Relativity 3

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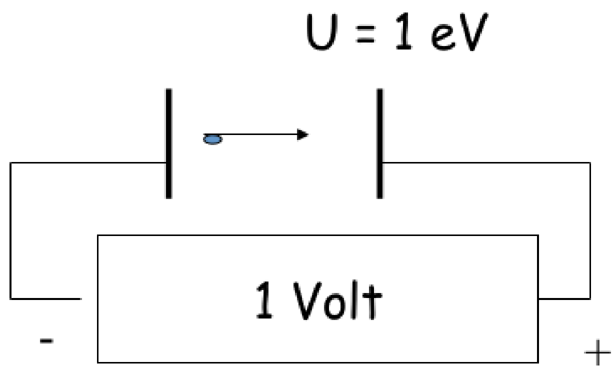
- Units for high-energy physics
- Examples of relativistic kinematics: scattering, collisions, decays
- Experimentally observed quantities

Units for high-energy physics

- In relativistic physics, it is convenient to express masses as *energy*$/c^2$ and momenta as *energy*$/c$

- Energies are typically in units of *electronvolts*, eV, or multiples of eV:
  1 eV $=$ energy acquired by an electron crossing a 1 V potential difference



  1 eV $= (1.602 \times 10^{-19}$ C) (1 V) $= 1.602 \times 10^{-19}$ J

- Typical multiples used in particle physics:
  1 MeV $= 10^6$ eV, 1 GeV $= 10^9$ eV, 1 TeV $= 10^{12}$ eV

- Mass of an electron in energy units: 0.511 MeV$/c^2$
  Mass of a proton: 938.2 MeV$/c^2$; of a neutron: 939.5 MeV$/c^2$

Units for high-energy physics (2)

Using GeV for energies and 1 fm=10^{-15} m (approximately the size of a proton) for lengths:

Quantity	HEP Units	SI Units
Length	1 fm	10^{-15} m
Energy	1 GeV	1.602×10^{-10} J
Mass	1 GeV/c^2	1.78×10^{-27} Kg
c	2.998×10^{23} fm/s	2.998×10^{8} m/s
$\hbar$	6.59×10^{-25} GeV·s	1.055×10^{-34} J·s
$\hbar c$	0.1975 GeV·fm	3.162×10^{-26} J·m

Since $\frac{e^2}{\hbar c} = \alpha \simeq \frac{1}{137}$, fine structure constant, $e^2 = \alpha\hbar c = 1.44$ MeV·fm

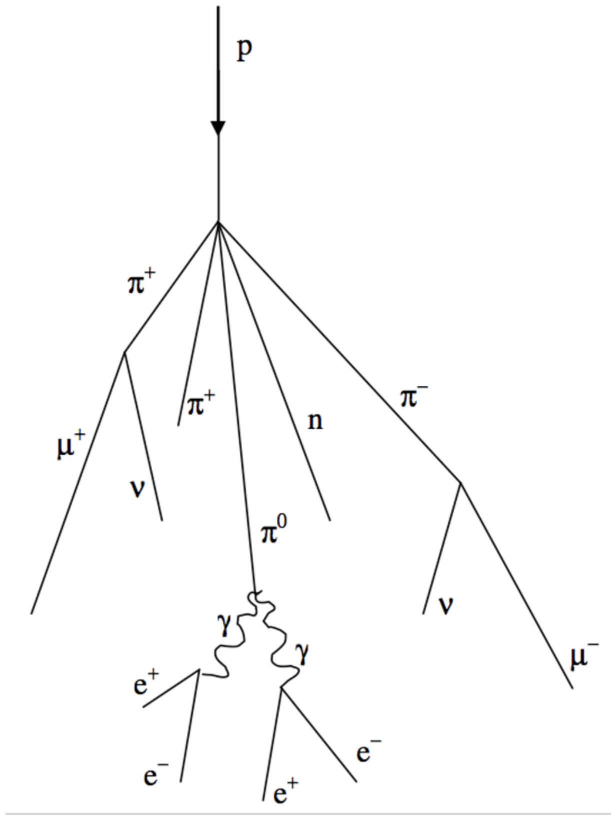
One often sets $c = 1$ so masses and momenta are also measured in GeV.

