Relativity 2

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- Space-time and Lorentz transforms
- Four-vectors formalism
- Relativistic kinematics: the energy-momentum four-vector

The concept of Space-Time

- Let us define an *event* as a point in the space (x, y, z) at time t (in a given inertial reference frame)

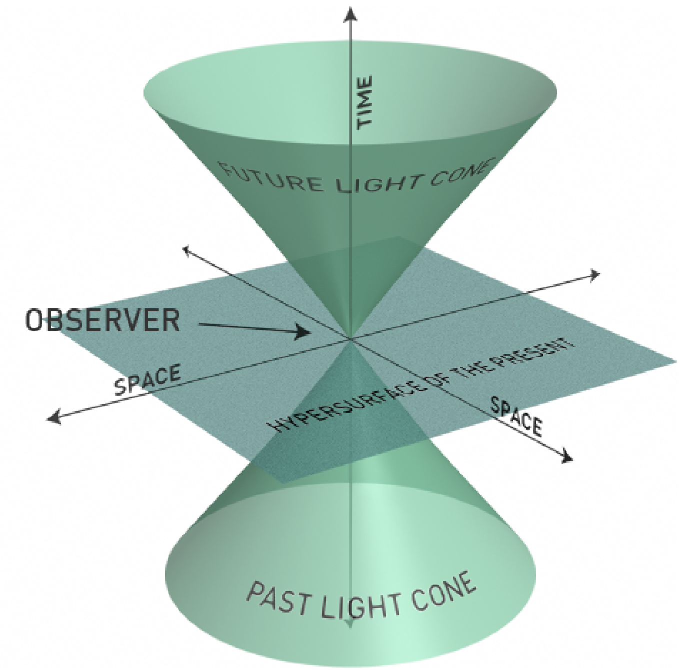
- Let us represent an event as a vector in a four-dimensional space: a *four-vector*. It is convenient to use (ct, x, y, z) as dimensionally consistent coordinates

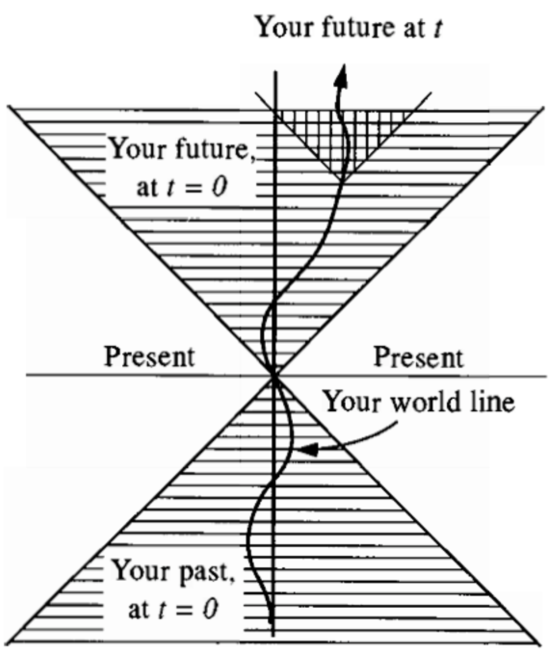
- A moving particle describes a line in the space-time, called *world line*

- Light rays passing through origin at $t = 0$ define a surface called *light cone*.

Light cone

In the figure, a typical example of a light cone, projected over two space coordinates and with the time axis in the vertical direction (*Minkowski diagram*)





The world line of a physical object always stays "inside" the cone; at time $t = 0$ it goes from the "cone of the past" (below) to the "cone of the future" (above); the tangent to the curve in any point is always "inside" the cone (because the speed $v < c$). This is called a "time-like" world line.

