



Fully NP, CL can be taken everywhere, $\mathrm{L} \to 2\mathrm{L}$ via Step Scaling Functions

$$\Phi_i^{\mathrm{HQET}}(2\mathrm{L}) = \sigma_i\left(\left\{\Phi_j^{\mathrm{HQET}}(\mathrm{L}), j = 1, \ldots, N_{\mathrm{HQET}}\right\}\right) \qquad 2\mathrm{L} = 2\mathrm{L}_1 \approx 1.0\,\mathrm{fm}$$

Step scaling to $\mathrm{L} = \mathrm{L}_2$



Finite-size scaling to $\mathrm{L}_2 = 2\mathrm{L}_1$:

- Amounts to solve a matrix equation to obtain the HQET parameters at larger lattice spacings . . .
- . . . corresponding to β–values for simulations in large volume, "L_∞", where a B-meson in HQET fits comfortably

1.) Continuum limit $a = 0.025\,\mathrm{fm}, \ldots, 0.012\,\mathrm{fm}$

$$\Phi_i(L_1, M, 0) = \lim_{a/L_1 \to 0} \Phi_i^{\mathrm{QCD}}(L_1, M, a)$$

2.) HQET parameters for $a = 0.05\,\mathrm{fm}, \ldots, 0.025\,\mathrm{fm}$

$$\begin{aligned} \omega(M, a) &\equiv \phi^{-1}(L_1, a)\,[\,\Phi(L_1, M, 0) - \eta(L_1, a)\,] \\ &= \begin{pmatrix} L_1^{-1}\Phi_1(L_1, M, 0) - \Gamma^{\mathrm{stat}}(L_1, a) \\ \Phi_2(L_1, M, 0) - \zeta_A(L_1, a) \\ \ldots \end{pmatrix} \end{aligned}$$

3.) Insert into $\Phi(L_2, M, a)$

$$\begin{aligned} \Phi(L_2, M, 0) &= \lim_{a/L_2 \to 0} [\eta(L_2, a) + \phi(L_2, a)\,\omega(M, a)] \\ &= \lim_{a/L_2 \to 0} \underbrace{\begin{pmatrix} L_2\,[\,\Gamma^{\mathrm{stat}}(L_2, a) - \Gamma^{\mathrm{stat}}(L_1, a)\,] \\ \zeta_A(L_2, a) - \zeta_A(L_1, a) \\ \ldots \end{pmatrix}}_{\text{finite SSFs}} + \underbrace{\begin{pmatrix} \frac{L_2}{L_1}\,\Phi_1(L_1, M, 0) \\ \Phi_2(L_1, M, 0) \\ \ldots \end{pmatrix}}_{\text{QCD mass dependence}} \end{aligned}$$

4.) Repeat 2.) for $L_1 \to L_2$ to obtain $\omega(M, a)$ for $a = 0.1\,\mathrm{fm}, \ldots, 0.05\,\mathrm{fm}$

$$\omega(M, a) \equiv \phi^{-1}(L_2, a)\,[\,\Phi(L_2, M, 0) - \eta(L_2, a)\,]$$

Use of the HQET parameters

These HQET parameters can finally be exploited for phenomenological applications in the $B_{(s)}$–meson system, e.g. to

- calculate the b-quark mass and the $B_{(s)}$–meson decay constant:

$$m_B = m_{\rm bare} + E_{\rm stat} + \omega_{\rm kin} E_{\rm kin} + \omega_{\rm spin} E_{\rm spin}$$

$$\frac{\Phi}{\sqrt{2}} \equiv F_B\sqrt{m_B/2} = Z_A^{\rm HQET}\left(1 + b_A^{\rm stat} a m_q\right) p_{\rm stat} \times \left(1 + c_A^{\rm HQET} p_{\delta A} + \omega_{\rm kin} p_{\rm kin} + \omega_{\rm spin} p_{\rm spin}\right)$$