Time Dilatation in action (2)

A typical energy of a muon produced in the high atmosphere is $E \simeq 50$ GeV. Its mass is $m_\mu = 106$ MeV, so $\gamma = E/mc^2 \simeq 500$, $\beta \simeq 1$.

The half life of a muon is $\tau_0 = 2.2 \times 10^{-6}$ s in its reference frame.

In our reference frame: $\tau = \gamma\tau_0 \simeq 1.1$ ms.
In this time, the muon travels a distance:
$s \simeq c\tau = (1.1 \times 10^{-3}\text{s})(3 \times 10^{8}\text{m/s})$=330 km



Relativistic kinematics: summary

The energy-momentum four-vector p^μ for a particle of mass m moving with velocity $\mathbf{v}$ is

$$p^\mu = (\frac{E}{c}, \mathbf{p}) = (m\gamma c, m\gamma \mathbf{v}), \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}$$

with Lorentz invariant norm $p^\mu p_\mu = -p_0^2 + |\mathbf{p}|^2 = -m^2c^2$.

For a photon of wave vector $\mathbf{k}$ and frequency $\nu = \omega/2\pi$ ($\omega = ck$):

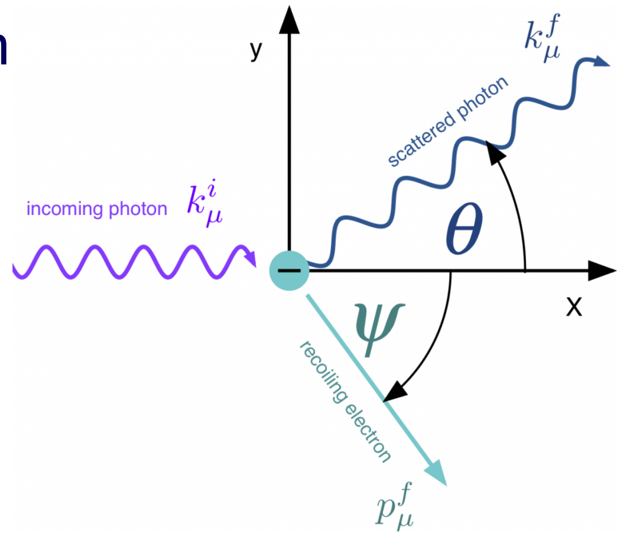
$$p^\mu = (\frac{E}{c}, \mathbf{p}) = (\frac{\hbar\omega}{c}, \hbar\mathbf{k}) = (\hbar k, \hbar\mathbf{k})$$

with Lorentz invariant norm $p^\mu p_\mu = 0$. Note that the relation between E and the frequency ν, $E = h\nu$, is purely quantum-mechanical.

Relativistic kinematics: collisions

During a collision process, *the sum of energy-momentum four-vectors of all particles* $P^\mu = \sum_i p_i^\mu$ is conserved.

A simple example: a photon hitting an electron at rest (*Compton scattering*).



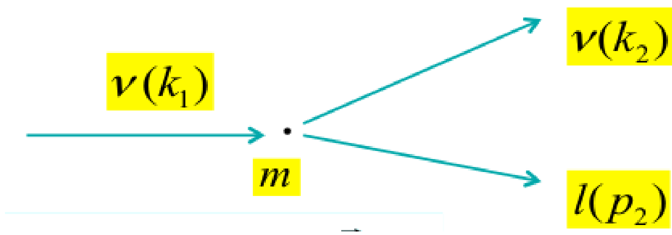
$$
\begin{aligned}
\hbar k + mc &= \hbar k' + E/c \\
\hbar \mathbf{k} &= \hbar \mathbf{k}' + \mathbf{p}
\end{aligned}
$$

One derives $\mathbf{p} = \hbar(\mathbf{k} - \mathbf{k}')$ and $E = \hbar(k - k')c + mc^2$. Using $E^2 = m^2c^4 + p^2c^2$, one finds $\hbar k k'(1 - \cos\theta) = mc(k - k')$. In terms of the wavelength $\lambda = \dfrac{2\pi}{k}$, one can finally write $\boxed{\lambda' - \lambda = \dfrac{h}{mc}(1 - \cos\theta)}$

