Cosmological Perturbation Theory and Primordial Gravitational Waves

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- Cosmological Perturbation Theory. Gauge transformations: scalar, vector and tensor modes.
- Generation of gravitational waves during inflation.
- Second-order tensor modes. Linear evolution of GW. Upper bounds on the gravitational-wave background (GWB). Anisotropy and non-Gaussianity of the GWB.

Bibliography

- J.E. Lidsey, A.R. Liddle, E.W. Kolb, E.J. Copeland, T. Barreiro and M. Abney, Reconstructing the inflation potential : An overview, Rev. Mod. Phys. **69**, 373 (1997) [astro-ph/9508078].
- M. Maggiore, Gravitational wave experiments and early universe cosmology, Phys. Rept. 331, 283 (2000) [gr-qc/9909001].
- Bartolo N., Komatsu E., Matarrese S. & Riotto A. 2004, Non-Gaussianity from inflation: Theory and observations, Phys. Rept. **402**, 103 (2004) [astro-ph/0406398]
- Baumann D., TASI lectures on inflation, arXiv:0907.5424 (2009)
- M.C. Guzzetti, N. Bartolo, M. Liguori and S. Matarrese, Gravitational waves from inflation, Riv. Nuovo Cim. **39**, 399 (2016) [arXiv:1605.01615].
- N. Bartolo et al., Science with the space-based interferometer LISA. IV: Probing inflation with gravitational waves, JCAP **1612**, 026 (2016) [arXiv:1610.06481].
- C. Caprini and D.G. Figueroa, Cosmological Backgrounds of Gravitational Waves, Class. Quant. Grav. **35**, 163001 (2018) [arXiv:1801.04268]

The Big Bang “crisis”

i. Horizon problem: does our Universe belong to … a set of measure zero?
ii. Flatness problem: do we need to fine-tune the initial conditions of our Universe?
iii. Cosmic fluctuation problem: how did perturbations come from?

Particle horizon (Rindler 1956)

The particle horizon (=past event horizon) at cosmic time t is the distance travelled by a photon (pr other null-like particle) since the beginning of time to the considered time. This cosmological horizon measures the proper distance from which one could possibly retrieve information, hence the distance of causal corelation among events.

- By integrating the FRW line element with ds=0, one finds a proper horizon distance

$$d_{particle\ horizon}(t) = a(t)c\int_0^t dt'/a(t') \approx c/H(t)$$

- where in the last <u>approximate</u> equality we introduced the so-called Hubble radius $d_H(t) = c/H(t)$ *which yields an effective estimate of the distance of causal correlation*

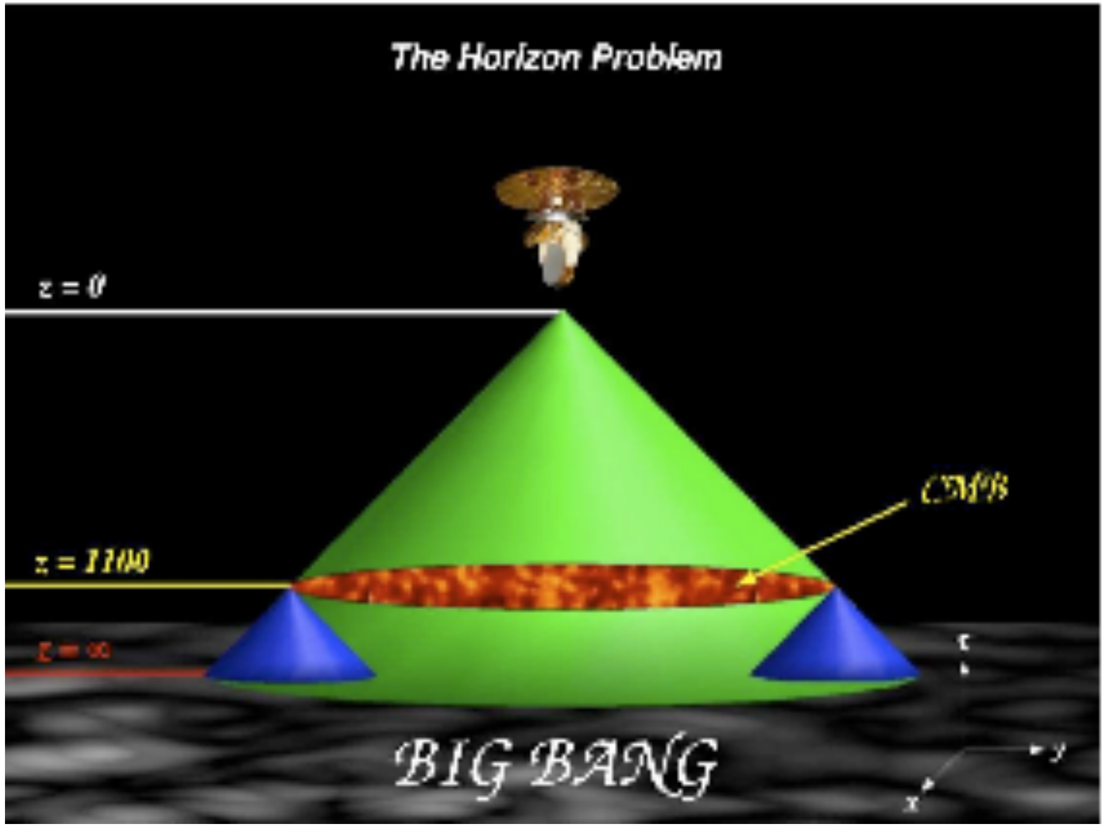
The horizon problem

The comoving scale

of causal correlation

$r_H(t) = 1 / a(t)H(t)$

grows with time



Kinney 2003

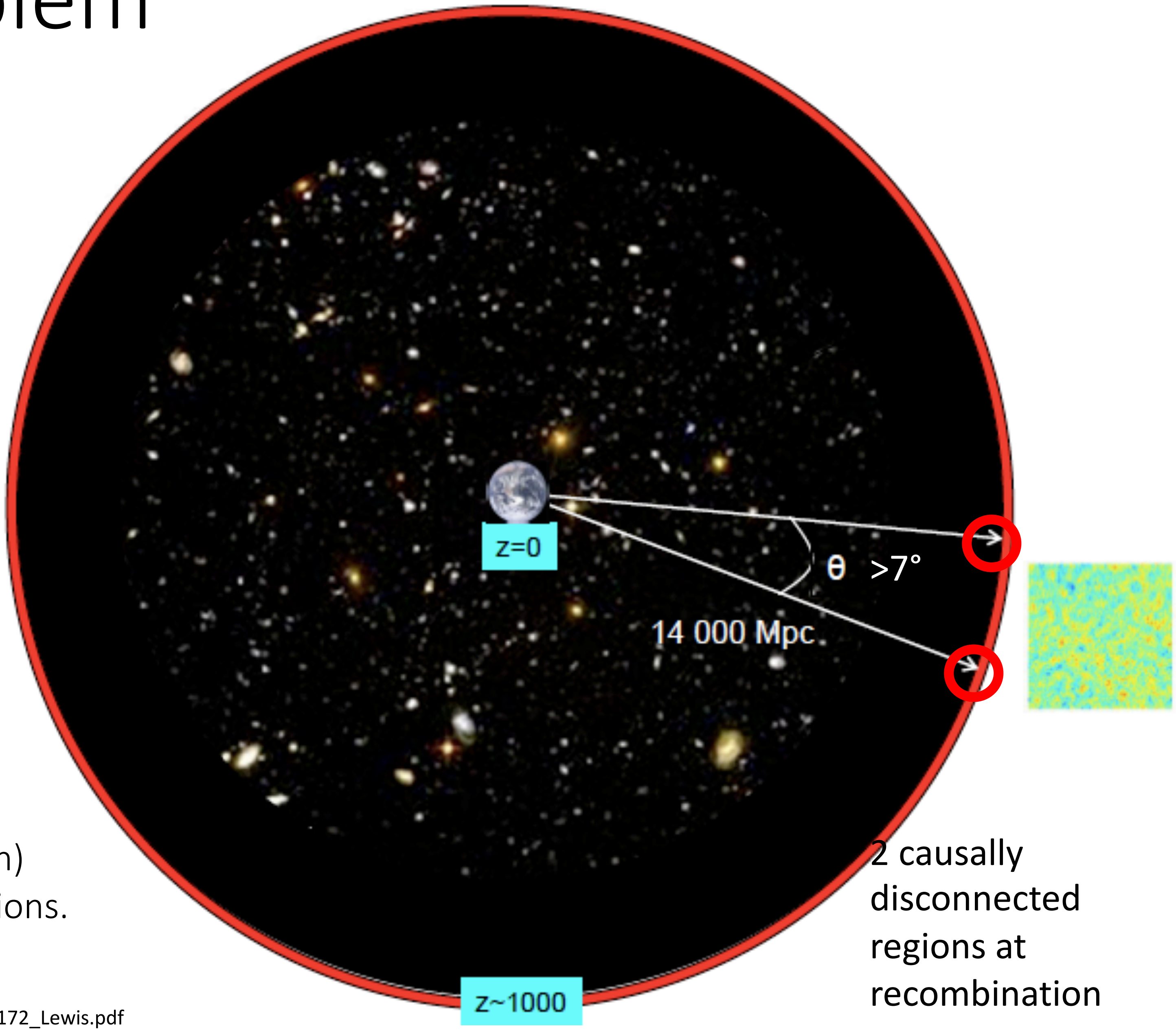
The horizon problem

Last scattering surface: CMB photons decouple at recombination epoch $z \sim 1100$ ($t_r \sim$ 400,000 years)

Light could have travelled
$L \sim c\, t_r$ (horizon)
Such a distance would subtend an angle of about 1 degree.

We actually see CMB photons with nearly the same T at bigger angular scales >7° (e.g. COBE satellite).

The CMB at last scattering (recombination) Consists of 10^5 causally disconnected regions.



