Designing Matrix Models for Zeta Functions

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This talk is based on a collaboration with
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arXiv:1807.07342 [math-ph]

Matrix model for Riemann zeta via its local factors

and some ongoing work

Disclaimer: We are **not** trying to prove the Riemann hypothesis,
lest you think we've lost it!

Outline

Zeta function: infinite sum and product

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- extended to real $s > 1$ by Chebyshev.
- analytically continued by Riemann to the complex s-plane as a meromorphic function.

  Useful in regularising infinities in Physics.

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The symmetric zeta-function $\xi(s) = \frac{1}{2}\pi^{-s/2} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is an entire function that satisfies $\xi(s) = \xi(1-s)$. Its zeroes are at the non-trivial zeroes of $\zeta(s)$, at $s = \gamma_m = \frac{1}{2} + it_m$.

Some related functions

n	γ_n
1	$\frac{1}{2} \pm i\,14.1347\ldots$
2	$\frac{1}{2} \pm i\,21.0220\ldots$
3	$\frac{1}{2} \pm i\,25.0108\ldots$
4	$\frac{1}{2} \pm i\,30.4248\ldots$
5	$\frac{1}{2} \pm i\,32.9350\ldots$
6	$\frac{1}{2} \pm i\,37.5861\ldots$
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First few zeroes

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$$\mathcal{J}(x) = Li(x) - \sum_n Li\left(x^{\gamma_n}\right)$$

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Dyson pointed out that this has the same behaviour as the two-point correlator of the eigenvalues of an ensemble of random hermitian matrices.

Numerical evidence and extension

Odlyzko confirmed this behaviour from his numerical computation of Riemann zeroes.

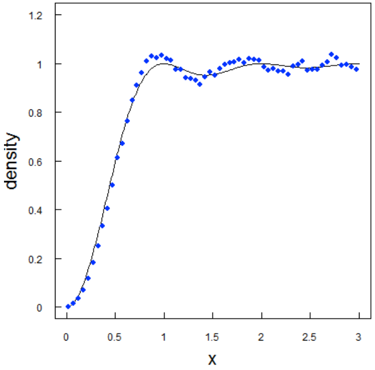


Figure : Blue dots describe the normalized spacings of the first 10^5 non-trivial zeros of the Riemann zeta function. The solid line describes the two-point correlation function of GUE of random matrices. (source: Wikipedia)

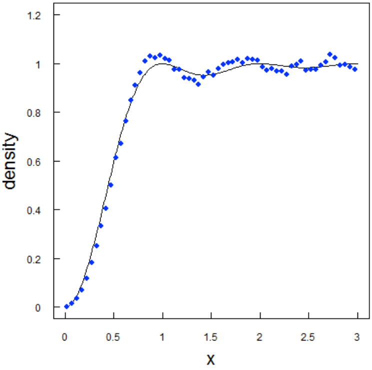
Numerical evidence and extension

Odlyzko confirmed this behaviour from his numerical computation of Riemann zeroes.

Rudnick-Sarnak extended it to higher correlators.

Özlük extended to the zeroes of Dirichlet L-functions:

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p\in\text{primes}} \frac{\chi(p)}{(1-p^{-s})}$$



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Search for the Hamiltonian

Berry-Keating (and Connes) proposed the quantised form of the classical xp Hamiltonian : $H_{\text{BK}} = (xp + px) = -2i\hbar\left(x\frac{d}{dx} + \frac{1}{2}\right)$.

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$$\begin{array}{ccc} \text{Riemann zeroes} & & \text{Energy eigenvalues} \\ \updownarrow & \sim & \updownarrow \\ \text{Primes} & & \text{Periods} \\ & & \text{(of primitive periodic orbits)} \end{array}$$

Intriguing similarities

The fluctuating part of the distribution function $J(x) = \sum_m \Theta(x - t_m)$ has the form

$$J_{\text{fl}}(x) = -\frac{1}{\pi} \sum_p \sum_{n \in \mathbb{N}} \frac{1}{n} e^{-n \ln p/2} \sin(xn \ln p)$$

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This is to be compared with the fluctuating part of the energy eigenvalues in the Gutzwiller trace formula relating to the periods of a chaotic dynamical system:

$$\varrho_{\text{fl}}(E) = \frac{1}{\pi} \sum_p \sum_{n\in\mathbb{N}} \frac{1}{n} e^{-n\lambda_p \tau_p/2} \sin\left(nS_p(E) - \frac{\pi}{2} n\mu_p\right)$$

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  Suggestion: Restrict the values of x and p.

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New ingredients: p-adic analysis, Hilbert space over p-adic numbers, in particular, wavelets as a basis, and operators on this space.

Outline

Conformal map

The eigenvalues of large $N \times N$ unitary matrices gives a density $\rho(\theta) = \sum \delta(\theta - \theta_i)$ (distribution function) on the unit circle.