In this case, the nature of the particles does not change in the collision

Relativistic kinematics: collisions (2)

A more general case: a neutrino hits an electron and produces a muon



$$\nu(k_1) + e(p_1) \to \nu(k_2) + l(p_2)$$

In the following, we use units in which $\hbar = 1$ and $c = 1$ **and the sign convention** $p^2 = p_0^2 - \mathbf{p}^2 = -p_\mu p^\mu$ **for square module of four-vectors**

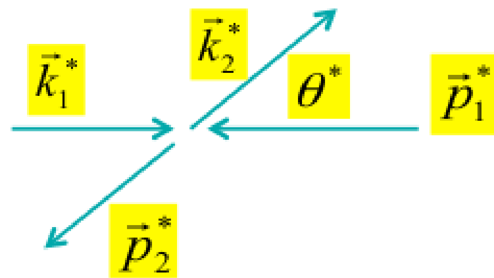
$k_i = (\omega_i, \mathbf{k}_i)$ with $\omega_i^2 - \mathbf{k}_i^2 = \epsilon^2$ (very small for neutrinos!)
$p_1 = (E_1, \mathbf{p}_1)$ with $E_1^2 - \mathbf{p}_1^2 = m^2$; $p_2 = (E_2, \mathbf{p}_2)$ with $E_2^2 - \mathbf{p}_2^2 = m_\mu^2$ (mass of the muon). In the laboratory (LAB) reference frame:

$$p_1 + k_1 = p_2 + k_2 \longrightarrow \begin{cases} \omega_1 + E_1 & = & \omega_2 + E_2 \\ \mathbf{k}_1 + \mathbf{p}_1 & = & \mathbf{k}_2 + \mathbf{p}_2 \end{cases}$$

Relativistic kinematics: collisions (3)

In the Center of Mass (CM) reference frame:

$$\begin{cases} \omega_1^* + E_1^* & = & \omega_2^* + E_2^* \\ \mathbf{k}_1^* + \mathbf{p}_1^* & = & \mathbf{k}_2^* + \mathbf{p}_2^* \end{cases}$$



The norm is conserved and is a Lorentz invariant:

$$s = (k_1^* + p_1^*)^2 = (\omega_1^* + E_1^*)^2 - (\mathbf{k}_1^* + \mathbf{p}_1^*)^2 \longrightarrow \sqrt{s} = \omega_1^* + E_1^*$$

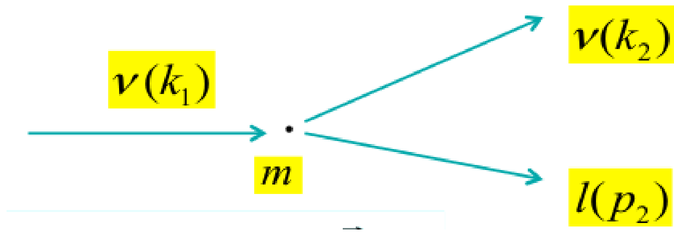
because $\mathbf{k}_1^* + \mathbf{p}_1^* = 0$ (we are in the CM). $\sqrt{s}$ is the maximum energy that can be transformed into mass: $s = (\omega_2 + E_2)^2 \geq m_\mu^2$.

In general, for the sum over masses in the final state, $\boxed{\sum_f m_f \leq \sqrt{s}}$.

$M = \sqrt{s}$ is also called the *effective* or *invariant* mass of a process.