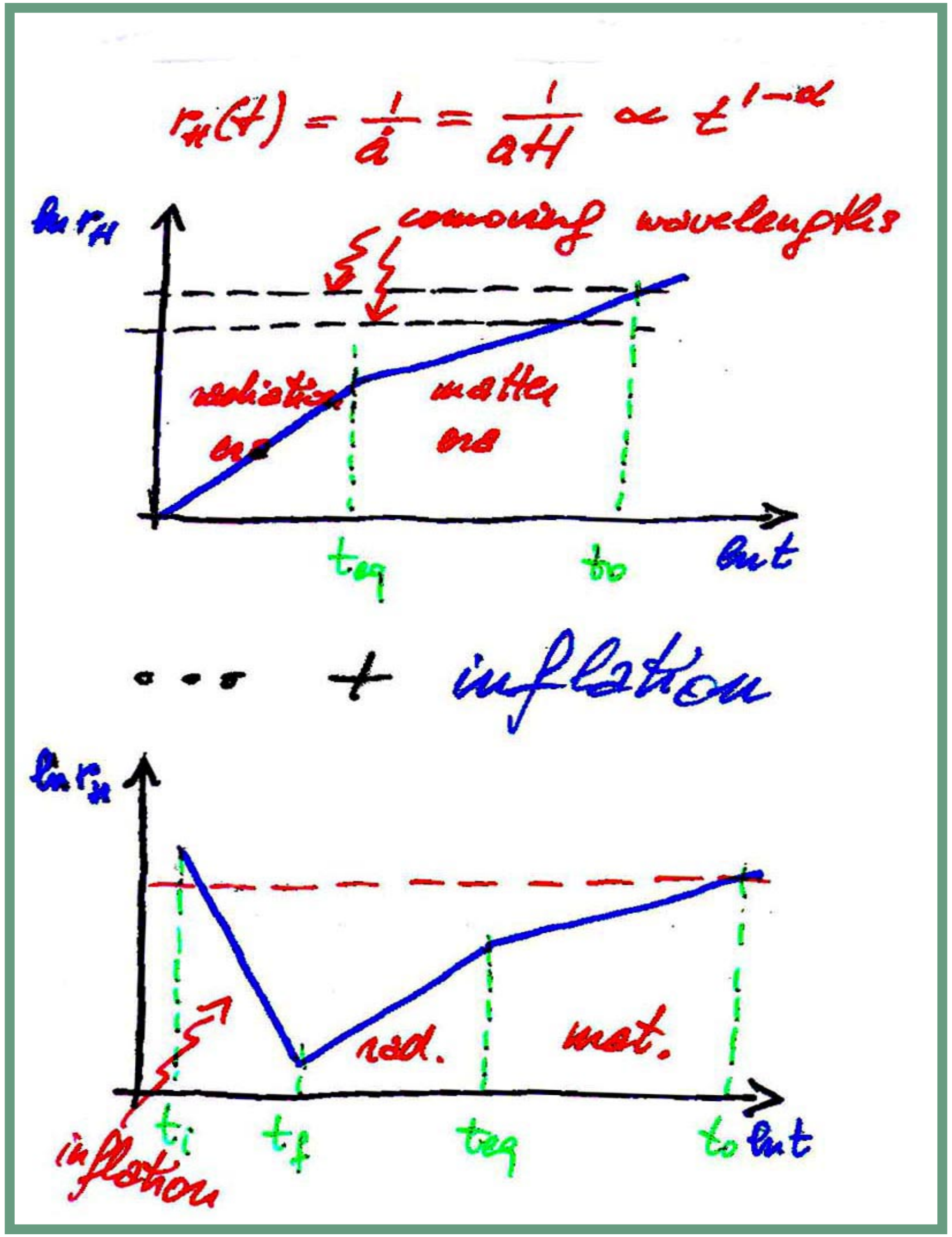
Adapted from http://benasque.org/2011Astrophysics/talks_contr/172_Lewis.pdf

Evolution of the comoving Hubble radius

To solve the horizon problem one needs a period when r_H decreases with time, i.e. a period of ***accelerated expansion:***

$$\ddot{a} > 0$$

i.e. the comoving scale of causal correlation $\mathbf{r_H(t)}$ decays with time during inflation



Solution of the horizon problem

The horizon problem is solved if a region that was causally connected at the beginning of inflation, t_i, whose typical size is $d_H(t_i) = a(t_i)\, r_H(t_i)$ after inflating by a factor

$$Z \equiv a(t_f)/a(t_i) = \exp\int_{t_i}^{t_f} dt H(t) \equiv \exp N_{\text{inf}}$$

is able to contain the present Hubble radius scaled back to the end of inflation t_f:

$$r_H(t_i) \geq r_H(t_0)$$

This is possible only if $r_H(t)$ decreases with time during inflation:

$$\dot{r}_H(t) < 0 \Leftrightarrow \ddot{a}(t) > 0$$

for a suitable time-interval

"Minimal inflation": $Z=Z_{min}$

Take $Z=Z_{min}$ such that $r_H(t_0)=r_H(t_i)$. From the definition of Z one finds

$$Z_{\min} \approx \left(10^{30} T_f / T_{Planck}\right)^{2/|1+3w_{\inf}|} \qquad w_{\inf} = p_{\inf} / \rho_{\inf}$$

which, for inflation final temperature not far from the Planck energy (T_{Planck}~10^{19}GeV), and for a nearly de Sitter equation of state w_{inf}=-1, leads to a minimum number of inflation e-folds

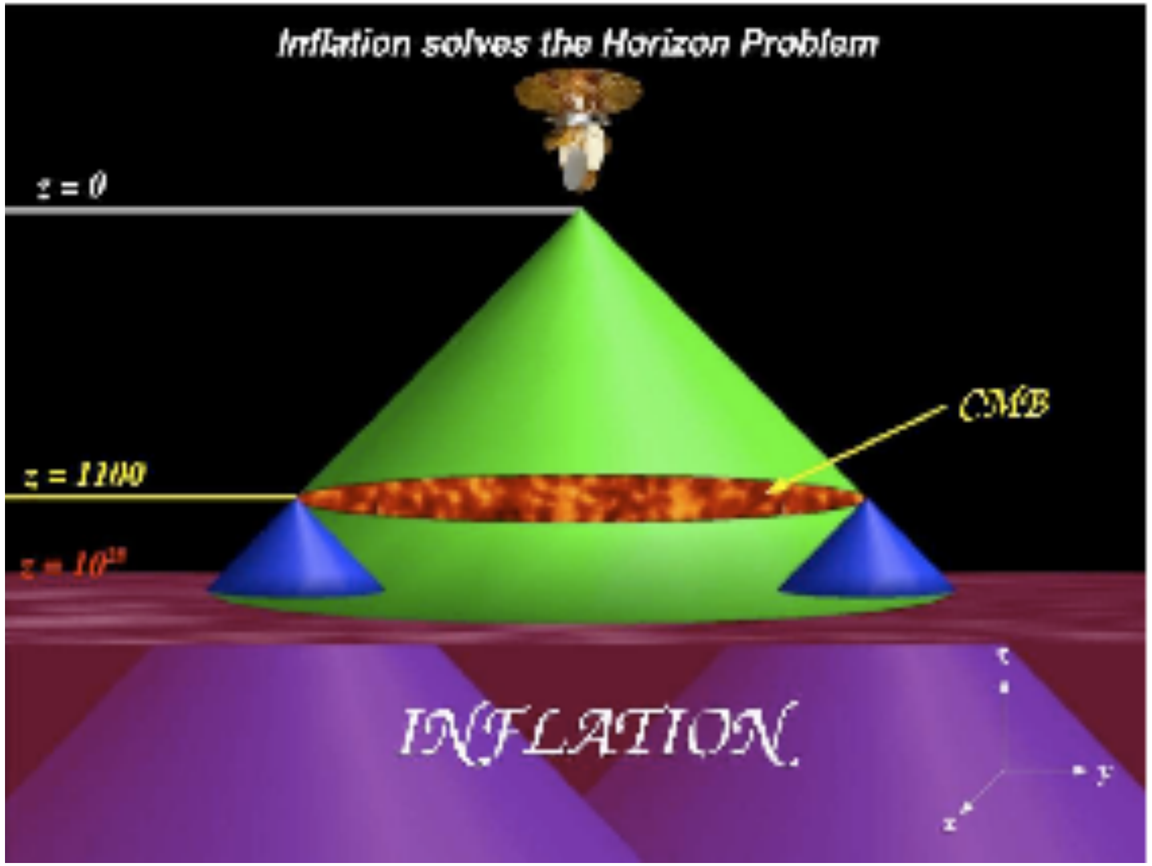
$$N_{inf} \sim 60$$

With such a choice $\Omega_0=\Omega_i$ which automatically solves the flatness problem. More in general

$$(\Omega_0^{-1}-1)/(\Omega_i^{-1}-1)=(Z/Z_{min})^{-|1+3w_{inf}|}$$

Solution of the horizon problem

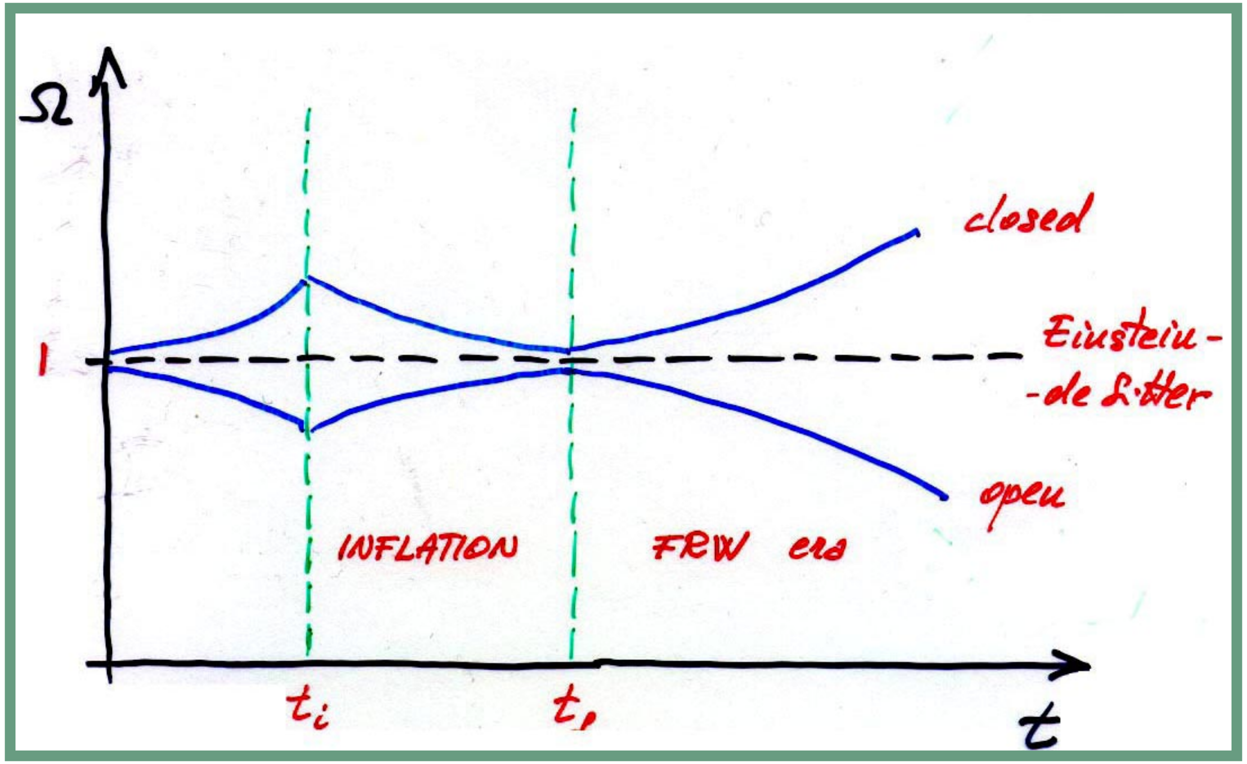
About ***60 e-folds*** of inflation suffice to solve the horizon and flatness problems. Inflation usually lasts much much longer.