- Mass splittings, such as (radial) excitation energies of $B_{(s)}$–states and the $B_{(s)} - B^*_{(s)}$ mass difference to $O(1/m_b)$:

$$\Delta E^{\rm HQET}_{n,1} = \left(E^n_{\rm stat} - E^1_{\rm stat}\right) + \omega_{\rm kin}\left(E^n_{\rm kin} - E^1_{\rm kin}\right) + \omega_{\rm spin}\left(E^n_{\rm spin} - E^1_{\rm spin}\right)$$

$$\Delta E_{P-V} = \tfrac{4}{3}\,\omega_{\rm spin} E^1_{\rm spin}$$

E^i_y, p_y : plateau averages of (bare) effective HQET energies and matrix elements in large volume

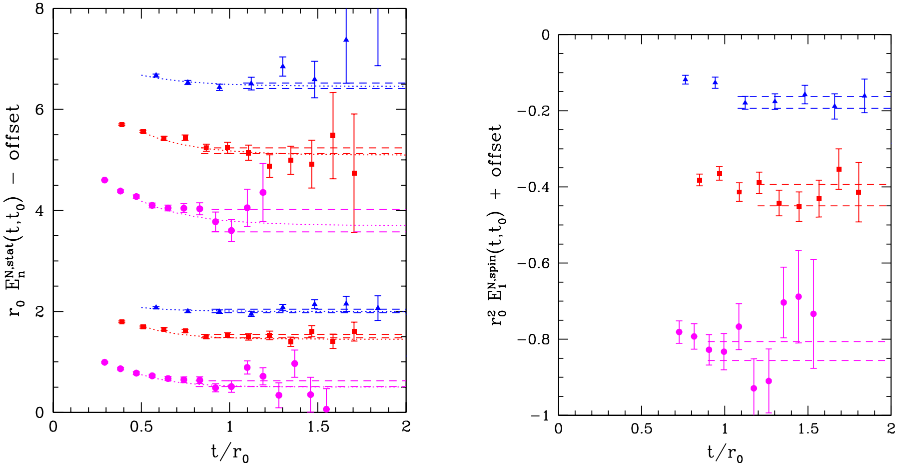
- Note: The power-divergent δm drops out in energy *differences*

Some examples of $\mathrm{N_f} = 0$ results

Blossier, Della Morte, Garron, von Hippel, Mendes, Simma & Sommer, arXiv:1004.2661

Excited state energy levels, $a \approx (0.1, 0.08, 0.05)$ fm, $L \approx 1.5$ fm, $T = 2L$

- ► CF matrices $C^{\mathrm{stat}}_{ij}(t) = \sum_{x,\mathbf{y}} \langle O_i(x_0 + t, \mathbf{y})\, O^*_j(x) \rangle_{\mathrm{stat}}$ & $\mathcal{O}_{\mathrm{spin/kin}}$ insertions
- ► GEVP: all-to-all propagators, t– dilution, Gaussian smeared variational basis



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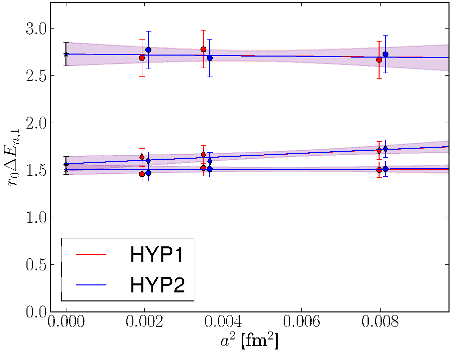
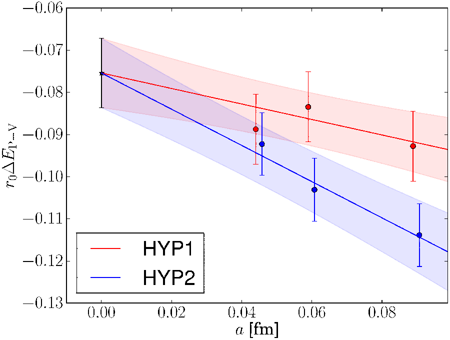




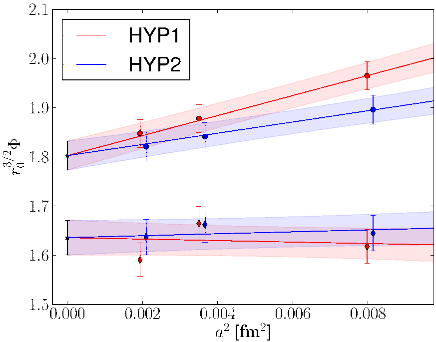
- Linear a–term suppressed by $1/m_b$, physical $O(1/m_b)$ corrections are small
- Divergences cancel after proper NP renormalization