Lorentz transforms in space-time

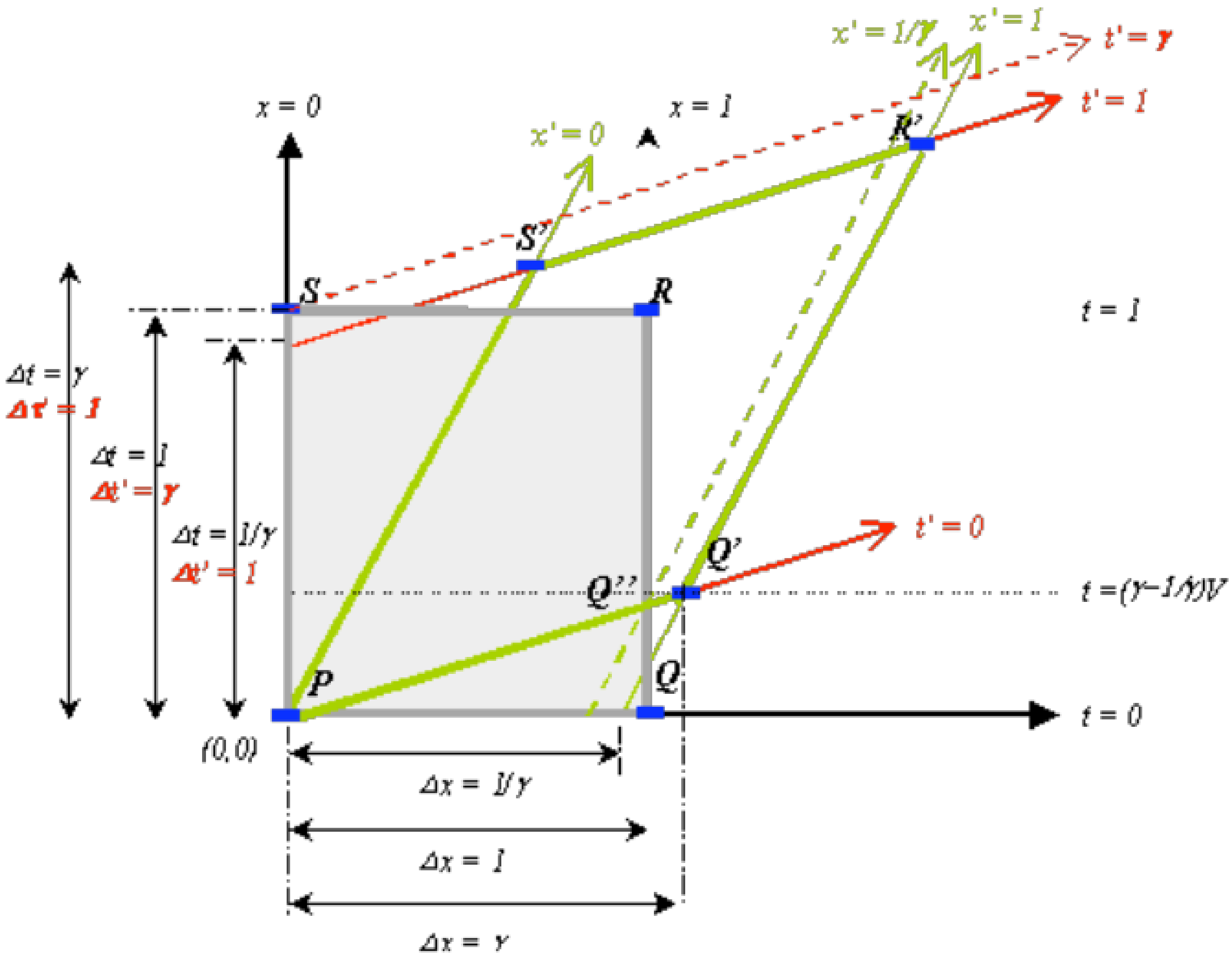
- Lorentz transforms are generalized rotations in space-time, that modify the relative directions of the axis, expand or contract them.

- Contrary to usual rotations, they do not leave the usual square module of vectors: $x^2 + y^2 + z^2 + (ct)^2$, unchanged. The invariant quantity is $I = x^2 + y^2 + z^2 - (ct)^2$: the *square module of four-vectors*

- Lorentz transforms move a point over the set of points with constant I: an hyperbole in t and x, a rotation hyperboloid in t, x, y, with asymptotes on the light cone.

Lorentz transforms in space-time (2)

A graphical representation of a Lorentz transform in the (xt) plane:



Events in space-time

- Let us consider two events in space-time:

$$\mathbf{x}_1 = (ct_1, x_1, y_1, z_1), \quad \mathbf{x}_2 = (ct_2, x_2, y_2, z_2)$$

and their four-vector difference (*interval*): $\Delta\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$. Depending upon the value of $I = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$, we can distinguish the interval into

  – *Type space*: $I > 0$. The two events may be simultaneous in some reference frame
  – *Type light*: $I = 0$. The two events are “connected” by a ray of light
  – *Type time*: $I < 0$. The two events cannot be simultaneous in any reference frame

Four-vector formalism

Let us introduce notations: $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$$\begin{cases} (x^0)' &= \gamma(x^0 - \beta x^1) \\ (x^1)' &= \gamma(x^1 - \beta x^0) \\ (x^2)' &= x^2 \\ (x^3)' &= x^3 \end{cases}$$

where $\beta = V/c$. Lorentz transforms in matrix form:

$$\begin{pmatrix} (x^0)' \\ (x^1)' \\ (x^2)' \\ (x^3)' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

In general, *a four-vector is an object whose components follow Lorentz transforms*