Fixed target vs colliding targets

Assuming that the target is fixed
($\mathbf{p}_1 \simeq 0$ in this case):



$$
\begin{aligned}
s &= (k_1+p_1)^2 = k_1^2 + p_1^2 + 2k_1 \cdot p_1 = \epsilon^2 + m^2 + 2\omega_1 E_1 - 2\mathbf{k}_1 \cdot \mathbf{p}_1 \\
&= \epsilon^2 + m^2 + 2\omega_1 m
\end{aligned}
$$

$\epsilon \sim 0$ and at high energies, m is also negligible: $\boxed{\sqrt{s} \simeq \sqrt{2\omega_1 m}}$

The production of a muon (mass: $m_\mu = 106$ MeV) is possible if the invariant mass $\sqrt{s}$ exceeds the mass of the muon m_μ:

$$
\epsilon^2 + m^2 + 2\omega_1 m \geq m_\mu^2 \longrightarrow \omega_1 \geq \frac{m_\mu^2 - m^2}{2m} = \frac{11200 - 0.26}{1.02}\text{MeV} \simeq 11\text{GeV}
$$

Fixed target vs colliding targets (2)

Let us assume now a head-to-head collision as in the picture below.

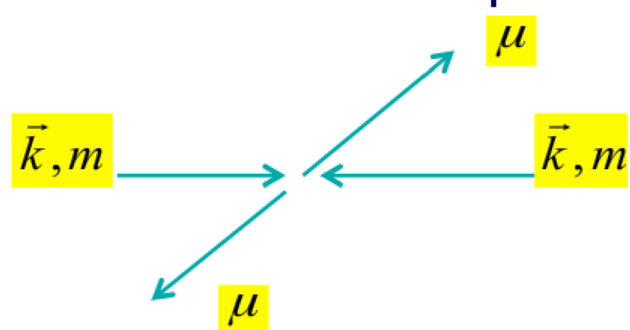
We have $k_1 = (\omega_1, \mathbf{k}_1)$, $k_2 = (\omega_2, \mathbf{k}_2)$, with $\omega_i = \sqrt{\mathbf{k}_i^2 + m_i^2}$. Invariant mass:

$$s = (k_1 + k_2)^2 = (\omega_1 + \omega_2)^2 - (\mathbf{k}_1 + \mathbf{k}_2)^2 = (\omega_1 + \omega_2)^2$$

because $\mathbf{k}_1 + \mathbf{k}_2 = 0$. At high energies $\omega_i \simeq |\mathbf{k}_i|$ and $\boxed{\sqrt{s} \simeq 2|\mathbf{k}_1|}$

Pairs of muons can be produced from an electron and a positron if

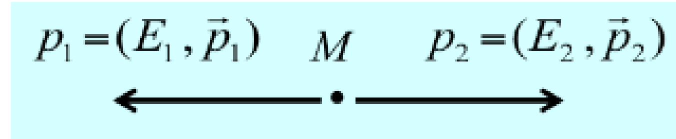
$$2|\mathbf{k}_1| \geq 2m_\mu \longrightarrow |\mathbf{k}_1| \sim m_\mu = 106\text{MeV}$$



Note the difference wrt the previous case!

Decay of a particle into two particles

An unstable particle decays into two particles. In the CM:



$$
\begin{aligned}
0 &= \mathbf{p}_1 + \mathbf{p}_2, \qquad p = |\mathbf{p}_1| = |\mathbf{p}_2| \\
M &= E_1 + E_2 = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2}, \qquad M \geq m_1 + m_2
\end{aligned}
$$

From $E_1 = \sqrt{m_1^2 + p^2}$ one finds $p^2 = E_1^2 - m_1^2$ and $E_2 = M - E_1 = \sqrt{m_2^2 - m_1^2 + E_1^2}$. After some algebra, one finds

$$
E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}, \qquad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M},
$$

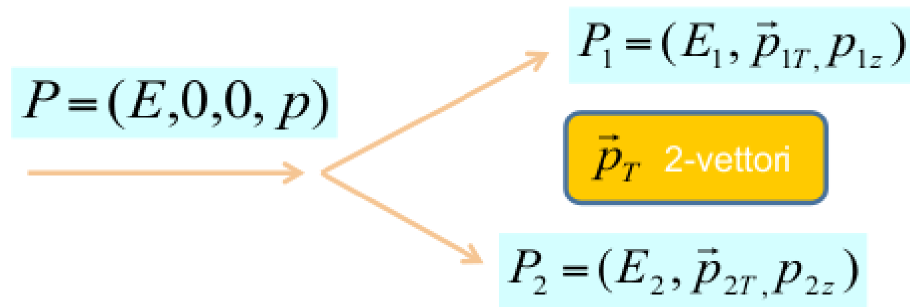
$$
p = \frac{\sqrt{\left(M^2 - (m_1 - m_2)^2\right)\left(M^2 - (m_1 + m_2)^2\right)}}{2M}
$$