  $\Rightarrow$ Strong *numerical* evidence for the renormalizability of HQET

Some examples of $N_f = 0$ results

Blossier, Della Morte, Garron, von Hippel, Mendes, Simma & Sommer, arXiv:1006.5816

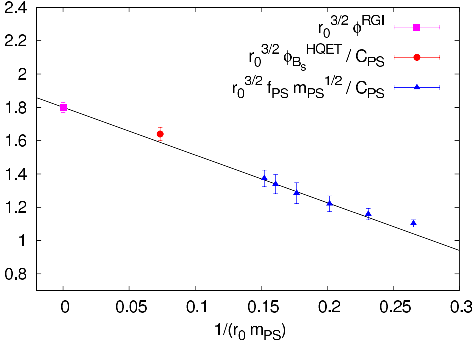
Matrix elements in the B-meson system via applying the same techniques



- Important remark:
  Here, the full factor $Z_A^{stat} = Z_{A,RGI}^{stat} C_{PS}(M_b/\Lambda)$ is implicitly evaluated non-perturbatively, i.e., C_{PS} *irrelevant* in the context of NP matching!
- HYP & GEVP lead to (2 – 3)% precision for F_{B_s} in the continuum limit
  $r_0 = 0.5$ fm: $F_{B_s}^{stat} = 229(3)$ MeV, $F_{B_s}^{stat+1/m} = 212(5)$ MeV
  (using $r_0 = 0.45$ fm leads to $\simeq 15\%$ increase, but $O(1/m_b^2)$ corrections are small)

Some examples of $N_f = 0$ results

Computation of F_{B_s} in HQET matches at m_{B_s} with a straight interpolation between the QCD charm sector (around F_{D_s}) and $F_{B_s}^{stat}$

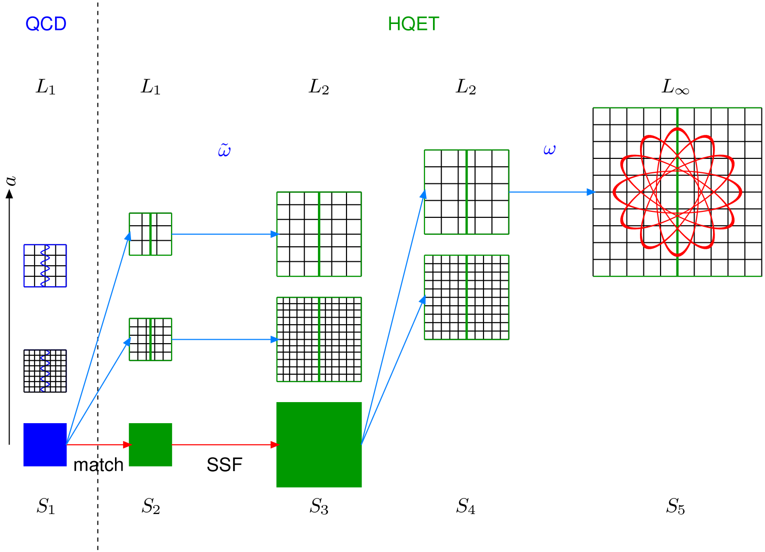


- In this comparison, C_{PS} just enters to compensate for the logarithmic scaling of Φ with m_b, i.e., $C_{PS} =$ perturbative "relic" in interpolation strategies
- Given the unclear precision of PT, interpolation methods to be taken with care, as the inherent perturbative $[\alpha_s(m_b)]^3$–errors are difficult to estimate
- Anyway, data points beyond charm computationally challenging for $N_f > 0$

First physical results in the two-flavour theory

Which ingredients are needed ?

Recall the strategy . . .



First physical results in the two-flavour theory

Which ingredients are needed ?

