

Matrix model and β -deformed $\mathcal{N} = 4$ SYM

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- Enlarged parameter space: $\{g\} \rightarrow \{g, h, \beta\}$
- Preserves $\mathcal{N} = 1$ supersymmetry and can be made finite on a certain submanifold of the parameter space.

For instance, for β real the condition for finiteness up to two loops reads

$$h^2 \left[1 - \frac{2}{N_c^2} (1 - \cos \beta) \right] = g^2$$

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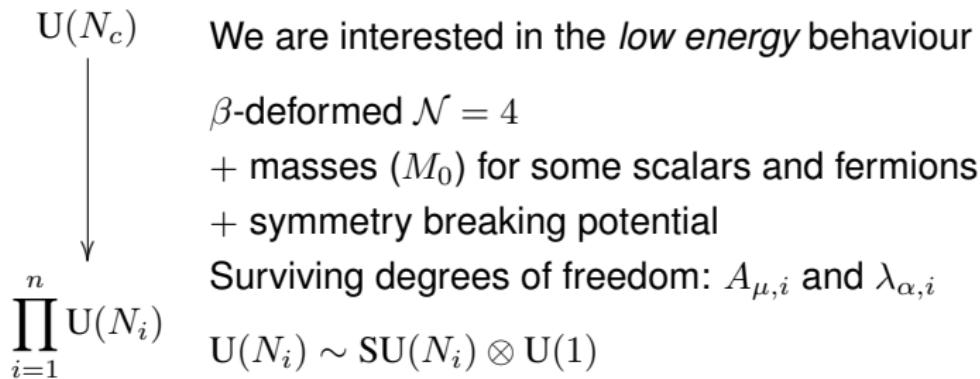
+ masses (M_0) for some scalars and fermions

+ symmetry breaking potential

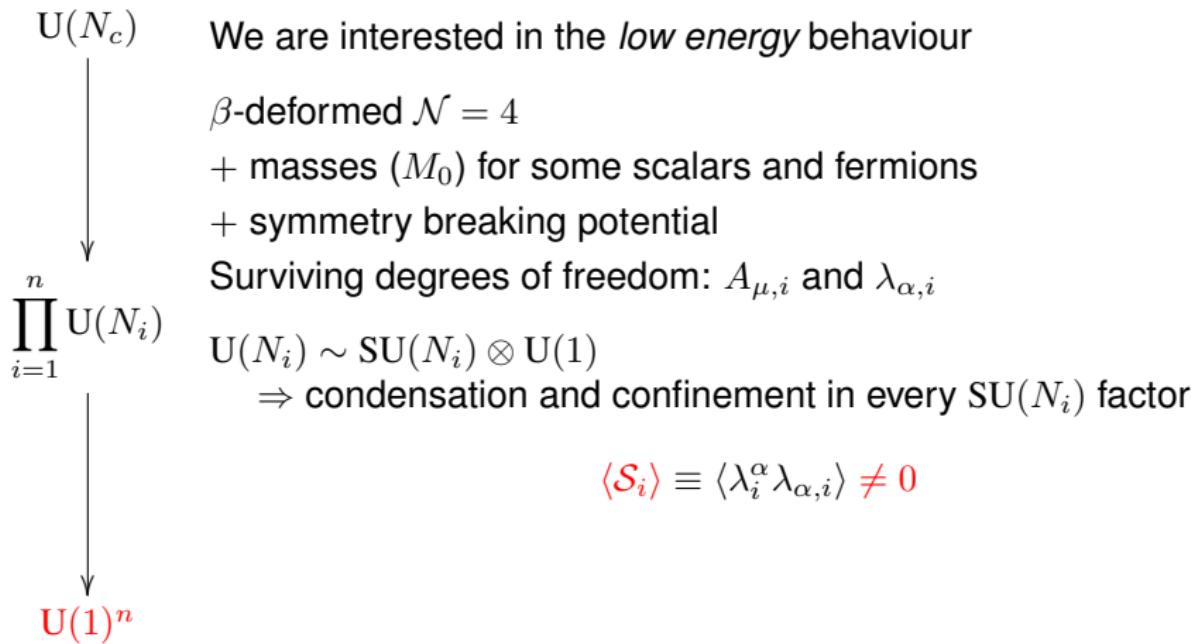
Surviving degrees of freedom: $A_{\mu,i}$ and $\lambda_{\alpha,i}$

