# *Matrix model and $\beta$-deformed $\mathcal{N} = 4$ SYM*


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- Enlarged parameter space: $\{g\} \to \{g, h, \beta\}$
- Preserves $\mathcal{N}=1$ supersymmetry and can be made finite on a certain submanifold of the parameter space.

For instance, for $\beta$ real the condition for finiteness up to two loops reads

$$h^2\left[1-\frac{2}{N_c^2}(1-\cos\beta)\right]=g^2$$

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At the end of the day, $n$ $\mathrm{U}(1)$ “photons” (and *“photinos”*) $w_i^{\alpha}$

$$\boxed{\mathcal{L} = \boldsymbol{\tau_{ij}(\varphi)} w_i^{\alpha} w_{j\alpha}} \qquad i, j = 1, \ldots, n \qquad \varphi = \langle \phi_1 \rangle + \imath \langle \phi_2 \rangle$$

# The Matrix Model

Back in 2002, Dijkgraaf and Vafa *conjectured* a relation between $\mathrm{U}(N_c)$ supersymmetric gauge field theories and zero-dimensional bosonic matrix model

| $\mathcal{N}=1$ gauge theory | | Matrix Model |
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| $W_{\text{tree}}(\Phi_i)$ | $\Longleftrightarrow$ | $S_m(\hat{\Phi}_i)$ |
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The microscopic superpotential of the gauge theory is taken as the action for the random matrices:

G.T. $$Z=\int\mathcal{D}\Phi\mathcal{D}\bar{\Phi}\exp-\mathrm{tr}_{\mathrm{U}(N_c)}\left[\mathcal{W}^2+\bar{\Phi}\,\mathrm{e}^V\,\Phi+W_{\text{tree}}(\Phi)+\text{h.c.}\right]$$

M.M. $$Z_m=\mathrm{e}^{-\frac{\hat{N}^2}{g_m^2}\mathcal{F}}=\int\mathrm{d}\hat{\Phi}\exp-\left[\frac{\hat{N}}{g_m}\,\mathrm{tr}\,W_{\text{tree}}(\hat{\Phi})\right]$$

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$\hat{N}$ and $N_c$ are completely **independent!**

# Black magic at work

The Matrix Model admits a 't Hooft large-$\hat{N}$ expansion,

$$-\log Z = \frac{\hat{N}^2}{g_m^2}\mathcal{F} = \exp\sum_{h\geq 0}\big[\frac{g_m}{\hat{N}}\big]^{2h-2}\mathcal{F}_h(\mathcal{S}_i) \qquad \mathcal{S}_i \equiv \lim_{\hat{N}\to\infty} g_m\frac{\hat{N}_i}{\hat{N}}$$

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The Dijkgraaf-Vafa prescription leads to the formula:

$$W_{\text{eff}}(\mathcal{S}_i, w_i^\alpha) = \sum_i N_i \frac{\partial \mathcal{F}_0}{\partial \mathcal{S}_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \mathcal{F}_0}{\partial \mathcal{S}_i \partial \mathcal{S}_j} w_i^\alpha w_{\alpha j}$$

$$\boldsymbol{\tau_{ij}} = \frac{\partial^2 \mathcal{F}_0}{\partial \mathcal{S}_i \partial \mathcal{S}_j} - \delta_{ij} \frac{1}{N_i} \sum_k N_k \frac{\partial^2 \mathcal{F}_0}{\partial \mathcal{S}_i \partial \mathcal{S}_k}$$

where $\tau_{ij}$ is the coupling constant matrix for the $\mathrm{U}(1)$ "photons" $w_i^\alpha$

# Result

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$$\tau(M_0 = 0) = -\log \frac{g^2(\varphi_1 - \varphi_2)^2}{h^2(\mathrm{e}^{\imath\beta/2}\,\varphi_1 - \mathrm{e}^{-\imath\beta/2}\,\varphi_2)(\mathrm{e}^{-\imath\beta/2}\,\varphi_1 - \mathrm{e}^{\imath\beta/2}\,\varphi_2)}$$

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$$+ \frac{8h^4 A^2 \sin^2\beta/2}{g^4\Delta^4(0)\Delta(\beta)\Delta(-\beta)}\Big[(-3 + 4\cos\beta - \cos 2\beta)\varphi_1^6 + 2(-5 + 6\cos\beta - \cos 2\beta)\varphi_1^5\varphi_2$$
$$+ (-13 + 16\cos\beta - 3\cos 2\beta)\varphi_1^4\varphi_2^2 + 4(-1 + 2\cos\beta - \cos 2\beta)\varphi_1^3\varphi_2^3 + (\varphi_1 \leftrightarrow \varphi_2)\Big]$$

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$$+ \frac{16h^8 A^4 \sin^4\beta/2}{g^8\Delta^8(0)\Delta^2(\beta)\Delta^2(-\beta)}\Big[(-139 + 166\cos\beta - 6\cos 2\beta - 26\cos 3\beta + 5\cos 4\beta)\varphi_1^{12}$$
$$+ 2(-258 + 368\cos\beta - 125\cos 2\beta + 30\cos 3\beta + 3\cos 4\beta)\varphi_1^{11}\varphi_2$$
$$- 2(446 - 767\cos\beta + 464\cos 2\beta - 89\cos 3\beta + 36\cos 4\beta)\varphi_1^{10}\varphi_2^2$$
$$+ 2(-606 + 1116\cos\beta - 709\cos 2\beta + 250\cos 3\beta + 39\cos 4\beta)\varphi_1^9\varphi_2^3$$
$$+ (-2513 + 3650\cos\beta - 870\cos 2\beta + 578\cos 3\beta - 125\cos 4\beta)\varphi_1^8\varphi_2^4$$
$$- 4(876 - 1220\cos\beta + 871\cos 2\beta + 6\cos 3\beta + 97\cos 4\beta)\varphi_1^7\varphi_2^5$$
$$+ 2(-827 + 1745\cos\beta - 694\cos 2\beta + 559\cos 3\beta + 99\cos 4\beta)\varphi_1^6\varphi_2^6 + (\varphi_1 \leftrightarrow \varphi_2)\Big]$$

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Leading order in accord with the literature

Dorey & Hollowood, 2005; Kuzenko & Tseytlin, 2005

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Higher orders

$$A^2 \propto \exp[-8\pi^2/g^2]$$

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Consistency check

$$\lim_{\beta \to 0} \tau = 0$$

$\Rightarrow$ We recover pure $\mathcal{N} = 4$!

# Conclusions

## Done!

- Versatility: $M_0 = 0, M_0 \neq 0$; $\mathrm{U}(N_c) \mapsto \mathrm{U}(N_1) \otimes \mathrm{U}(N_2) \otimes \dots$
- Also different models: $\mathcal{N} = 1$, $\mathcal{N} = 1^*$, $\mathcal{N} = 2$, ...

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## To Do

- More general deformations: $\mathrm{tr}\,\Phi_i^3$ and the like
- Instanton calculus in $\mathcal{N} = 4$ $\beta$-deformed
- $\mathrm{U}(N_c)$ vs. $\mathrm{SU}(N_c)$
- ...