S_1 NP matching of HQET to QCD in finite volume with a relativistic b, to perform the power-divergent subtractions

- Crucial element of this step:
  Calculation of the *heavy quark mass dependence* of heavy-light meson observables in the continuum limit of finite-volume QCD (L_1)
- . . . already discussed above

$S_{2,3,4}$ HQET computations in small & intermediate volumes

- Evaluation of the HQET step scaling functions to connect the small matching ($L_1 \approx 0.5\,\mathrm{fm}$) to the intermediate volume ($L_2 = 2L_1 \approx 1\,\mathrm{fm}$)
- Interpolation of the resulting HQET parameters to the large-volume "L_∞" lattice spacings ($\beta = 5.2, 5.3, 5.5$)

S_5 HQET computations in large volume

- Extract HQET energies & matrix elements, using $N_f = 2$ dynamical configurations in large volume ("L_∞", periodic b.c.'s) produced by CLS
- Action: NP'ly $O(a)$ improved $N_f = 2$ Wilson ; algorithm: DD-HMC
- Problem of slowed sampling of topological modes with decreasing a less relevant, because HQET can afford to work with coarser lattices

HQET energies & matrix elements (preliminary)

Preliminary $N_f = 2$ HQET results in large volume

ALPHA Collaboration, in progress

CLS based

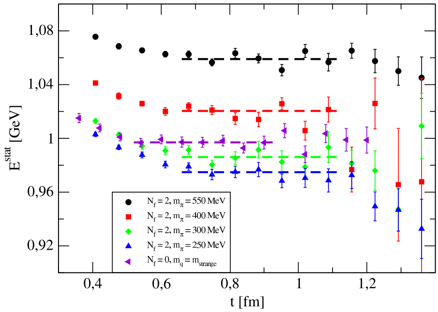
- Gauge configuration ensembles with $N_f = 2$ NP'ly $O(a)$ improved Wilson fermions generated within Coordinated Lattice Simulations (= community European team effort, employing Lüscher's DD-HMC)

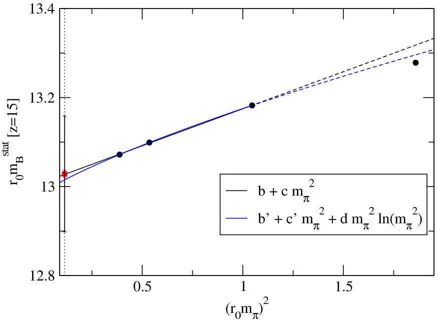
β	$a\,[\mathrm{fm}]$	$L^3 \times T$	$m_\pi\,[\mathrm{MeV}]$	#	traj. sep.
5.2	0.08	$32^3 \times 64$	700	110	16
		$32^3 \times 64$	370	160	16
5.3	0.07	$32^3 \times 64$	550	152	32
		$32^3 \times 64$	400	600	32
		$48^3 \times 96$	300	192	16
		$48^3 \times 96$	250	350	16
5.5	0.05	$32^3 \times 64$	430	250	20
		$48^3 \times 96$	430	30	16

- High numerical accuracy of lattice HQET thanks to technical advances:
  [Hasenfratz & Knechtli, 2001; Lüscher & Wolff, 1990; Foley et al., 2005; ALPHA Collaboration 2004-2009]
  - HYP-smeared static actions, giving improved statistical precision
  - solve the Generalized EigenValue Problem for a correlator matrix to cleanly quantify systematic errors from excited state contaminations
  - Variant of the stochastic all-to-all propagator method for light quarks

HQET energies & matrix elements (preliminary)

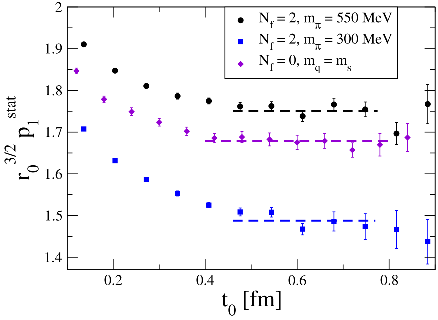
Static energies ($\beta = 5.3$, $a \approx 0.07\,\mathrm{fm}$) & extrapolation to the chiral limit, where the uncertainty due to r_0/a is still large [Scale setting preliminary]