$$\prod_{i=1}^n U(N_i)$$

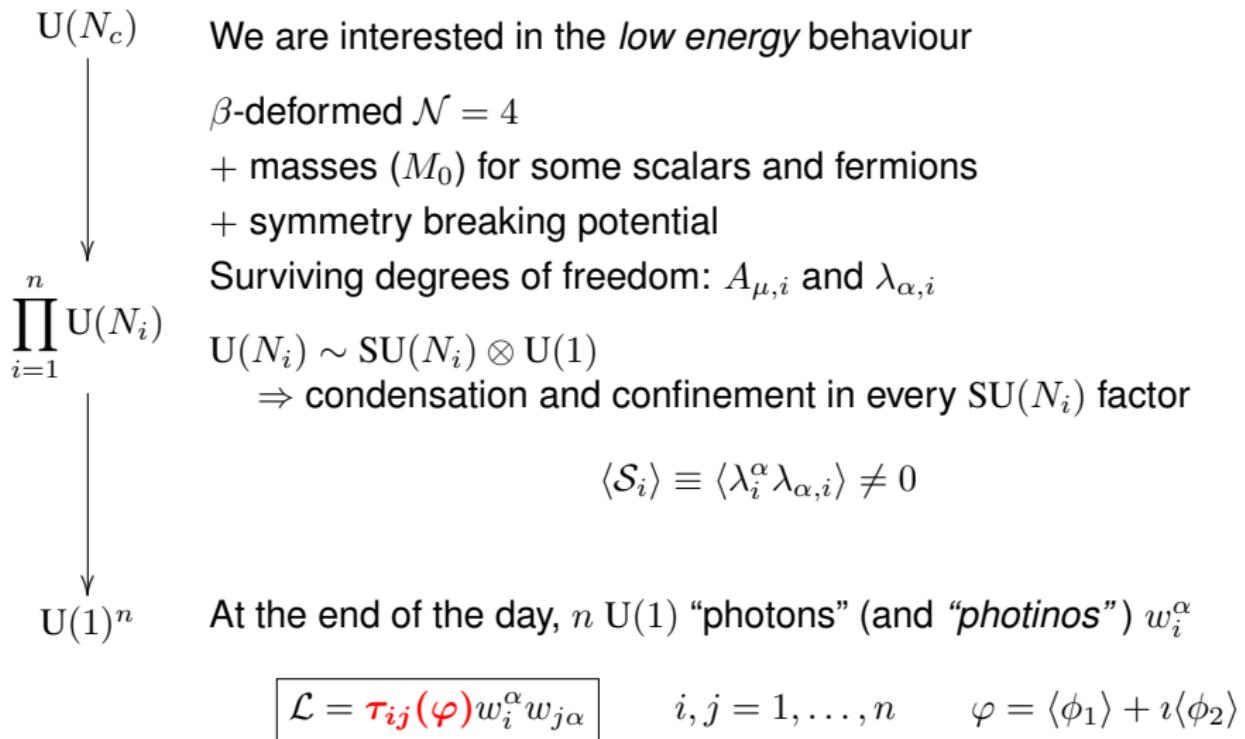
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The Matrix Model

Back in 2002, **Dijkgraaf** and **Vafa** conjectured a relation between $U(N_c)$ supersymmetric gauge field theories and zero-dimensional bosonic matrix model

$$\begin{array}{ccc} \mathcal{N} = 1 \text{ gauge theory} & & \text{Matrix Model} \\ W_{\text{tree}}(\Phi_i) & \iff & S_m(\hat{\Phi}_i) \\ \Phi_i : \text{chiral superfields} & & \hat{\Phi}_i : \hat{N} \times \hat{N} \text{ matrices} \end{array}$$