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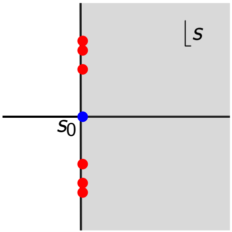
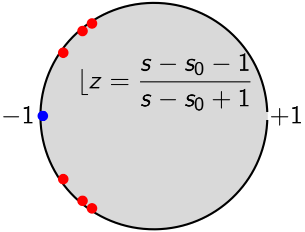
The eigenvalues of large $N \times N$ unitary matrices gives a density $\rho(\theta) = \sum \delta(\theta - \theta_i)$ (distribution function) on the unit circle. Given a distribution on the line $\mathrm{Re}\ s = s_0$, one can find a Gaussian Unitary Ensemble (GUE) such that its eigenvalue distribution is related to it.





$$s - s_0 = \frac{1+z}{1-z} = \frac{1+e^{i\theta}}{1-e^{i\theta}} = i \cot \frac{\theta}{2}$$

One-plaquette UMM

The partition function of the one-plaquette model is defined by:

$$\mathcal{Z} = \int \mathcal{D}U \exp\left[N \sum_{n=0}^{\infty} \frac{\beta_n}{n} \left(\text{Tr } U^n + \text{Tr } U^{\dagger n}\right)\right] = \int \prod_{i=1}^{N} \frac{d\theta_i}{2\pi}\, e^{-N^2 S_{\text{eff}}(\theta_i)}$$

where, $S_{\text{eff}}(\theta_i) = -\sum_{n=1}^{\infty} \sum_{i=1}^{N} \frac{2\beta_n}{n} \cos(n\theta_i) - \frac{1}{2} \sum_{i \neq j} \ln\left(4 \sin^2 \frac{\theta_i - \theta_j}{2}\right)$

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In the large N limit, $x = \frac{i}{N} \in [0,1]$ and $\theta_i \to \theta(x)$

$$S[\theta] = -\sum_{n=1}^{\infty} \int_0^1 dx\, \frac{2\beta_n}{n} \cos n\theta(x) - \frac{1}{2} \int_0^1 dx\, ⨍_0^1 dy\, \ln\left(4 \sin^2 \frac{\theta(x) - \theta(y)}{2}\right)$$

Saddle point

The saddle point of the action is determined by

$$\fint \frac{d\theta'}{2\pi}\rho(\theta')\cos\left(\frac{\theta-\theta'}{2}\right) = \sum_{n=1}^{\infty} 2\beta_n \sin n\theta \quad \left(= \frac{dV(\theta)}{d\theta}\right)$$

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$$2\pi\rho(\theta) = 2\mathrm{Re}\,[R(e^{i\theta})] - 1 = \lim_{\epsilon\to 0}\left[R\Big((1+\epsilon)e^{i\theta}\Big) - R\Big((1-\epsilon)e^{i\theta}\Big)\right]$$

UMM in terms of Irreps (Schematic)

The PF of a UMM can also be expanded in terms of the irreducible representions (irreps) of U(N)

$$\mathcal{Z} \sim \sum_{R\in\text{irreps}} \sum_{\vec{k},\vec{\ell}} \alpha(\vec{\beta},\vec{k})\alpha(\vec{\beta},\vec{\ell})\,\chi_R(C(\vec{k}))\chi_R(C(\vec{\ell}))$$

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(where $\chi_R(C(\vec{k}))$ is the character of the conjugacy class $C(\vec{k})$ of the permutation group $S_{K=\sum nk_n}$.). The following have been used

$$\prod_n (\text{Tr } U^n)^{k_n} = \sum_R \chi_R(C(\vec{k}))\text{Tr}_R(U)$$

$$\int \mathcal{D}U\,\text{Tr}_R(U)\text{Tr}_{R'}(U^\dagger) = \delta_{RR'}$$

Young diagrams and momenta

Irreps can be labelled by the number of boxes in Young diagrams.
In the large N limit

$$\mathcal{Z} = \int \mathcal{D}h(x) \int d\vec{k}\, d\vec{\ell}\, \exp\left(-N^2 S_{\text{eff}}[h(x), \vec{k}, \vec{\ell}]\right)$$

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The variables h are the momenta conjugate to the eigenvalues θ.

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There is a density $\Omega(\theta, h)$ in the phase space, such that

$$\int dh\, \Omega(\theta, h) = \rho(\theta) \quad \text{and} \quad \int d\theta\, \Omega(\theta, h) = u(h)$$

Expectation: Phase space description may lead to a Hamiltonian.

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- Write the partition function of the UMM in phase space: $Z \sim \int d\theta\, d\mathfrak{h}\, e^{-H(\theta,\mathfrak{h})}$.
- Try to realise this as the trace of some operator: $Z \sim \mathrm{Tr}\,\hat{\varrho}$.