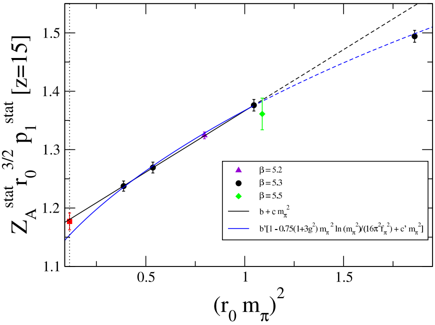




HQET energies & matrix elements (preliminary)

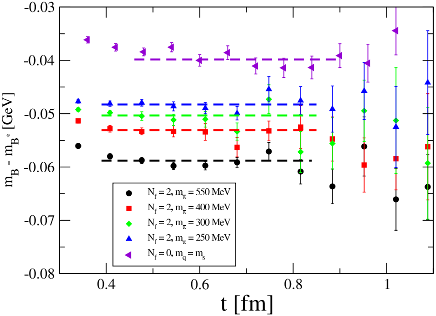
B-meson decay constant (F_B): renormalized (not $O(a)$ improved) matrix element of A_0^{stat}, data well described by HMχPT

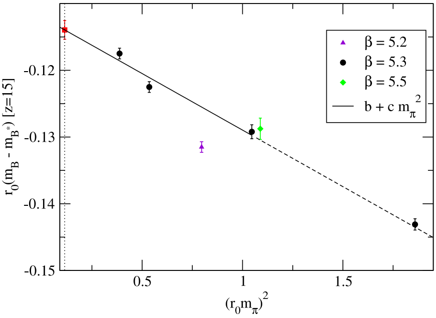




HQET energies & matrix elements (preliminary)

Spin-splitting: situation for $\mathrm{O}(1/m)$ terms of energies is encouraging





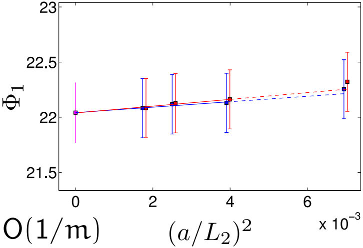
HQET parameters (preliminary)

ALPHA Collaboration, in progress

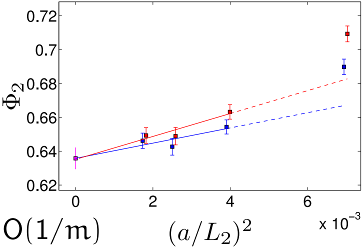
After evolution to L_2 where $5.3 \lesssim \beta \lesssim 5.8$

$\Phi_1 = L \langle B(L) | \mathbb{H} | B(L) \rangle$ $\Phi_2 = \ln \left(L^{3/2} \langle \Omega(L) | A_0 | B(L) \rangle \right)$

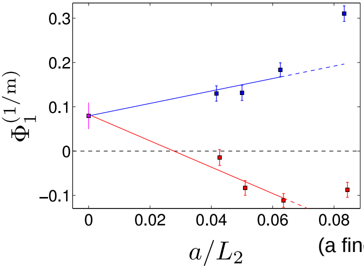
$O(m)$



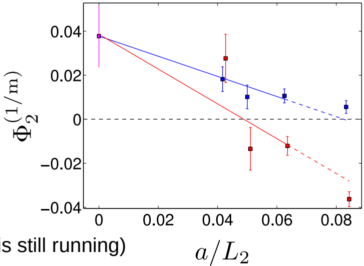
$O(1)$



$O(1/m)$



$O(1/m)$



(a finer lattice resolution is still running)

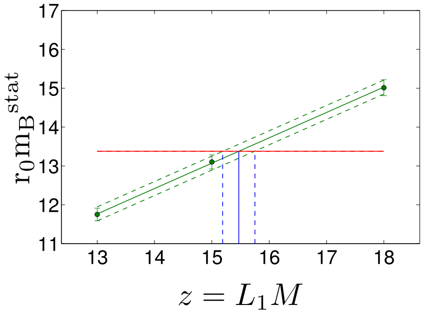
b-quark mass interpolation (preliminary)