Four-vector formalism (2)

Alternatively, Lorentz transforms may be written as:

$$(x^i)' = \sum_{j=0}^{3} \Lambda^i_j x^j, \qquad i = 0 \div 3$$

where Λ^i_j is the matrix earlier defined (the reason for “high” and “low” indices will be clarified soon; note that the matrix has unit determinant)

It is easily verified that such transform conserves the square module I of four-vectors:

$$I = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^0)^2$$

and in general, the four-vector analogue of the scalar product:

$$\mathbf{x} \cdot \mathbf{y} \equiv x^1 y^1 + x^2 y^2 + x^3 y^3 - x^0 y^0$$

Covariant and contravariant indices

It is practical to introduce *covariant* components:

$$x_0 = -ct, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z$$

in addition to those (known as *contravariant*) already introduced. The only difference is in the sign of the time component. The square module and scalar product of four-vectors become:

$$I = \sum_{i=0}^{3} x_i x^i, \qquad \mathbf{x} \cdot \mathbf{y} = \sum_{i=0}^{3} x_i y^i = \sum_{i=0}^{3} x^i y_i$$

The *Einstein convention* is used: repeated indices are understood to be summed. In all physical quantities, covariant indices are summed with contravariant indices. This guarantees both the correct form and the correct invariance properties with respect to a change of reference frame.

Relativistic kinematics

Let us consider a particle moving with velocity $\mathbf{v}$ in an inertial frame $\mathcal{S}$. The time dt in $\mathcal{S}$ and the time $d\tau$ in a reference frame moving with the particle are connected by

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt = \frac{dt}{\gamma}$$

The quantity $d\tau$ is called *proper time*.

The *proper velocity* $\mathbf{u}$ is thus defined as

$$\mathbf{u} \equiv \frac{d\mathbf{r}}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\mathbf{r}}{dt} = \gamma \mathbf{v}$$

Note that $d\mathbf{r}$ refers to the frame $\mathcal{S}$, but the proper velocity $\mathbf{u}$ differs from the usual definition of velocity, $\mathbf{v} = d\mathbf{r}/dt$, by a factor γ

Proper time and velocity

What is the rationale behind the introduction of these quantities?

- The proper time $d\tau$ is *invariant* (or scalar), by construction
- One may define a four-vector: the *four-velocity* $u^\mu = \dfrac{dx^\mu}{d\tau}$, where:

$$u^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma c$$

$u = (\gamma c, \gamma \mathbf{v})$ transforms according to Lorentz rules, by construction: it is a four-vector, divided by an invariant

Note that dx^μ/dt follows transformation laws that are actually more complex than those for the proper velocity $dx^\mu/d\tau$!

Relativistic momentum

In Classical Mechanics one defines the *momentum* $\mathbf{p}_{cl}$:

$$\mathbf{p}_{cl} = m\mathbf{v}, \qquad \frac{d\mathbf{p}_{cl}}{dt} = \mathbf{F}$$

What is the equivalent of $\mathbf{p}$ in the relativistic case? A good candidate for the space part is

$$\mathbf{p} = m\mathbf{u} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(note that m is an invariant)

Both Newton's second law and the conservation of momentum are still valid if we use the relativistic expression for the four-vector momentum

Energy-momentum four-vector

$\mathbf{p}$ is the space part of a four-vector: what is p^0?

$$p^0 = mu^0 = \frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \frac{E}{c}$$

where $E = \gamma mc^2$ plays the role of *relativistic energy*. If $v = 0$, we obtain the famous Einstein's formula for the energy of a particle at rest:

$$E_0 = mc^2$$

What is the relation between relativistic energy and classical kinetic energy?

$$E - E_0 \simeq \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + ..., \qquad \frac{v}{c} << 1$$

For an isolated system *the energy-momentum four-vector* p^μ *is conserved*

Energy-momentum four-vector (2)

The square module of the energy-momentum four-vector is of course a Lorentz invariant and is related to the mass of the particle via:

$$p_\mu p^\mu = -(p^0)^2 + \mathbf{p} \cdot \mathbf{p} = -m^2 c^2$$

alternatively: $E^2 - p^2c^2 = m^2c^4$, from which one obtains $E(p)$:

$$E = \sqrt{m^2c^4 + p^2c^2} = c\sqrt{m^2c^2 + p^2}$$

Classical limit: $p << mc$ and $E(p) \simeq mc^2 + \frac{p^2}{2m}$.

Ultrarelativistic limit: $p >> mc$ and $E(p) \simeq pc$

The latter expression is exactly true, $E(p) = pc$, in the case of massless $(m = 0)$ particles traveling at the speed of light: the *photons*.