Outline

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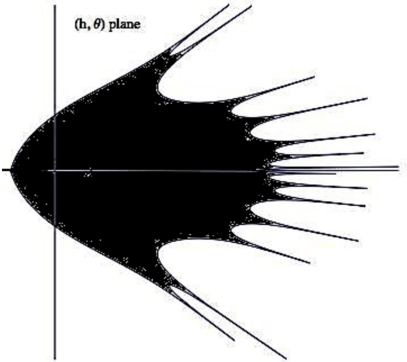
in terms of the Keiper-Li numbers[‡]

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n} s^{n-1} \ln \xi(s)\Big|_{s=1} = \sum_i \left[1 - \left(1 - \frac{1}{\gamma_i}\right)^n\right]$$

Phase space density of the UMM of $\xi(s)$

The density in the phase space is



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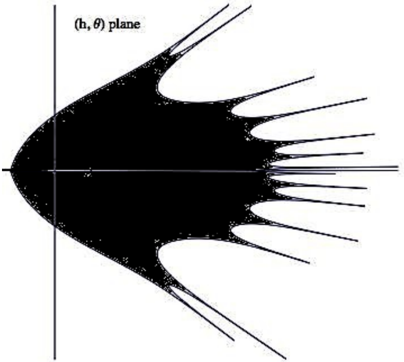

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The prime power counting function $J(x)$ jumps by $1/n$ at every p^n:

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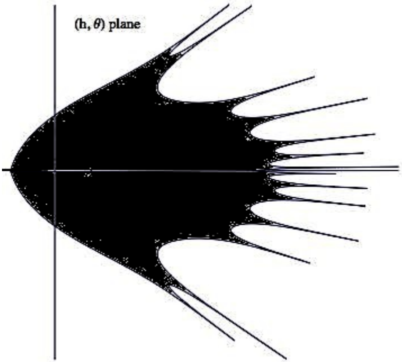

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Turns out that the momentum density $h(x) \sim J_{\text{fl}}(x)$, the fluctuating part of the counting function.

Outline

Local and adelic zeta

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Singling out a prime: Leaving rationality

A p-adic number $\xi = p^N\left(\xi_0 + \xi_1 p + \xi_2 p^2 + \cdots\right)$, where N is an integer, $\xi_k = \{0, 1, \cdots p-1\}$ but $\xi_0 \neq 0$, and

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Local zeta and the resolvent

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These poles can be brought on the unit circle on $z = \frac{s-1}{s+1}$ plane.

$$R_<(z) = 1 + \frac{z}{(1-z)^2} \frac{p^{-s(z)}}{1 - p^{-s(z)}}, \qquad R_>(z) = -\frac{z}{(1-z)^2} \frac{p^{s(z)}}{1 - p^{s(z)}}$$

The resolvent above satisfies all the properties ($R_<(0) = 1$, $R_>(z \to \infty) = 0$ and $R_<(z) + R_>(1/z) = 1$).

$\star$(Caveat)

A well-known measure

The following is easily computed:

$$\int_{p\mathbb{Z}_p} |h|_p^{s-1}\, dh = \frac{(1-p^{-1})p^{-s}}{(1-p^{-s})}, \qquad p\mathbb{Z}_p = \left\{ h \in \mathbb{Q}_p :\ |h|_p < 1 \right\}$$

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So $2R_<(z) - 1 = p \int_{p\mathbb{Z}_p} dh \left(1 + \frac{2z}{(p-1)(1-z)^2}\, |h|_p^{\frac{1+z}{1-z}-1} \right)$

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This is suggestive of a phase space density

$$\Omega_p(\theta, h) = p - \frac{p}{2(p-1)\sin^2\left(\frac{\theta}{2}\right)} \, |h|_p^{-i\cot\left(\frac{\theta}{2}\right)-1} \sim p - \frac{p^{-in\cot\left(\frac{\theta}{2}\right)}}{2(p-1)\sin^2\left(\frac{\theta}{2}\right)}$$

Vladimirov derivative and Kozyrev wavelets

The totally disconnected topology of $\mathbb{Q}_p$, makes differentiation difficult. [Vladimirov] defined derivative as an integral kernel:

$$\left(D^{\alpha}_{(p)}f\right)(x) = \frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p} dx\, \frac{f(x) - f(y)}{|x-y|_p^{\alpha}}, \qquad \alpha \in \mathbb{C}$$

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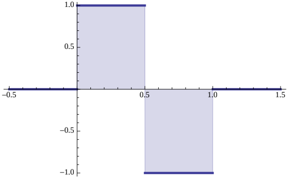