ALPHA Collaboration, in progress

Now insert $\omega_1 \in \omega(M, a)$ for $N_f = 2$:

$$\begin{aligned} m_B \;=\;& \omega_1 + E_{stat} = m_{bare} + E_{stat} = \omega_1 + E_{stat} \\ =\;& \lim_{a\to 0}\left[E_{stat} - \Gamma^{stat}(L_2, a)\right] && a = (0.1 - 0.05)\,\mathrm{fm} \\ &+ \lim_{a\to 0}\left[\Gamma^{stat}(L_2, a) - \Gamma^{stat}(L_1, a)\right] && a = (0.05 - 0.025)\,\mathrm{fm} \\ &+ \frac{1}{L_1}\lim_{a\to 0}\Phi_1(L_1, M_b, a) && a = (0.025 - 0.012)\,\mathrm{fm} \end{aligned}$$

Analysis with $r_0 m_B^{(exp)}$, $r_0 = (0.475 \pm 0.025)\,\mathrm{fm}$ [Scale setting preliminary]



- $\overline{m}_b^{\overline{MS}}(\overline{m}_b)^{stat} = 4.255(25)_{r_0}(50)_{stat+renorm}(?)_a$ GeV
- NP renormalization; no CL yet in the large volume part (only $\beta = 5.3$)
- Error dominated by $\approx 1\%$ on Z_M in $L_1 M = Z_M\, Z\,(1 + b_m a m_q) \times L_1 m_q$
- Dependence on the matching kinematics is very small

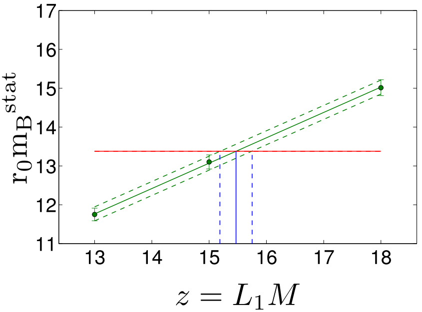
b-quark mass interpolation (preliminary)

ALPHA Collaboration, in progress

Now insert $\omega_1 \in \omega(M, a)$ for $N_f = 2$:

$$\begin{aligned} m_B &= \omega_1 + E_{stat} = m_{bare} + E_{stat} = \omega_1 + E_{stat} \\ &= \lim_{a\to 0}\left[E_{stat} - \Gamma^{stat}(L_2, a)\right] && a = (0.1 - 0.05)\,\mathrm{fm} \\ &\quad + \lim_{a\to 0}\left[\Gamma^{stat}(L_2, a) - \Gamma^{stat}(L_1, a)\right] && a = (0.05 - 0.025)\,\mathrm{fm} \\ &\quad + \frac{1}{L_1}\lim_{a\to 0}\Phi_1(L_1, M_b, a) && a = (0.025 - 0.012)\,\mathrm{fm} \end{aligned}$$

Analysis with $r_0 m_B^{(exp)}$, $r_0 = (0.475 \pm 0.025)\,\mathrm{fm}$ [Scale setting preliminary]



- $\overline{m}_b^{\overline{MS}}(\overline{m}_b)^{stat+1/m} =$ $4.276(25)_{r_0}(50)_{stat+renorm}(?)_a$ GeV
- NP renormalization; no CL yet in the large volume part (only $\beta = 5.3$)
- Error dominated by $\approx 1\%$ on Z_M in $L_1 M = Z_M\, Z\,(1 + b_m a m_q) \times L_1 m_q$
- Dependence on the matching kinematics is very small

b-quark mass interpolation (preliminary)

ALPHA Collaboration, in progress

Now insert $\omega_1 \in \omega(M, a)$ for $N_f = 2$:

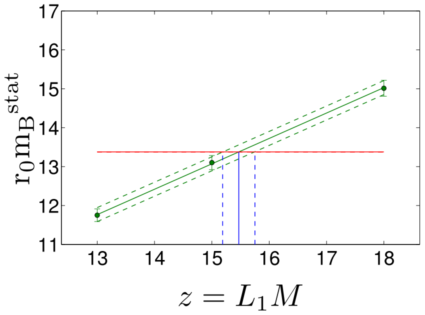
$$m_B = \omega_1 + E_{stat} = m_{bare} + E_{stat} = \omega_1 + E_{stat}$$

$$= \lim_{a\to 0}\left[E_{stat} - \Gamma^{stat}(L_2, a)\right] \qquad a = (0.1 - 0.05)\,\mathrm{fm}$$

$$+ \lim_{a\to 0}\left[\Gamma^{stat}(L_2, a) - \Gamma^{stat}(L_1, a)\right] \qquad a = (0.05 - 0.025)\,\mathrm{fm}$$

$$+ \frac{1}{L_1}\lim_{a\to 0}\Phi_1(L_1, M_b, a) \qquad a = (0.025 - 0.012)\,\mathrm{fm}$$