Kinney 2003

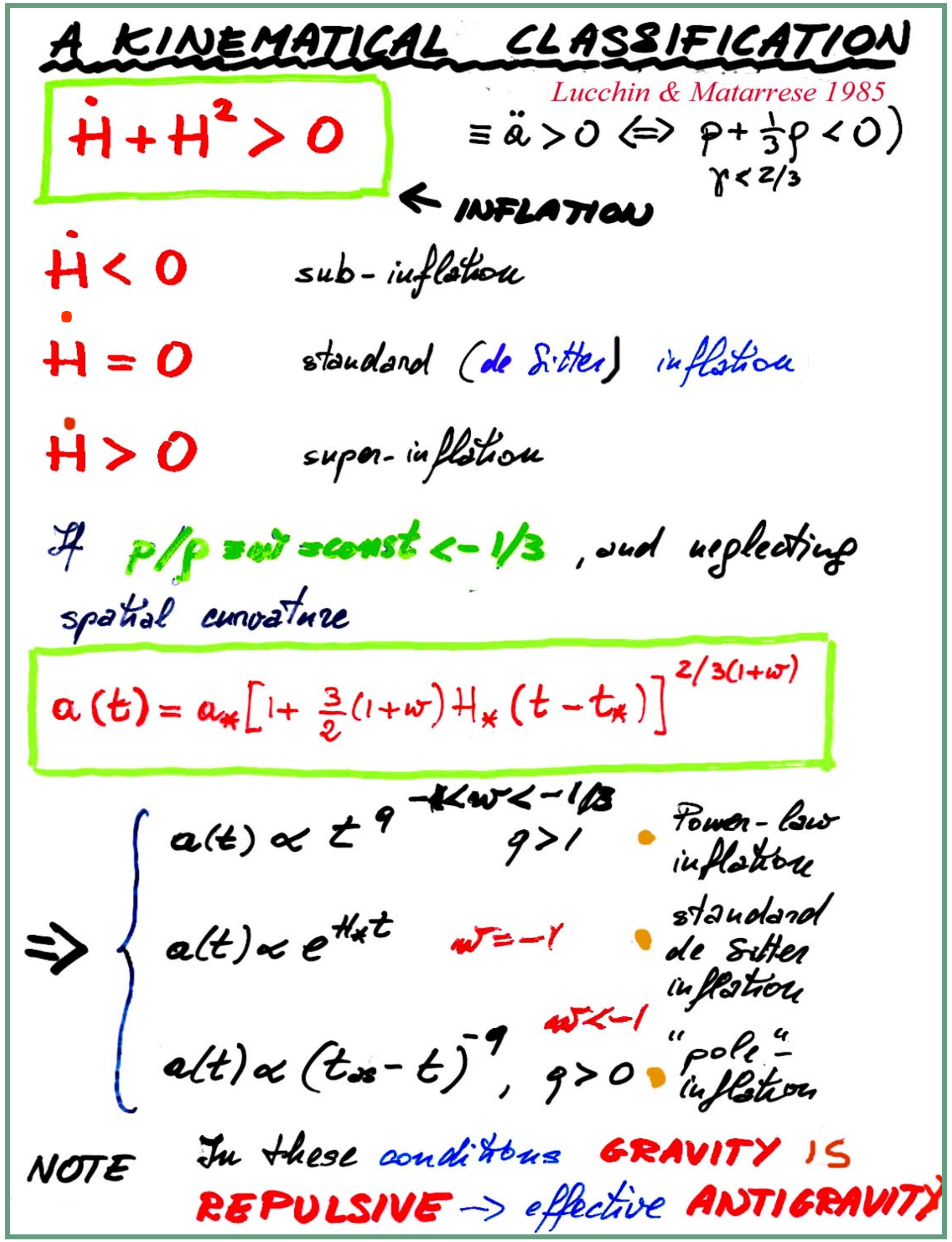
Evolution of the density parameter: flatness problem

The density parameter decreases with time if the Universe expansion is decelerated. One needs a fine-tuning of ~ 60 orders of magnitude (!) at the Planck time in order to allow for a density parameter of order unity today! A period of ***accelerated expansion*** automatically solves the problem.



Kinematics of inflation

The accelerated expansion can be realized by many different types of scale factor time-dependence, originating from different equations of state (w=p/ρ) during inflation. For slow-roll inflation this in turn comes from a choice of the inflaton potential V(φ).



The rise and fall … of the comoving Hubble horizon

(late-time dark energy dominance neglected for simplicity)

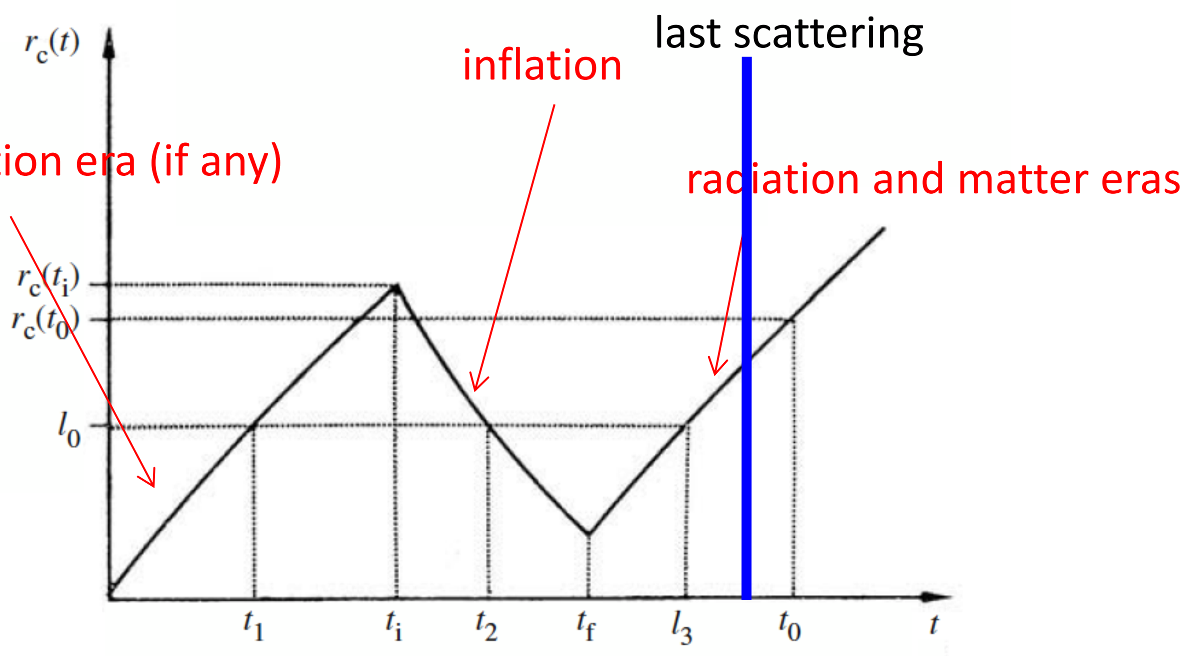


Figure 7.4 Evolution of the comoving cosmological horizon $r_c(t)$ in a universe characterised by a phase with an accelerated expansion (inflation) from t_i to t_f. The scale l_0 enters the horizon at t_1, leaves at t_2 and re-enters at t_3. In a model without inflation the horizon scale would never decrease so scales entering at t_0 could never have been in causal contact before. The horizon problem is resolved if $r_c(t_0) \leqslant r_c(t_i)$.

credits: Coles & Lucchin 2002

A brief history of the inflationary model

- 1979 – 1980 A. Starobinski shows that a de Sitter stage in the early Universe is driven by trace anomalies of quantum fields in an external gravitational field; this phase is later terminated by the generation of scalar field fluctuations ("scalarons") ➔ a stochastic background of gravity-waves should be the observable relic of this early phase.

- 1981 A. Guth shows that "Inflation" (i.e. a quasi-de Sitter expansion phase in the early Universe) is caused by a first-order phase transition ("Old Inflation"); this phase solves the "monopole overproduction problem" of phase transitions at high temperature ➔ the horizon and flatness problems find automatic solution. However, the phase transition never gets to an end ("graceful exit problem"). Similar ideas were in (Brout et al. 1978; Suto 1981; Kazanas 1981;).

- 1982 A dynamical symmetry-breaking mechanism is invoked to avoid the graceful exit problem of Guth's model in two independent analyses (Albrecht & Steinhardt 1982; Linde 1982). The "New Inflation" model is based on the slow-rolling of a scalar field along an almost flat potential. This scalar field is initially associated to the Higgs sector of GUT

A brief history of the inflationary model

- 1982 – 1983 Many independent groups (Guth & Pi; Starobinski; Hawking; Bardeen, Steinhardt & Turner) show that during slow-rolling inflation scalar perturbations are created by quantum vacuum oscillations of the scalar field, leading to density fluctuations dr/r ~ $l^{1/2}$ where l is the self-coupling constant of the scalar field. Consistency with the observed isotropy of the CMB constrains l to be much less than 10^{-4}. This leads to two classes of problems: the thermal initial conditions problem and the nature of the scalar field which needs to be very weakly coupled with the rest of the world (it must be a "singlet")

- 1983 A. Linde proposes a new class of models, called "Chaotic Inflation", where thermal initial conditions (i.e. a metastable state) are replaced by an unstable initial scalar-field state motivated by Heisenberg uncertainty relation near the Planck scale.

- 1983 Particle-physics theorists argue that Supersymmetry might be the natural environment for such a weakly coupled scalar field, which can be easily added as a novel scalar sector to the theory: the "Inflaton".

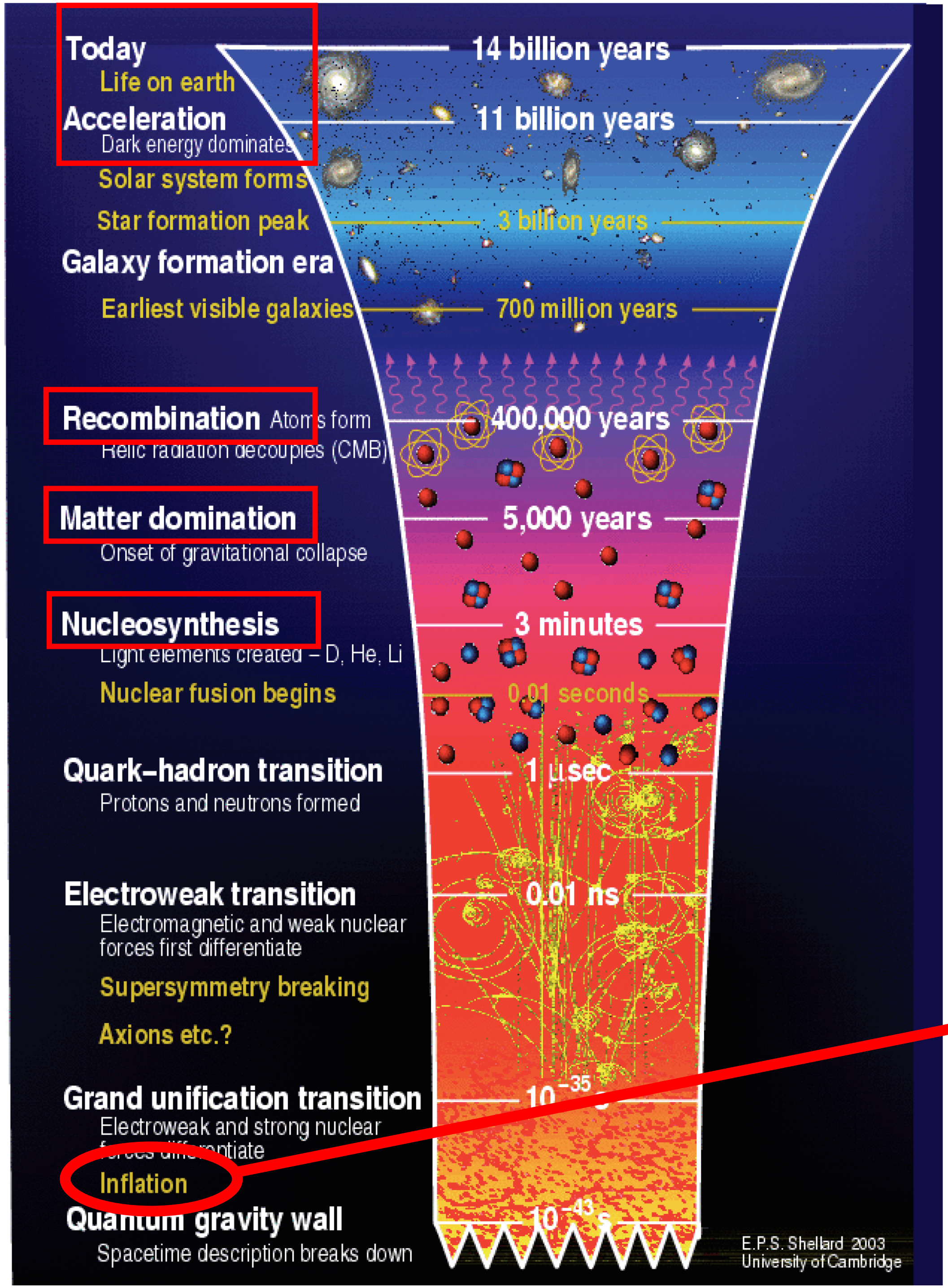
A brief history of the inflationary model