Example: $\pi^- \to \mu^- + \nu_\mu$, $m_\pi = 140$ MeV, $E_\mu \simeq \dfrac{m_\pi^2 + m_\mu^2}{2m_\pi} \simeq 110$ MeV

Decay of a particle into two particles (2)

As before, in the laboratory frame, with particle momentum along z:

$$\begin{aligned} p &= p_{1z} + p_{2z} \\ 0 &= \mathbf{p}_{1T} + \mathbf{p}_{2T} \\ E &= E_1 + E_2 \end{aligned}$$



Let us use Lorentz transforms between LAB and CM. Note that: $\gamma = E/M$; $\beta = \sqrt{\gamma^2 - 1}/\gamma = p/E$

$$E_1 = \gamma\left(E_1^{CM} + \beta p_{1z}^{CM}\right); \qquad E_2 = \gamma\left(E_2^{CM} + \beta p_{2z}^{CM}\right)$$

$$p_{1z} = \gamma\left(p_{1z}^{CM} + \beta E_1^{CM}\right); \qquad p_{2z} = \gamma\left(p_{2z}^{CM} + \beta E_2^{CM}\right)$$

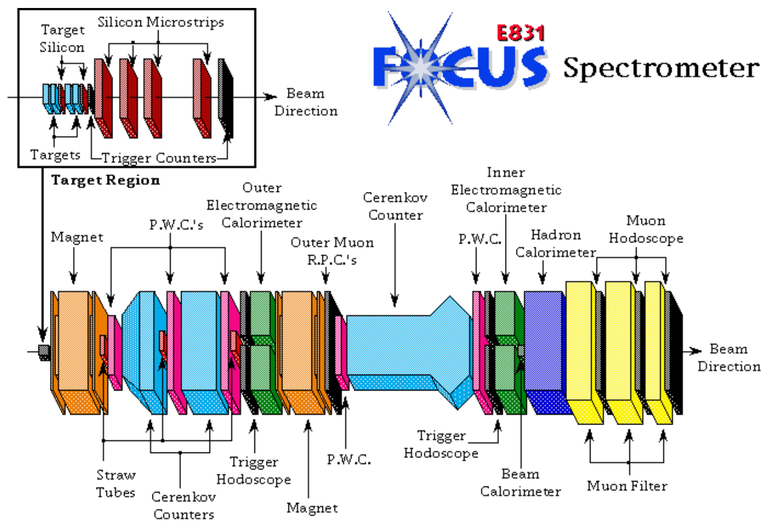
$$\mathbf{p}_{1T} = \mathbf{p}_{1T}^{CM}; \qquad \mathbf{p}_{2T} = \mathbf{p}_{2T}^{CM}$$

More cases of decay

Example: decay of a pion into two photons, $\pi^0 \to \gamma\gamma$. Invariant mass:
$M^2 = (p_{1\gamma} + p_{2\gamma})^2 = p_{1\gamma}^2 + p_{2\gamma}^2 + 2p_{1\gamma}p_{2\gamma} = 2E_1E_2(1 - \cos\theta_{12})$

Decay of a particle into three (or any number) particles:

A series of detectors can distinguish different particles, measure E and $\mathbf{p}$ of all particles



$$\begin{cases} P_1 = (E_1, \mathbf{p}_1) \\ P_2 = (E_2, \mathbf{p}_2) \\ P_3 = (E_3, \mathbf{p}_3) \end{cases}$$