Analysis with $r_0 m_B^{(exp)}$, $r_0 = (0.475 \pm 0.025)\,\mathrm{fm}$ [Scale setting preliminary]



- $\overline{m}_b^{\overline{MS}}(\overline{m}_b)^{stat+1/m} = 4.347(40)_{r_0}(48)\,\mathrm{GeV}$ ($N_f = 0$!)
- NP renormalization; no CL yet in the large volume part (only $\beta = 5.3$)
- Error dominated by $\approx 1\%$ on Z_M in $L_1 M = Z_M\, Z\,(1 + b_m a m_q) \times L_1 m_q$
- Dependence on the matching kinematics is very small

Unquenching effect is presently not significant

b-quark mass interpolation (preliminary)

ALPHA Collaboration, in progress

Now insert $\omega_1 \in \omega(M, a)$ for $N_f = 2$:

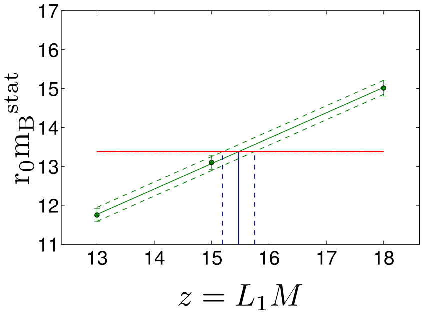
$$m_B = \omega_1 + E_{stat} = m_{bare} + E_{stat} = \omega_1 + E_{stat}$$

$$= \lim_{a\to 0}\left[E_{stat} - \Gamma^{stat}(L_2, a)\right] \qquad a = (0.1 - 0.05)\,\mathrm{fm}$$

$$+ \lim_{a\to 0}\left[\Gamma^{stat}(L_2, a) - \Gamma^{stat}(L_1, a)\right] \qquad a = (0.05 - 0.025)\,\mathrm{fm}$$

$$+ \frac{1}{L_1}\lim_{a\to 0}\Phi_1(L_1, M_b, a) \qquad a = (0.025 - 0.012)\,\mathrm{fm}$$

Analysis with $r_0 m_B^{(exp)}$, $r_0 = (0.475 \pm 0.025)\,\mathrm{fm}$ [Scale setting preliminary]



- $\overline{m}_b^{\overline{MS}}(\overline{m}_b)^{stat+1/m} = 4.276(25)_{r_0}(50)_{stat+renorm}(?)_a$ GeV
- NP renormalization; no CL yet in the large volume part (only $\beta = 5.3$)
- Error dominated by $\approx 1\%$ on Z_M in $L_1 M = Z_M\, Z\,(1 + b_m a m_q) \times L_1 m_q$
- Dependence on the matching kinematics is very small

Unquenching effect is presently not significant

Conclusions & Outlook

- Lattice heavy flavour physics is becoming a precision field
- Lattice QCD inputs have to be pushed to few-% level (incl. a reliable assessment of all systematics), to contribute to uncovering signals for BSM physics in CKM analyses and resolve / support current tensions
- Dynamical quark simulations ($N_f = 2, 2+1, 2+1+1$) are routine: $m_\pi \sim 500\,\text{MeV}$ (2001) $\;\longrightarrow\;$ $m_\pi \lesssim 250\,\text{MeV}$ (2010), but the behaviour of algorithms at small lattices spacings needs to be understood
- An entirely non-perturbative renormalization & matching in HQET is doable with considerable accuracy
  - Pert. functions C_X not needed altogether within our NP HQET strategy
  - Physics goals of lattice HQET with $1/m$–corrections:
    b-quark mass, decay constants $F_{B_{(s)}}$ (1st $O(1/m)$ computation ever !), mass splittings, semi-leptonic form factors
  - Continuum limit of the large volume part for $N_f = 2$ finished soon
  - $N_f = 4$ in the longer run: add also strange & charm sea quark flavours