- 1986 A. Linde notices that chaotic inflation is the most probable state of the Universe. Only a tiny fraction of the inflated Universe ends the accelerated phase undergoing reheating, thus leading to a post-inflation phase resembling the observed Universe. The vast majority of the Universe volume undergoes "Eternal Inflation". This necessarily calls for the use of the "Anthropic Principle" in cosmology.

- 1999 The detection of the first Doppler peak in the CMB anisotropies by the *BOOMERanG* and *MAXIMA* collaborations gives strong support to the inflationary prediction of a flat (W = 1) Universe.

- 2003 WMAP yields spectacular support to all the most important predictions of inflation: flat Universe, adiabatic and nearly scale-invariant density perturbations, T-E cross-correlation, …! Only the inflation generated gravitational-wave background is yet undetected.

- 2013-2018 *Planck* provides the stringent constraints on models of inflation and constrains primordial non-Gaussianity to small values, giving support to the simplest inflation models.



We are here

$Z_{rec} \sim 1100$

$Z_{eq} \sim 3500$

T~1 MeV

We seek information about very early times and very high energies $E \sim 10^{16}$ GeV ... or lower

Major events in the cosmic history

	Time	Energy	Redshift
Planck Epoch?	$< 10^{-43}$ s	10^{18} GeV	
String Scale?	$\gtrsim 10^{-43}$ s	$\lesssim 10^{18}$ GeV	
Grand Unification?	$\sim 10^{-36}$ s	10^{15} GeV	
Inflation?	$\gtrsim 10^{-34}$ s	$\lesssim 10^{15}$ GeV	
SUSY Breaking?	$< 10^{-10}$ s	> 1 TeV	
Baryogenesis?	$< 10^{-10}$ s	> 1 TeV	
Electroweak Unification	10^{-10} s	1 TeV	
Quark-Hadron Transition	10^{-4} s	10^2 MeV	
Nucleon Freeze-Out	0.01 s	10 MeV	
Neutrino Decoupling	1 s	1 MeV	
BBN	3 min	0.1 MeV	
Matter-Radiation Equality	10^4 yrs	1 eV	10^4
Recombination	10^5 yrs	0.1 eV	1,100
Dark Ages	$10^5 - 10^8$ yrs		> 25
Reionization	10^8 yrs		$25 - 6$
Galaxy Formation	$\sim 6 \times 10^8$ yrs		~ 10
Dark Energy	$\sim 10^9$ yrs		~ 2
Solar System	8×10^9 yrs		0.5
Albert Einstein born	14×10^9 yrs	1 meV	0

Credits: Baumann 2012

Inflation in the early Universe

- Inflation is an epoch of accelerated expansion in the early Universe (~ 10^{-34} s after the "Big Bang") which allows to solve two inconsistencies of the standard Big Bang model (horizon: why is the Universe so homogeneous and isotropic on average + flatness: why is the Universe spatial curvature so small even ~ 14 billion years after the Big Bang?).

- Inflation (Brout et al. 1978; Starobinski 1980; Kazanas 1980; Sato 1981; Guth 1981; Linde 1982, Albrecht & Steinhardt 1982; etc. ...) is based upon the idea that the vacuum energy of a scalar quantum field, dubbed the "inflaton", dominates over other forms of energy, hence giving rise to a quasi-exponential (de Sitter) expansion, with scale-factor

$$a(t) \approx \exp(Ht)$$

Inflation predictions

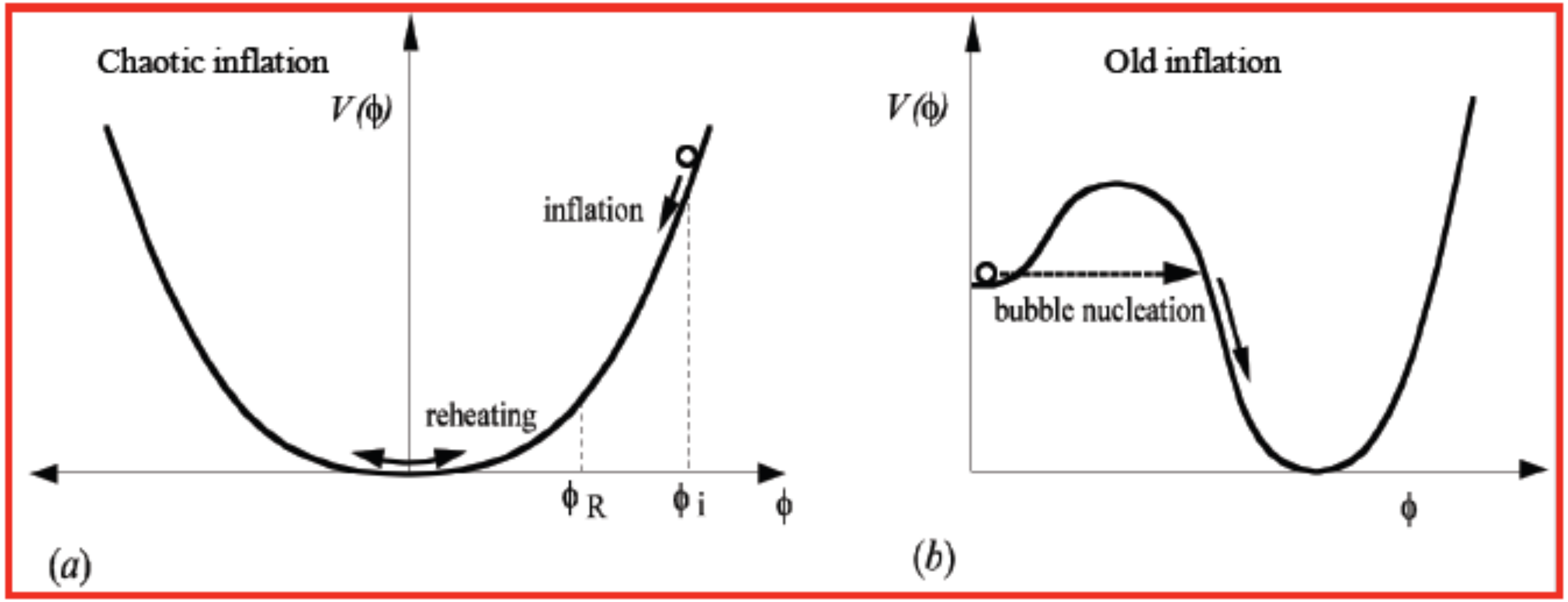
- Quantum vacuum oscillations of the inflaton (or other scalar fields, such as the “curvaton”) give rise to classical fluctuations in the energy density, which provide the seeds for Cosmic Microwave Background (CMB) radiation temperature anisotropies and polarization, as well as for the formation of Large Scale Structures (LSS) in the present Universe.

- All the matter and radiation which we see today must have been generated after inflation (during “reheating”), since all previous forms of matter and radiation have been tremendously diluted by the accelerated expansion (“Cosmic no-hair conjecture”).

- In full generality a stochastic gravitational-wave background is predicted, whose amplitude is related to the energy scale during inflation.

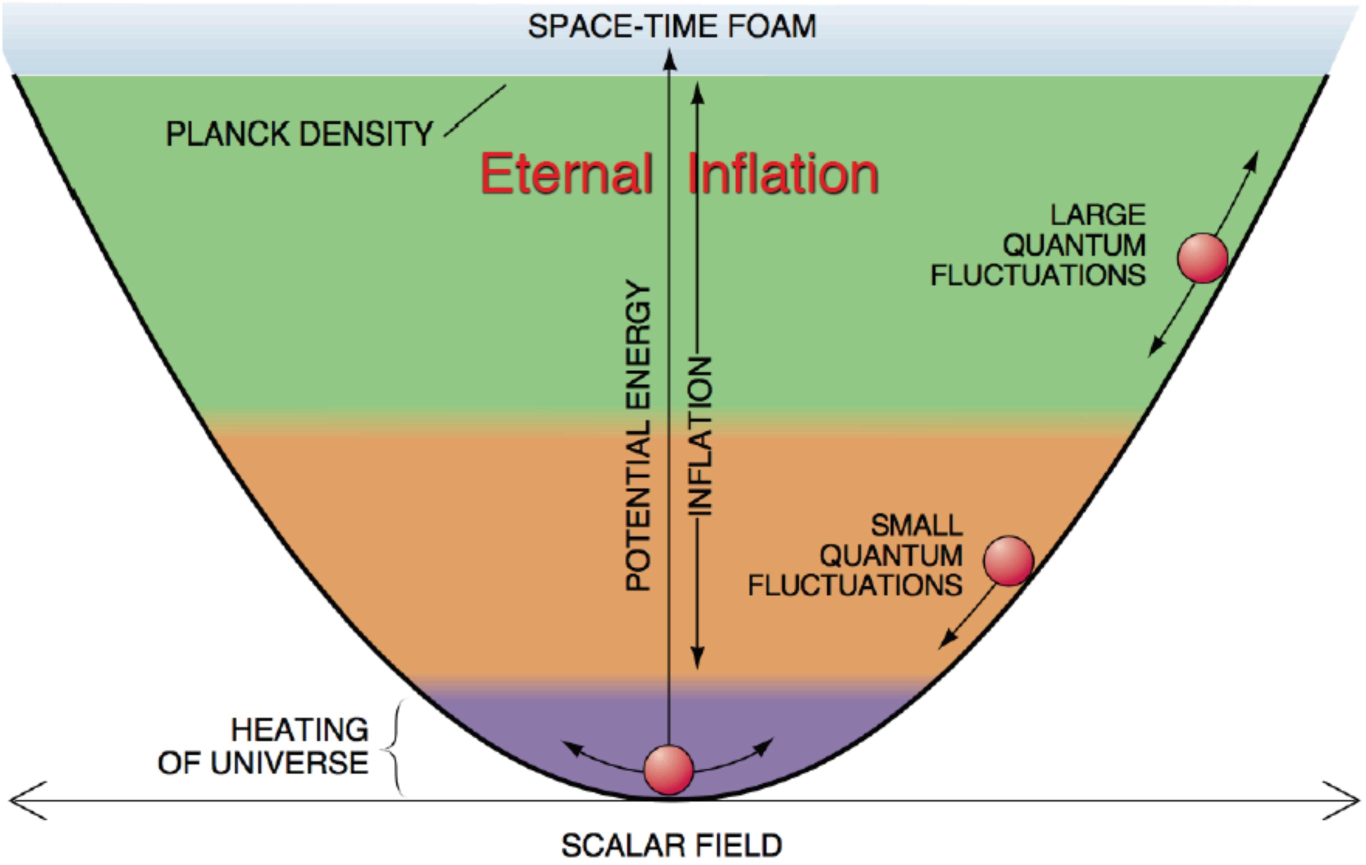
Inflation dynamics

- Different models of inflation derive from different potentials and different initial conditions. Old inflation (*Guth 1981) assumes thermal initial conditions* (which are very difficult to achieve). Chaotic inflation (*Linde 1983) is based* on the application of the uncertainty principle at Planck energies.