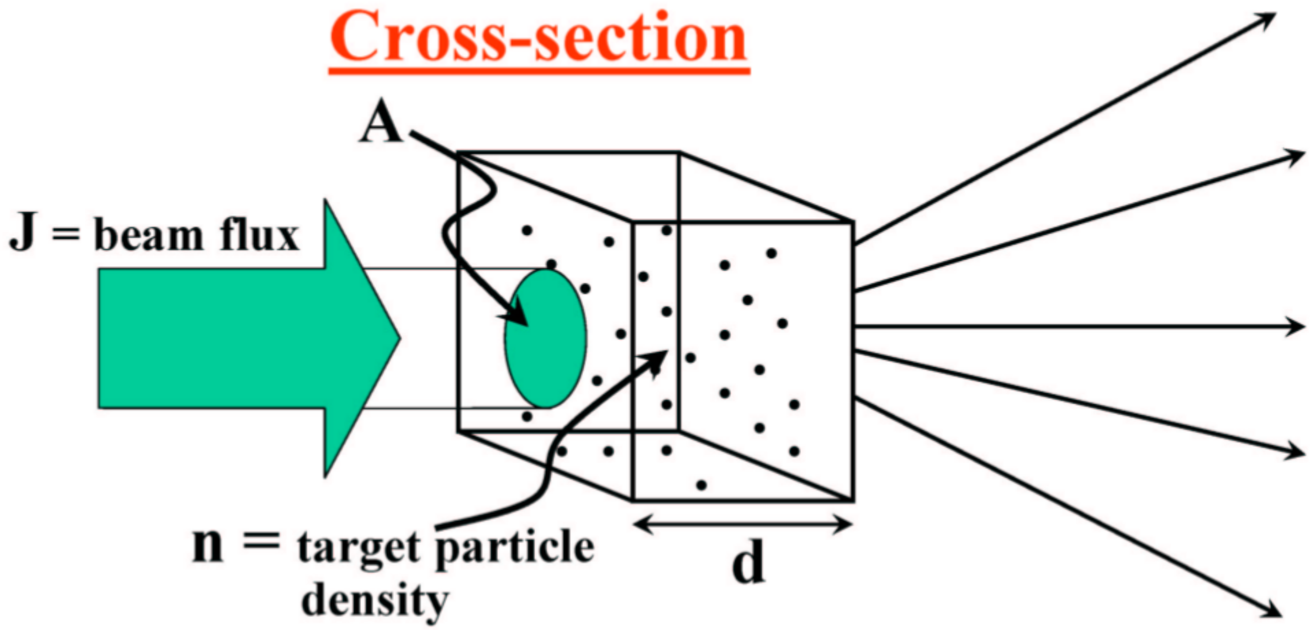
Let us build the quantity $s = (E_1 + E_2 + E_3)^2 - (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)^2$. This is a Lorentz invariant, that in the rest frame of the decaying particle can be easily evaluated: $s = M^2$, the *invariant mass*. Then one searches for peaks in histograms of invariant mass.

Observables: cross sections, decay rates

- For scattering processes, the relevant quantity to be measured in experiments is the *cross section* σ. The cross section has the units of a surface (m^2, or cm^2; also used in high-energy physics, the *barn*, 10^{-28} m^2, or 100 fm^2)

- For decay processes, the relevant quantity to be measured in experiments is the *decay rate* λ. The decay rate has the units of an inverse of a time

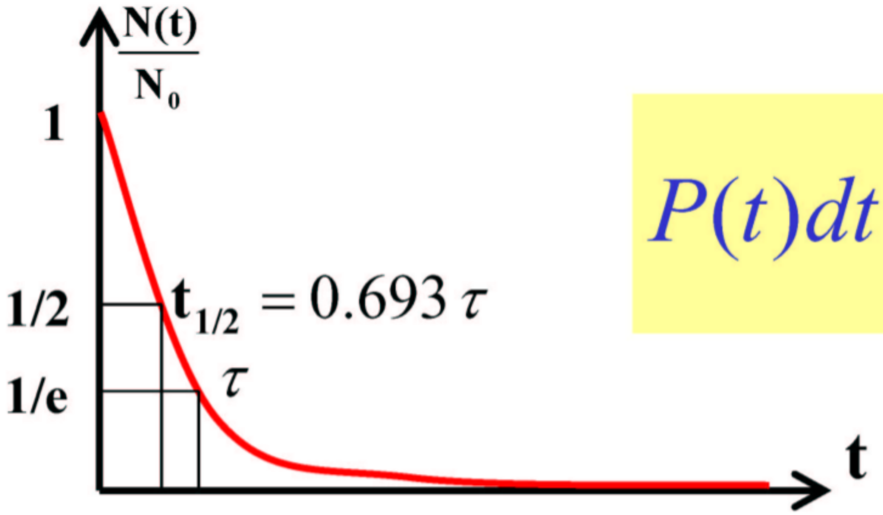
Both quantities are related to the *probability* that the considered process occurs