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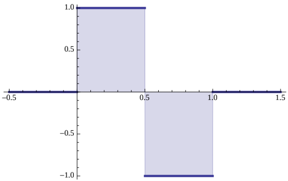

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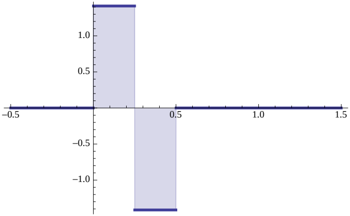

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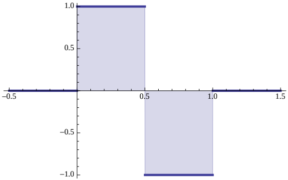

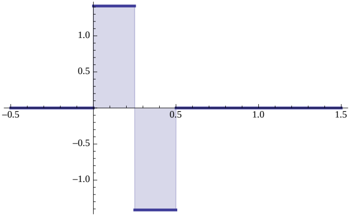

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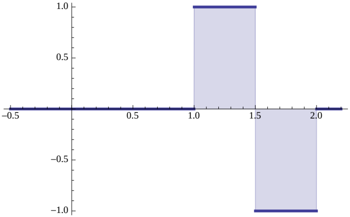

translated

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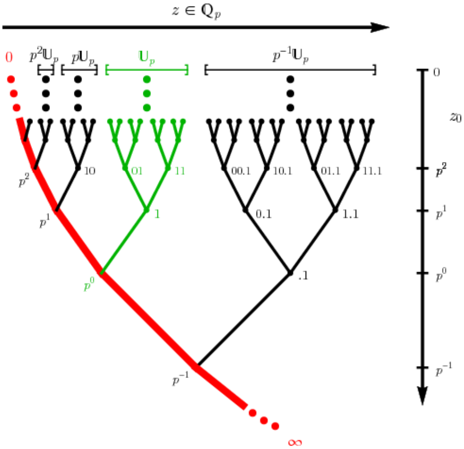

(Fig. from Gubser *et al*)

Wavelets on $\mathbb{Q}_p$

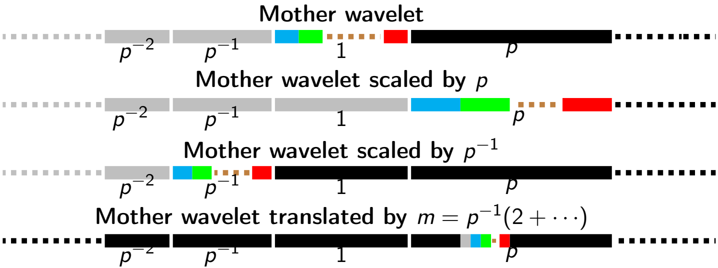

Figure : A schematic representation of the wavelets. The sets are ordered by the values $|\xi|_p = p^n$. (Colour code: grey = 1, black = 0, other colours correspond to primitive roots of unity.) [Dutta,-DG-Lala (2018)]

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The Hilbert space $\mathcal{H}_{(p)}$ of the quantum Hamiltonian is spanned by a subset of the Kozyrev wavelets (which are eigenfunctions of the Vladimirov derivative.)

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$$\sum_p \ln p \, \frac{dJ_p(\xi)}{d\xi} = \frac{d\psi(\xi)}{d\xi} = 1 - \underbrace{\sum_i \xi^{\gamma_i - 1}}_{\text{non-trivial zeroes}} - \sum_n \xi^{2n-1}$$

Divergence

Keeping only the non-trivial zeroes γ_i

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Clearly μ has to be independent of i. The reflection symmetry of ζ-function implies that $\mu > 1$ and *assuming* Riemann hypothesis $\mu > \frac{1}{2}$.

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Leads to a one-parameter family of Hamiltonians

$$H_\mu \sim H - \mu P$$

Trivial zeroes and local zeta at infinity

Consider $\zeta_{\mathbb{A}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\prod_p \zeta_p(s) \equiv \zeta_\infty(s)\prod_p \zeta_p(s)$, the *adelic zeta function*. It involves $\mathbb{Q}_p$ for all primes and $\mathbb{R}$, and satisfies $\zeta_{\mathbb{A}}(s) = \zeta_{\mathbb{A}}(1-s)$.

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We construct the Wigner functions on the local phase space $\mathbb{R} \times \mathbb{Q}_p$ for $\rho^{(p)}_{x'} = |x'\rangle\langle x'|$

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After integrating over position (momentum) Wigner function gives the counting function (resp. eigenvalue density) upto an infinite factor.

Wigner function in the large phase space

Issues with coordinates:

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should be equivalent. ([Neretin] Canonical bijection between the spaces of locally constant complex valued functions on $\otimes_p \mathbb{Q}_p$ and distributions on $\mathbb{R}$.) Integrated Wigner function:

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Thank you!