Chaotic inflation

No need for false vacua, thermal equilibrium, and phase transitions. Just use any sufficiently flat potential.



Credits: Andrei Linde 2013

Two simple but very important examples

“Large field” models

$$V(\phi) \propto \phi^{\alpha}$$

typical of ``caothic inflation scenario’’ (Linde `83)

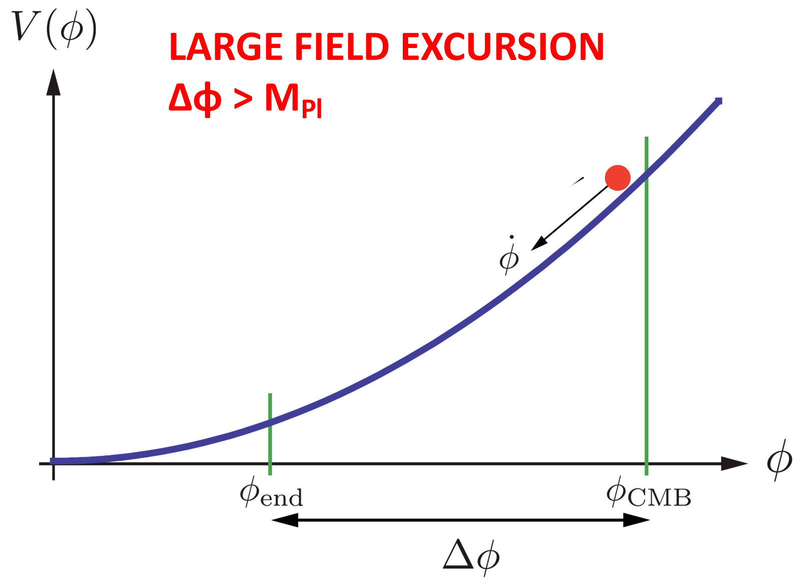
$$V(\phi) \propto \exp[\phi/\mu]$$

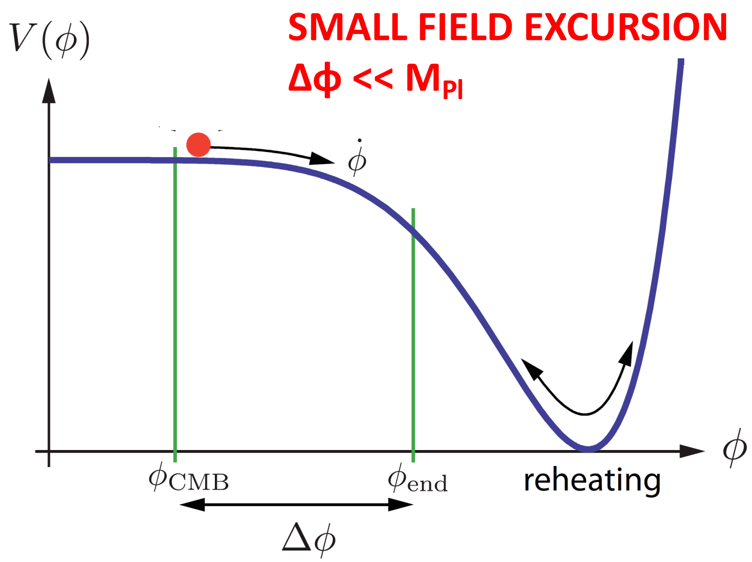
``power law inflation’’ (Lucchin & Matarrese‘85)

“Small field” models

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^p\right] \qquad \phi < \mu < M_{\rm Pl}$$

from spontaneous symmetry breaking or Goldstone, axion models (Linde; Albrecht, Steinhardt `82; Freese et al ‘90)





Inflation and the inflaton

Consider a simple real scalar field:

$$\boxed{S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right]} = S_{\rm EH} + S_\phi \, .$$

3 ingredients:

- The scalar field (the so called inflaton field)
- the gravitational field (i.e., the metric)
- the ``rest of the world'': fermions, gauge bosons, other scalars….(I will come back later to this)

Inflation and the Inflaton

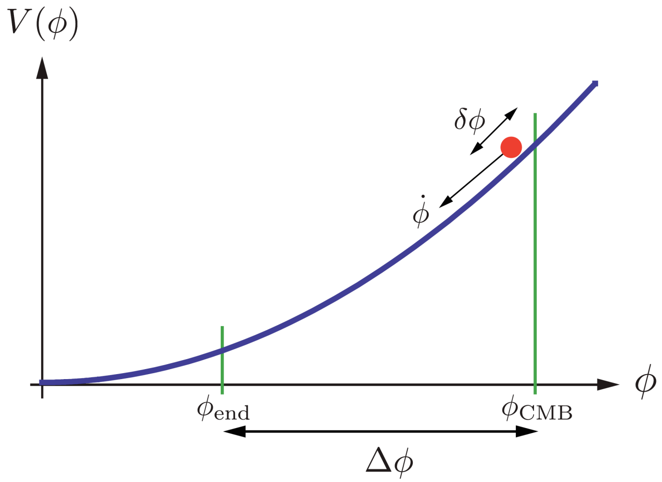
$$\mathcal{L}_\phi[\phi, g_{\mu\nu}] = -\frac{1}{2} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} - V(\phi)$$

Standard kinetic term

Inflaton potential: describes the self-interactions of the inflaton field and its interactions with the rest of the world

Think the inflaton mean field as a particle moving under a force induced by the potential V

Ex: $$V(\phi) = \frac{m^2}{2}\phi^2$$



Why does a scalar field work well in driving inflation?
Because a scalar field can provide an energy density that remains (almost) constant in time

Energy momentum tensor

$$T^{\phi}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \left[\frac{\partial(\sqrt{-g}\mathcal{L}_\phi)}{\partial g^{\mu\nu}} - \partial_\alpha \frac{\partial(\sqrt{-g}\mathcal{L}_\phi)}{\partial\partial_\alpha g^{\mu\nu}}\right]$$

For a real scalar field, minimally coupled (i.e. without coupling to gravity like $\xi R \phi^2$)

$$T^{\phi}_{\mu\nu} = 2\frac{\partial \mathcal{L}_\phi}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_\phi = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - V(\phi)\right)$$

Scalar field eq. of motion

Let us look in more details into the dynamics of a scalar field in a curved space-time

$$\frac{\delta S_\phi}{\delta \phi} = 0 \rightarrow \Box\phi = -\frac{\partial V}{\partial \phi} \rightarrow \qquad \boxed{\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2 \phi}{a^2} = -\frac{\partial V}{\partial \phi}}$$

This is our master equation: the Klein-Gordon equation for a scalar field in a FRW metric

Background evolution

Split the scalar field as a (classical) background + a (quantum) fluctuation

$$\phi(t) = \phi_0(t) + \delta\phi(\mathbf{x}, t)$$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,.$$

Dropping the fluctuation we obtain

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

$$\boxed{w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}},$$

If the potential energy dominates over the kinetic term we obtain acceleration ($w<1/3$) and, in the limit $w \rightarrow -1$ we get de Sitter evolution

$$V(\phi) \gg \dot{\phi}^2 \quad \Rightarrow \quad p_\phi \simeq -\rho_\phi \quad \Rightarrow \quad \boxed{a(t) \approx \exp(Ht)}$$

Quantifying slow-roll dynamics

Using the approximate Friedmann and Klein-Gordon eqs.:

$$H^2 \simeq \frac{8\pi G}{3} V(\phi), \qquad \dot{\phi}^2 \ll V(\phi) \Longrightarrow \frac{(V')^2}{V} \ll H^2$$

$$3H\dot{\phi} = -V'(\phi) \qquad \ddot{\phi} \ll 3H\dot{\phi} \Longrightarrow V'' \ll H^2.$$

Define two (first-order) slow-roll parameters, which need to be << 1 for successful slow-roll inflation to occur

$$\epsilon = -\frac{\dot{H}}{H^2} = 4\pi G \frac{\dot{\phi}^2}{H^2} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2,$$

$$\eta = \frac{1}{8\pi G}\left(\frac{V''}{V}\right) = \frac{1}{3}\frac{V''}{H^2},$$

$$\delta = \eta - \epsilon = -\frac{\ddot{\phi}}{H\dot{\phi}}.$$

← also useful

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)\,H^2,$$

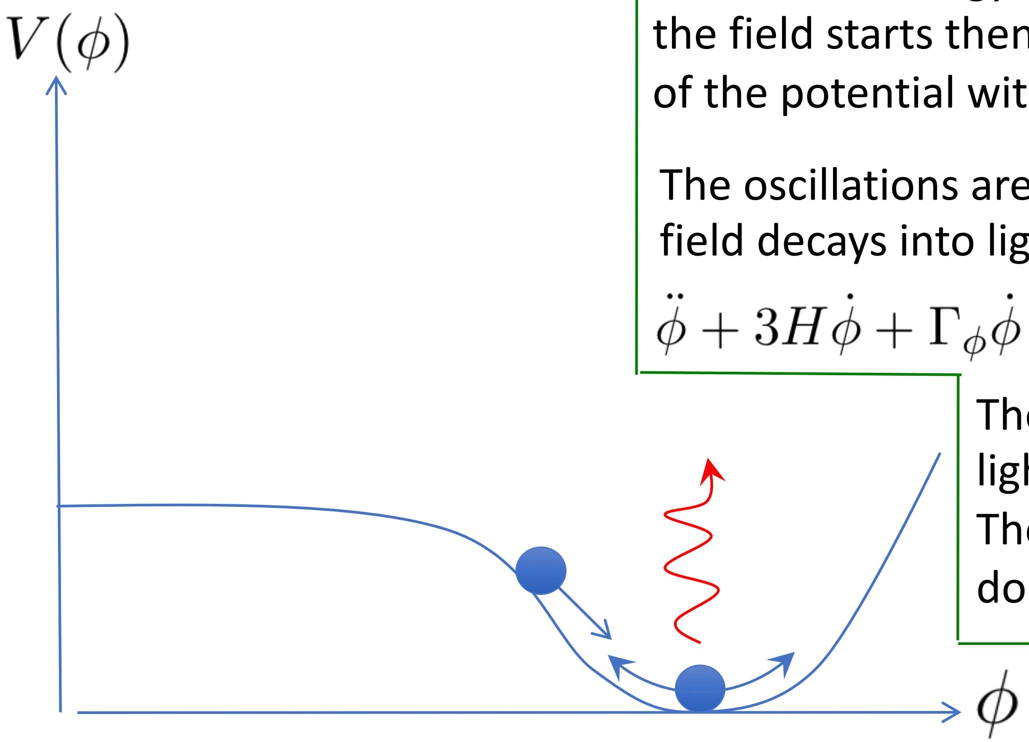
→ Inflation demands $\epsilon < 1$

Inflation ends when the inflaton field starts to ``feel'' the curvature of the potential