Definition of the (total) cross section: the number of scattering events during time dt is $dN = \sigma(JAnd)dt$, where J is the number of incident particles per unit time per unit surface, A, n, d as in the picture.

The quantity $L = JAnd$ is known as the *luminosity* (units: $[\ell]^{-2}[t]^{-1}$).

Decay time and Lifetime



$$P(t)dt = \frac{1}{\tau} e^{-t/\tau} dt$$

If the decay rate is λ, there are $dN = N(t)\lambda dt$ particle that decay in time dt. This leads to the $N(t) = N_0 e^{-\lambda t}$ law.

We define $\tau = \lambda^{-1}$ as the *mean life*. τ refers to the reference frame where the particle is at rest, that is, to the *proper time* of the particle! See the discussion about muons and cosmic rays.

For a particle with speed $v = \beta c$, $\tau' = \gamma\tau$ in the LAB reference frame

Mean life and decay width

If a particle has a finite lifetime with mean life τ, i.e. it decays with probability $1/\tau$, we can define the *decay width* Γ as $\boxed{\Gamma = \frac{\hbar}{\tau}}$.

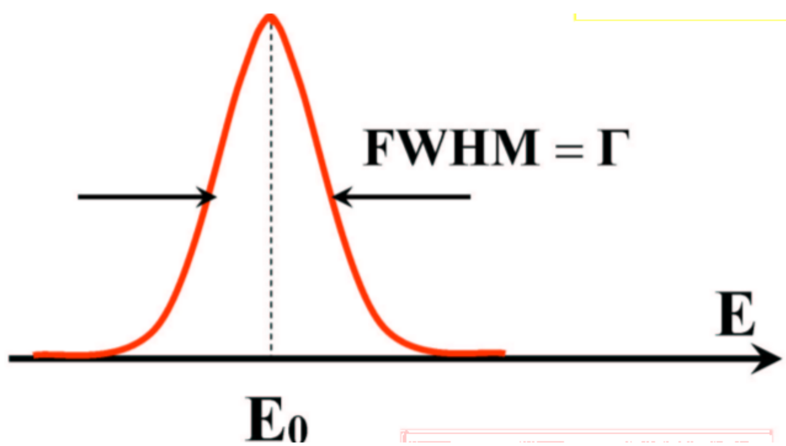
The decay width has the dimension of an energy ($\hbar =$ energy x time). In high-energy units, $\hbar = 6.58 \cdot 10^{-22}$ MeV s.

Strongly decaying particles have short lifetimes and hence large decay widths, e.g. the $\rho(770)$ has $\tau = 4.4 \cdot 10^{-24}$ s and $\Gamma = 151$ MeV.

Weakly decaying particles have long lifetimes and small decay widths, e.g. the K$^\circ$ meson has $\tau = 0.9 \cdot 10^{-10}$ s and $\Gamma = 7.3 \cdot 10^{-12}$ MeV

Mean life and decay width (2)

The decay width is a purely quantum phenomenon that can be interpreted as a manifestation of the indeterminacy principle $\Delta E \Delta t \geq \hbar$. It appears as a finite width in experiments as a function of energy:



$$\frac{\Gamma^2/4}{(E-E_0)^2+\Gamma^2/4}$$

Lorentzian line

with a typical Lorentzian form (FWHM=Full Width at Half Maximum). If there are more possible decay processes with probability $B_r(i)$ (normalized: $\sum_i B_r(i) = 1$), then $\boxed{\Gamma = \sum_i \Gamma_i}$, where $\Gamma_i = \Gamma B_r(i)$