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Φ_i : chiral superfields $\hat{\Phi}_i$: $\hat{N} \times \hat{N}$ matrices

The microscopic superpotential of the gauge theory is taken as the action for the random matrices:

G.T. $Z = \int \mathcal{D}\Phi \mathcal{D}\bar{\Phi} \exp - \text{tr}_{U(N_c)} \left[\mathcal{W}^2 + \bar{\Phi} e^V \Phi + W_{\text{tree}}(\Phi) + \text{h.c.} \right]$

M.M. $Z_m = e^{-\frac{\hat{N}^2}{g_m^2} \mathcal{F}} = \int d\hat{\Phi} \exp - \left[\frac{\hat{N}}{g_m} \text{tr } W_{\text{tree}}(\hat{\Phi}) \right]$

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\hat{N} and N_c are completely independent!

Black magic at work

The Matrix Model admits a 't Hooft large- \hat{N} expansion,

$$-\log Z = \frac{\hat{N}^2}{g_m^2} \mathcal{F} = \exp \sum_{h \geq 0} \left[\frac{g_m}{\hat{N}} \right]^{2h-2} \mathcal{F}_h(\mathcal{S}_i) \quad \mathcal{S}_i \equiv \lim_{\hat{N} \rightarrow \infty} g_m \frac{\hat{N}_i}{\hat{N}}$$

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The Dijkgraaf-Vafa prescription leads to the formula:

$$W_{\text{eff}}(\mathcal{S}_i, w_i^\alpha) = \sum_i N_i \frac{\partial \mathcal{F}_0}{\partial \mathcal{S}_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \mathcal{F}_0}{\partial \mathcal{S}_i \partial \mathcal{S}_j} w_i^\alpha w_{\alpha j}$$
$$\tau_{ij} = \frac{\partial^2 \mathcal{F}_0}{\partial \mathcal{S}_i \partial \mathcal{S}_j} - \delta_{ij} \frac{1}{N_i} \sum_k N_k \frac{\partial^2 \mathcal{F}_0}{\partial \mathcal{S}_i \partial \mathcal{S}_k}$$

where τ_{ij} is the coupling constant matrix for the U(1) "photons" w_i^α

Result

$$\mathrm{U}(2) \hookrightarrow \mathrm{U}(1)^2 \quad \tau_{ij} = \tau \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

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$$\tau(M_0 = 0) = -\log \frac{g^2(\varphi_1 - \varphi_2)^2}{h^2(e^{i\beta/2}\varphi_1 - e^{-i\beta/2}\varphi_2)(e^{-i\beta/2}\varphi_1 - e^{i\beta/2}\varphi_2)}$$

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$$\tau = \color{blue}{\tau^{(0)}}$$

Leading order in accord with the literature

Dorey & Hollowood, 2005; Kuzenko & Tseytlin, 2005

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Higher orders

$$A^2 \propto \exp[-8\pi^2/g^2]$$

\Rightarrow one-instanton action!

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Consistency check

$$\lim_{\beta \rightarrow 0} \tau = 0$$

⇒ We recover pure $\mathcal{N} = 4$!

Conclusions

Done!

- Versatility: $M_0 = 0, M_0 \neq 0; \mathrm{U}(N_c) \mapsto \mathrm{U}(N_1) \otimes \mathrm{U}(N_2) \otimes \dots$
- Also different models: $\mathcal{N} = 1, \mathcal{N} = 1^*, \mathcal{N} = 2, \dots$

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To Do

- More general deformations: $\mathrm{tr} \Phi_i^3$ and the like
- Instanton calculus in $\mathcal{N} = 4$ β -deformed
- $\mathrm{U}(N_c)$ vs. $\mathrm{SU}(N_c)$
- ...