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)H^2$$

So when $\epsilon \sim 1$ (and in general $\eta \sim 1$) inflation comes to an end

REHEATING PHASE



The kinetic energy is not negligible anymore;
the field starts then to oscillate around the minimum
of the potential with frequency $V_{,\phi\phi} \gg H^2$

The oscillations are damped because in this regime the inflaton field decays into lighter particles with a decay rate Γ_ϕ

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi\dot{\phi} = -V_{,\phi} \longrightarrow \dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi\rho_\phi$$

The kinetic energy is transferred to other lighter particles (release of latent heat). These particles thermalizes and start to dominate → ***the standard FRW universe starts***

credits: Nicola Bartolo

Physical meaning of inflaton fluctuations

Following Guth & Pi (1982), let's consider a scalar field φ evolving in de Sitter space – time with scale - factor $a(t) = e^{Ht}$

$\ddot{\phi} + 3H\dot{\phi} = -V' + e^{-2Ht}\nabla^2\phi$

and let $\varphi(t)$ be a homogeneous solution, then $\phi(\vec{x},t) = \varphi(t) + \delta\phi(\vec{x},t)$.

The quantity $\delta\phi$ satisfies the perturbed Klein - Gordon equation

$\delta\ddot{\phi} + 3H\delta\dot{\phi} \approx -V''\,\delta\phi + \underline{e^{-2Ht}\nabla^2\delta\phi}$ while $\dot{\varphi}$ obeys to the equation $\dddot{\varphi} + 3H\ddot{\varphi} = -V''\dot{\varphi}$

The term $\underline{e^{-2Ht}\nabla^2\delta\phi}$ decays with time and becomes soon negligible, hence $\delta\phi(\vec{x},t)$ and $\dot{\varphi}(t)$ satisfy the same equation. Moreover, the Wronskian $W\{\delta\phi,\dot{\varphi}\}$ obeys to

$\dot{W} = -3HW \qquad \Rightarrow W = W_0 e^{-3Ht} \to 0$ (large times).

Hence, at large times the most general solution reads $\delta\phi(\vec{x},t) \sim -\delta\tau(x)\dot{\varphi}(t)$

which, to first order in $\delta\tau$ yields $\phi(\vec{x},t) \sim \varphi(t - \delta\tau(\vec{x}))$

The effect of inflaton fluctuations is to produce a ***<u>space-dependent time-delay</u>*** *$\delta\tau$ in the evolution of the homogeneous mode φ. We would then expect density fluctuations $\delta\rho/\rho \sim H\delta\tau$.*

Perturbing geometry

To allow for deviations from homogeneous and isotropic universe perturb the FRW line-element (scalar perturbations only)

$$ds^2 = a^2((-1-2A)d\tau^2 + 2\,\partial_i B\, d\tau\, dx^i + ((1-2\psi)\delta_{ij} + D_{ij}E)\, dx^i\, dx^j).$$

Equivalently, write the covariant metric tensor as:

$$g_{\mu\nu} = a^2 \begin{pmatrix} -1 - 2A & \partial_i B \\ \partial_i B & (1 - 2\psi)\,\delta_{ij} + D_{ij}E \end{pmatrix}, \qquad D_{ij} = \left(\partial_i\partial_j - \tfrac{1}{3}\,\delta_{ij}\,\nabla^2\right).$$

whose inverse reads:

$$g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} -1 + 2A & \partial^i B \\ \partial^i B & (1 + 2\psi)\delta^{ij} - D^{ij}E \end{pmatrix}$$

Perturbed LHS of Einstein's eqs.

Expanding to first order in the metric perturbation the Christoffel symbols first and the Ricci tensor next one arrives at:

$$\delta G_{00} = -2\frac{a'}{a}\,\partial_i\partial^i\,B - 6\frac{a'}{a}\,\psi' + 2\,\partial_i\partial^i\,\psi + \frac{1}{2}\,\partial_k\partial^i\,D_i^k E\,;$$

$$\delta G_{0i} = -2\frac{a''}{a}\,\partial_i B + \left(\frac{a'}{a}\right)^2\,\partial_i B + 2\partial_i\,\psi' + \frac{1}{2}\partial_k\,D_i^k E' + 2\,\frac{a'}{a}\,\partial_i A$$

$$\begin{aligned}\delta G_{ij} \quad &= \quad \left(2\frac{a'}{a}\,A' + 4\frac{a'}{a}\,\psi' + 4\frac{a''}{a}\,A - 2\left(\frac{a'}{a}\right)^2\,A\right. \\ &+ \quad 4\frac{a''}{a}\,\psi - 2\left(\frac{a'}{a}\right)^2\,\psi + 2\psi'' - \partial_k\partial^k\,\psi\end{aligned}$$

Perturbed RHS of Einstein's & Klein-Gordon eqs.

Next consider the scalar field perturbation and look at its first-order effects on the stress-energy tensor and scalar field equation of motion

$$\phi(\mathbf{x},\tau) = \phi(\tau) + \delta\phi(\mathbf{x},\tau)$$

Stress-energy tensor

$$\delta T_{00} = \delta\phi'\,\phi' + 2\,A\,V(\phi)\,a^2 + a^2\,\frac{\partial V}{\partial\phi}\,\delta\phi\,;$$

$$\delta T_{0i} = \partial_i\,\delta\phi\,\phi' + \frac{1}{2}\,\partial_i B\,\phi'^2 - \partial_i B\,V(\phi)\,a^2\,;$$

$$\begin{aligned}\delta T_{ij} &= \left(\delta\phi'\,\phi' - A\,\phi'^2 - a^2\,\frac{\partial V}{\partial\phi}\,\delta^{(1)}\phi - \psi\,\phi'^2 + 2\,\psi\,V(\phi)\,a^2\right)\delta_{ij}\\ &+ \frac{1}{2}\,D_{ij}E\,\phi'^2 - D_{ij}E\,V(\phi)\,a^2\,.\end{aligned}$$

Klein-Gordon eq.

$$\begin{aligned}\delta\phi'' + 2\,\frac{a'}{a}\delta\phi' &- \partial_i\,\partial^i\,\delta\phi - A'\phi' - 3\,\psi'\phi' - \partial_i\,\partial^i B\,\phi'\\ &= -\delta\phi\,\frac{\partial^2 V}{\partial\phi^2}\,a^2 - 2\,A\,\frac{\partial V}{\partial\phi}.\end{aligned}$$

Gauge transformations

A gauge is a map between points of the physical (perturbed) space-time and points of the background. A gauge transformation is a change of such a map. This can be mimicked by a coordinate transformation

$$\widetilde{x^\mu} = x^\mu + \delta x^\mu$$

$$\delta x^0 = \xi^0(x^\mu);$$

$$\delta x^i = \partial^i \beta(x^\mu) + v^i(x^\mu); \quad \partial_i v^i = 0,$$

scalars

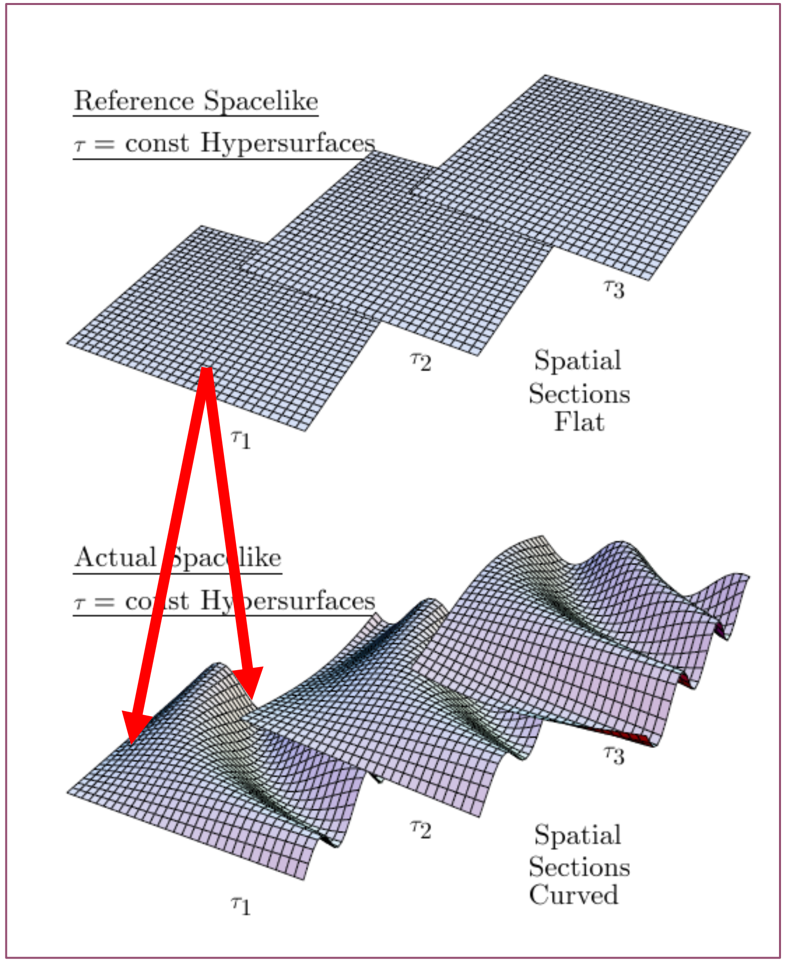
vector

Its effect on a tensor quantity Q is given by the Lie derivative of the quantity along the infinitesimal four-vector defining the coordinate transformation

$$\widetilde{\delta Q} = \delta Q + \pounds_{\delta x} Q_0$$

E.g., for a scalar f (e.g. the density) one gets:

$$\widetilde{\delta f} = \delta f - f' \xi^0$$



from: Riotto (2002)

Gauge transformations & gauge invariance

Under a gauge transformation our "scalar" perturbations transform as follows:

$$\tilde{A} = A - \xi^{0\prime} - \frac{a'}{a}\xi^0 ;$$
$$\tilde{B} = B + \xi^0 + \beta'$$
$$\tilde{\psi} = \psi - \frac{1}{3}\nabla^2\beta + \frac{a'}{a}\xi^0 ;$$
$$\tilde{E} = E + 2\beta .$$

→Beware of pure-gauge modes and of gauge artifacts ("tenacious myths")!

Two ways out: choose a gauge (but take care of residual gauge ambiguities)

Define gauge-invariant quantities (e.g. by linearly combining perturbations)

Bardeen's g.-i. potentials

$$\Phi = -A + \frac{1}{a}\left[\left(-B + \frac{E'}{2}\right)a\right]' ,$$
$$\Psi = -\psi - \frac{1}{6}\nabla^2 E + \frac{a'}{a}\left(B - \frac{E'}{2}\right) .$$

Useful gauge-invariant variables

The gauge-invariant (by construction) quantity

$$\zeta = \psi + \mathcal{H}\frac{\delta\rho}{\rho'} = \psi + H\frac{\delta\rho}{\dot{\rho}}$$

represents the curvature perturbation on spatial slices of uniform energy density.
For the scalar field fluctuation, one can analogously define the g.-i. variable.

$$\delta\phi^{(\mathrm{GI})} \;=\; -\delta\phi \;+\; \phi'\left(\frac{E'}{2} \;-\; B\right)$$

Alternatively, the so-called gauge-invariant Sasaki-Mukhanov variable

$$Q = \delta\phi + \frac{\phi'}{\mathcal{H}}\,\psi = \delta\phi + \frac{\dot{\phi}}{H}\psi \equiv \frac{\dot{\phi}}{H}\,\mathcal{R}$$

represents the inflaton perturbation on spatially flat gauges.
Hence: the inflaton perturbation and the curvature perturbation are related via a gauge-transformation!

Classical evolution of scalar perturbations

Before discussing the evaluation of the scalar-field fluctuations, let’s discuss how this information can be transferred to the post-inflationary evolution.

We need a gauge-invariant quantity that changes smoothly when the Universe changes its equation of state:

inflaton → radiation → matter → dark energy domination

The quantity ζ remains (approximately) constant outside the horizon, as long as non-adiabatic pressure terms do not appear (e.g. isocurvature perturbations)

$$\dot{\zeta} = -\frac{H}{p+\rho}\delta p_{\rm nad}$$

Scalar mode equation of motion in a quasi-de Sitter stage

In the longitudinal gauge (B=E=0), using Einstein's eqs., the i≠j components yield

$$\partial_i \partial_j \left(\psi - A\right) = 0 \Longrightarrow \psi = A$$

the 0-i components (momentum constraints) yield

$$\dot{\psi} + H\,\psi = 4\pi G \dot{\phi}\,\delta\phi = \epsilon\, H \frac{\delta\phi}{\dot{\phi}},$$

the 0-0 component (energy constraint) and the i-i components give

$$\begin{aligned} -3H\left(\dot{\psi} + H\psi\right) + \frac{\nabla^2\psi}{a^2} &= 4\pi G\left(\dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\psi + V'\delta\phi\right) \\ -\left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\psi - 3H\dot{\psi} - \ddot{\psi} &= -\left(\dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\psi - V'\delta\phi\right), \end{aligned}$$

Scalar mode equation of motion in a quasi-de Sitter stage

Combining the previous eqs. One obtains an equation only for (e.g.) the gravitational potential ψ

$$\ddot{\psi}_{\mathbf{k}} + \left(H - 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\dot{\psi}_{\mathbf{k}} + 2\left(\dot{H} - H\frac{\ddot{\phi}}{\dot{\phi}}\right)\psi_{\mathbf{k}} + \frac{k^2}{a^2}\psi_{\mathbf{k}} = 0$$

On super-horizon scales, using the background equation of motion and the definition of slow-roll parameters one can show that the gravitational potential is nearly constant, which upon replacement in the momentum constrain, gives

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \left(V'' + 6\epsilon H^2\right)\delta\phi_{\mathbf{k}} = 0.$$

This result can be used in the perturbed Klein-Gordon equation to obtain (still on super-horizon scales)

$$\psi_{\mathbf{k}} \simeq \epsilon\, H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}}$$

Rescaling the scalar field variable as $\delta\chi_{\mathbf{k}} = \delta\phi_{\mathbf{k}}/a$ we obtain

$$\begin{aligned} \delta\chi''_{\mathbf{k}} &- \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\delta\chi_{\mathbf{k}} = 0 \\ \nu^2 &= \frac{9}{4} + 9\epsilon - 3\eta. \end{aligned}$$

Solving for the scalar mode

For constant ν (which is consistent with being at first order in the slow-roll approximation the scalar field eq. of motion is solved by

$$u_{\mathbf{k}} = \frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \sqrt{-\tau}\, H_\nu^{(1)}(-k\tau)$$

and its complex conjugate. On super-horizon scales one has

$$u_{\mathbf{k}} = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} 2^{\left(\nu-\frac{3}{2}\right)} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2}-\nu}.$$

Reminding the definition of the curvature g.i. perturbation

$$\mathcal{R} \equiv -\Psi - \frac{H}{\phi'}\delta\phi^{(\mathrm{GI})} = -\frac{u}{z}$$

we finally find

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\,\pi^2}\left|\frac{u_{\mathbf{k}}}{z}\right|^2 = \frac{1}{2{m_{\mathrm{Pl}}}^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1} \equiv A_{\mathcal{R}}^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1}$$

With "scalar spectral index" $n_{\mathcal{R}} - 1 = 3 - 2\nu = 2\eta - 6\epsilon$

Action functional for cosmological GW

- Let's start from the action for Gravity + a neutral scalar field (which is what we need to describe standard inflation models)

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{1}{2} M_{\mathrm{pl}}^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right] ,$$

- Let's now focus on the gravitational part and perturb it up to second order in tensor modes, we find

$$S_{\mathrm{T}}^{(2)} = \frac{M_{\mathrm{pl}}^2}{8} \int \mathrm{d}^4x \, a^2(t) \left[\dot{h}_{ij} \dot{h}_{ij} - \frac{1}{a^2} \left(\nabla h_{ij} \right)^2 \right]$$

Equation of motion for (linear) GW

- By functionally differentiating the previous equation we easily find

$$\nabla^2 h_{ij} - a^2 \ddot{h}_{ij} - 3a\dot{a}\dot{h}_{ij} = 0.$$

- At this level the RHS can only vanish, as our scalar field cannot source tensor modes. Similarly, a perfect fluid will not give rise to tensor contributions in the RHS of Einstein's equations. Ford and Parker (1977) showed that this equation can be connected with the Klein-Gordon eq. for a minimally-coupled neutral scalar field once you Fourier transform and account for the GW polarozation tensor, namely:

$$v_{ij} \equiv \frac{aM_{\rm pl}}{2} h_{ij} \qquad v_{ij}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=(+,\times)} v_{\mathbf{k}}^{(\lambda)}(t) e_{ij}^{(\lambda)}(k) e^{i\mathbf{k}\cdot\mathbf{x}}$$

λ=+,x

where $e_{ij}^{(+,\times)}$ is a polarization tensor satisfying the conditions $e_{ij} = e_{ji}$, $k^i e_{ij} = 0$, $e_{ii} = 0$, with $+, \times$ the two GW polarization states

GWs have only (9→6-1-3=) two independent degrees of freedom, corresponding to the two polarization states of the graviton

Equation of motion for (linear) GW

The resulting e.o.m. reads

$$v_{\mathbf{k}}^{(\lambda)\,\prime\prime} + \left(k^2 - \frac{a''}{a}\right) v_{\mathbf{k}}^{(\lambda)} = 0. \quad (1)$$

Let us now study the qualitative behavior of its solutions. We can identify two main regimes depending on the relative magnitude of the second and third term. First, consider the case in which $a''/a \ll k^2$. Ignoring the second term in parentheses, the equation for v becomes that of a free harmonic oscillator, so that tensor perturbations oscillate with a damping factor 1/a. This approximation corresponds to overlook the effect of the expansion of the Universe. To make explicit the physical condition corresponding to this regime, notice that, since $a''/a \sim (a'/a)^2$, $a''/a \ll k^2$ corresponds to $k \gg aH$, i.e. to the sub-horizon behavior (check for example the case of a de Sitter space-time where $a(\tau) \sim 1/\tau$). Keeping in this regime, the solution of the above eq. reads

$$v_k(\tau) = Ae^{ik\tau}$$

which means that the amplitude of the modes of the original field h_{ij} decrease in time with the inverse of the scale-factor as an effect of the Universe expansion. Consider now the regime in which the second term is negligible with respect to the third one: $k^2 \ll a''/a$. There are two possible solutions

$$v_k(\tau) \propto a \quad \text{and} \quad v_k(\tau) \propto 1/a^2,$$

which corresponds to $h \propto$ const. and a decreasing in time solution, respectively. This situation clearly corresponds to the super-horizon regime. In particular we will be interested in the solutions with constant amplitude.

Quantization of cosmological GW

Now we calculate more accurately the power spectrum of tensor perturbations, solving (1) . We perform the standard quantization of the field writing

$$v_{\mathbf{k}}^{(\lambda)} = v_k(\tau)\hat{a}_{\mathbf{k}}^{(\lambda)} + v_k^*(\tau)\hat{a}_{-\mathbf{k}}^{(\lambda)\dagger},$$

where the modes are normalized so that they satisfy $v_k^* v_k' - v_k v_k'^* = -i$, and this condition ensures that $\hat{a}_{\mathbf{k}}^{(\lambda)}$ and $\hat{a}_{-\mathbf{k}}^{(\lambda)\dagger}$ behave as the canonical creation and annihilation operators. Following the simplest and most natural hypothesis, as initial condition, we assume that the Universe was in the vacuum state defined as $\hat{a}_{\mathbf{k}}^{(\lambda)}|0\rangle = 0$ at past infinity, that is the "Bunch-Davies vacuum state"

Equation (1) is a Bessel equation, which, in case of de Sitter spacetime, has the following exact solution

$$v_{\mathbf{k}}(\tau) = \sqrt{-\tau}\left[C_1 H_\nu^{(1)}(-k\tau) + C_2 H_\nu^{(2)}(-k\tau)\right],$$

Quantization of GW in de Sitter space

where C_1, C_2 are integration constants, $H_\nu^{(1)}$, $H_\nu^{(2)}$ are Hankel functions of first and second order and $\nu \simeq 3/2 + \epsilon$. Remember we have negative sign to τ because, from its definition, it lies in $-\infty < \tau < 0$. To determining C_1 and C_2, we impose that in the UV regime, that is sub-horizon scales, the solution matches the plane-wave solution $e^{-ik\tau}/\sqrt{2k}$ found before. This hypothesis is a direct consequence of the Bunch-Davies vacuum condition. Using the asymptotic form of Hankel functions

$$H_\nu^{(1)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}} e^{i\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)}, \qquad H_\nu^{(2)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}} e^{-i\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)},$$

the second term in the solution has negative frequency, so that we have to fit $C_2 = 0$, while matching the asymptotic solution to a plane wave leads to

$$C_1 = \frac{\sqrt{\pi}}{2} e^{i(\nu + \frac{1}{2})\frac{\pi}{2}}.$$

Then the exact solution becomes

$$v_{\mathbf{k}} = \frac{\sqrt{\pi}}{2} e^{i(\nu + \frac{1}{2})\frac{\pi}{2}} \sqrt{-\tau} H_\nu^{(1)}(-k\tau).$$

Note: this approach can be trivially extended to any inflation model. See Abbott and Wise (1984).

Quantization of cosmological GW

In particular, for our purpose we are interested in the super-horizon wavelength behaviour, where the Hankel function reads

$$H_\nu^{(1)}(x \ll 1) \sim \sqrt{2/\pi} e^{-i\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \left[\Gamma(\nu)/\Gamma(3/2)\right] x^{-\nu},$$

so that the fluctuations on such scales become

$$v_{\mathbf{k}} = e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} 2^{(\nu-\frac{3}{2})} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2}-\nu},$$

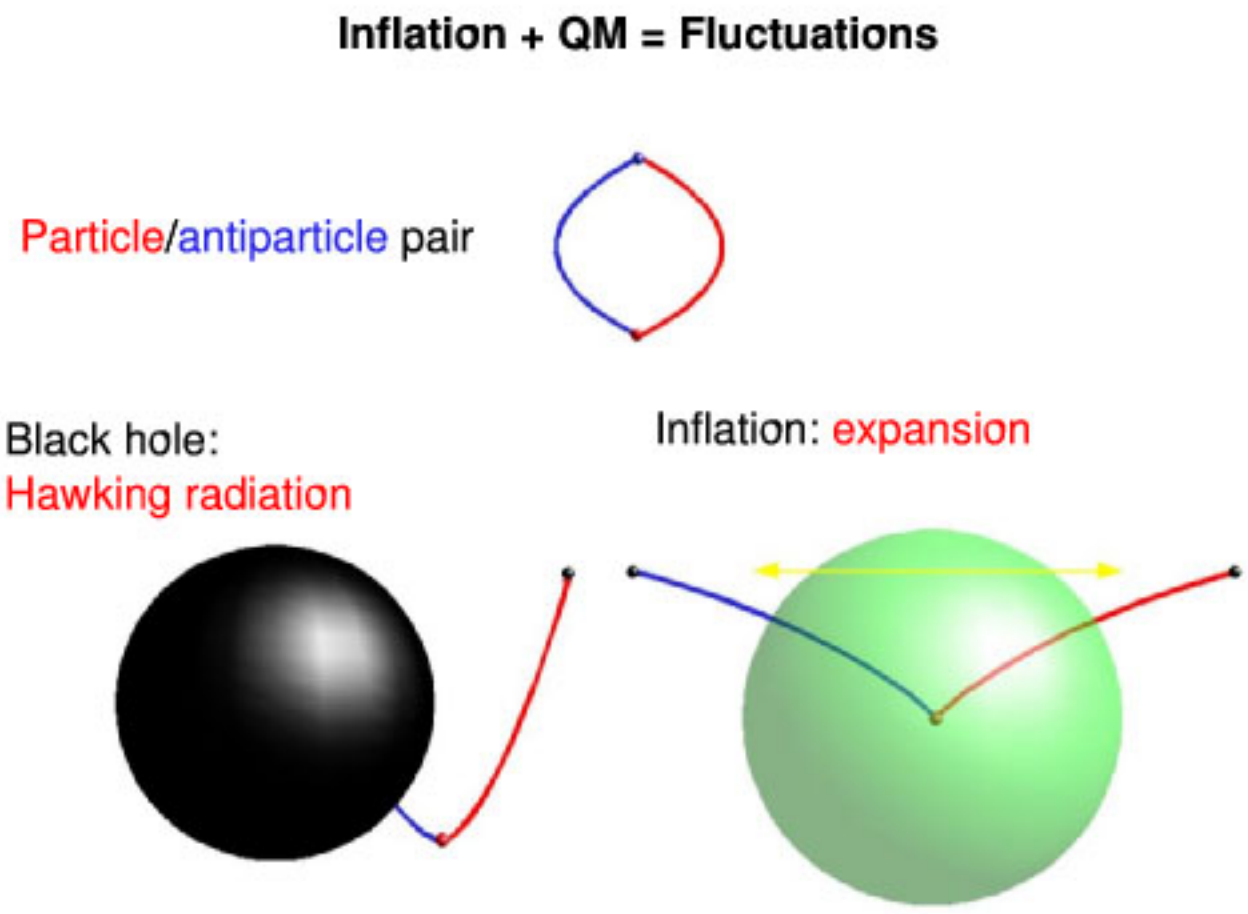
where Γ is the Euler function.

Generation of cosmological seeds

Particle creation in either strong (Hawking 1972) or rapidly varying (Parker 1969) gravitational fields

Schrödinger (1939): "*an alarming phenomenon*".

In QED the analogous effect in a strong electric field is known as "Klein paradox"



Kinney 2003

Power-spectrum of primordial GW

- We can now write the tensor-mode power-spectrum

$$P_{\rm T}(k) = \frac{k^3}{2\pi^2} \sum_\lambda \left| h_{\bf k}^{(\lambda)} \right|^2$$

so that on super-horizon scales the following power spectrum holds

$$P_{\rm T}(k) = \frac{8}{M_{\rm pl}^2} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon}$$ which we can rewrite as $$P_{\rm T}(k) = A_{\rm T} \left(\frac{k}{k_*}\right)^{n_{\rm T}}$$

- Notice that it is almost scale-invariant, which means that all the GW produced, nearly frozen on super-horizon scales, have all the same amplitude. Moreover, the tensor spectral index is negative (*"red spectrum"*) if $dH/dt < 0$ in agreement with the Null Energy Condition (NEC). For $n_T > 0$ (which requires violation of NEC!) it is indicated as "*blue spectrum"*. We refer to the case in which $n_T = 0$ as "scale-invariant" (historically "cosmological white-noise"). Here A_T is the tensor amplitude at some *pivot scale* k_* and n_T is the tensor spectral index.

Consistency relation

Consistency relation. In the considered inflationary scenario an interesting consistency relation holds between quantities which involve tensor perturbations. To get it, we introduce the *tensor-to-scalar ratio*

$$r(k_*) \equiv \frac{A_\mathrm{T}(k_*)}{A_\mathrm{S}(k_*)},$$

that yields the amplitude of the GW with respect to that of the scalar perturbations at some fixed pivot scale , this quantity depends on the time-evolution of the inflaton field, as

$$r = \frac{8}{M_\mathrm{pl}^2}\left(\frac{\dot{\varphi}}{H}\right)^2,$$

that is $r = 16\epsilon$. Furthermore, we have shown that a nearly scale-invariant spectrum of tensor modes is expected, being $n_\mathrm{T} = -2\epsilon$. Therefore at the lowest order in slow-roll parameters, one finds the following consistency relation

$$r = -8n_\mathrm{T}.$$

Energy density of GW

Let us now introduce some useful definitions, in particular to identify the GW energy-density. Consider the weak-field limit, where GW can be described as space-time ripples propagating on a fixed background. The vacuum field equations read $G_{\mu\nu} = 0$, which is equivalent to $R_{\mu\nu} = 0$. Making explicit the Ricci tensor as a sum of a background term and perturbative terms up to second order,

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu}(h) + R^{(2)}_{\mu\nu}(h) + \mathcal{O}(h^3)$$

one can deduce from the vacuum equations, how the presence of the GW affects the background (where, for example, $R^{(2)}_{\mu\nu}$ (h) indicates the contribution to the Ricci tensor which contains terms as $\sim h \cdot h$). The terms that play this role then can be interpreted as a stress-energy tensor $t_{\mu\nu}$ due to the presence of GW. In this direction it is useful to note that $R_{\mu\nu}$ can be written as a sum of two kinds of terms, those representing a smooth contribution and others which encode the fluctuating part. Each of the two contributions vanishes on its own. The background term varies only on large scales with respect to some coarse-graining scale, therefore we are interested in the equation for the smooth contributions. The only linear term $\bar{R}^{(1)}_{\mu\nu}(h)$ solves by itself $R^{(1)}_{\mu\nu}(h)=0$. Then, the remaining equation for the smooth part of the vacuum equation reads:

$$\bar{R}_{\mu\nu} + \langle R^{(2)}_{\mu\nu} \rangle = 0$$

where $\langle \ldots \rangle$ indicates the average over several wavelengths which extracts the smooth contribution with respect to the coarse-graining scale.

Energy density of GW

An analogous reasoning can be enlarged to the Einstein tensor, so that one gets the following Einstein equations, in vacuum:

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} = \langle R^{(2)}_{\mu\nu} \rangle - \frac{1}{2}\bar{g}_{\mu\nu}\langle R^{(2)} \rangle$$

The terms on the RHS tell how the presence of GW affects the background metric, then they can be interpreted as the GW stress-energy tensor $t_{\mu\nu}$, apart from a factor $8\pi G$. In terms of the tensor perturbations of the metric it reads:

$$t_{\mu\nu} = \frac{1}{32\pi G}\langle \partial_\mu h_{ij} \partial_\nu h^{ij} \rangle$$

Energy density if GW

From the previous equation, the GW energy-density, on a FRW background, reads

$$\rho_{\mathrm{gw}} = \frac{1}{32\pi G a^2} \langle h'_{ij}(\mathbf{x},\tau)\, h'^{ij}(\mathbf{x},\tau)\rangle .$$

Where primes denote differentiation w.r.t. conformal time τ and $(1/a)d/d\tau = d/dt$. However, more often one makes use of the GW energy-density per logarithmic frequency interval, normalized to the critical density

$$\Omega_{\mathrm{GW}}(k,\tau) \equiv \frac{1}{\rho_{\mathrm{c}}} \frac{\mathrm{d}\rho_{\mathrm{gw}}}{\mathrm{d}\ln k} . \qquad \rho_{\mathrm{c}} \equiv 3H^2/8\pi G$$

Gravity-wave background from inflation

- As originally noticed by Starobinski (1979) an early period of quasi-de Sitter evolution leaves its imprint in terms of a low-amplitude *stochastic background of gravitational waves* (see also Grishchuck 1975, Rubakov et al. 1982, Fabbri & Pollock 1982, Abbott & Wise 1984) which originated from quantum vacuum fluctuations of (linearized) spin-2 gravitational perturbations ("gravitons"), left the horizon during inflation (hence remaining frozen and unobservable) and re-entered the horizon recently, hence becoming potentially observable as classical tensor perturbations of space-time.

- The detection of these primordial gravitational waves represents the "smoking gun" proof of the validity of the inflationary theory, otherwise very hard to "falsify"; other crucial specific imprints being: the existence of perturbations with a super-horizon seed (***detected!***), specific non-Gaussian signatures of primordial perturbations (***strongly constrained by Planck, which supports the simplest inflation models***).

Slow-roll parameters and cosmological observables

$$\epsilon = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 \qquad \xi^2 = M_P^2 V' V''' / V^2 \qquad M_P \equiv (8\pi G_N)^{-1/2}$$

$$\eta = M_P^2 \left(\frac{V''}{V}\right)$$

Scalar (comoving curvature) perturbation power-spectrum

$$\begin{cases} n_R - 1 = -6\varepsilon + 2\eta \\ dn_R / d\ln k = -2\xi + 16\varepsilon\eta - 24\varepsilon^2 \end{cases}$$

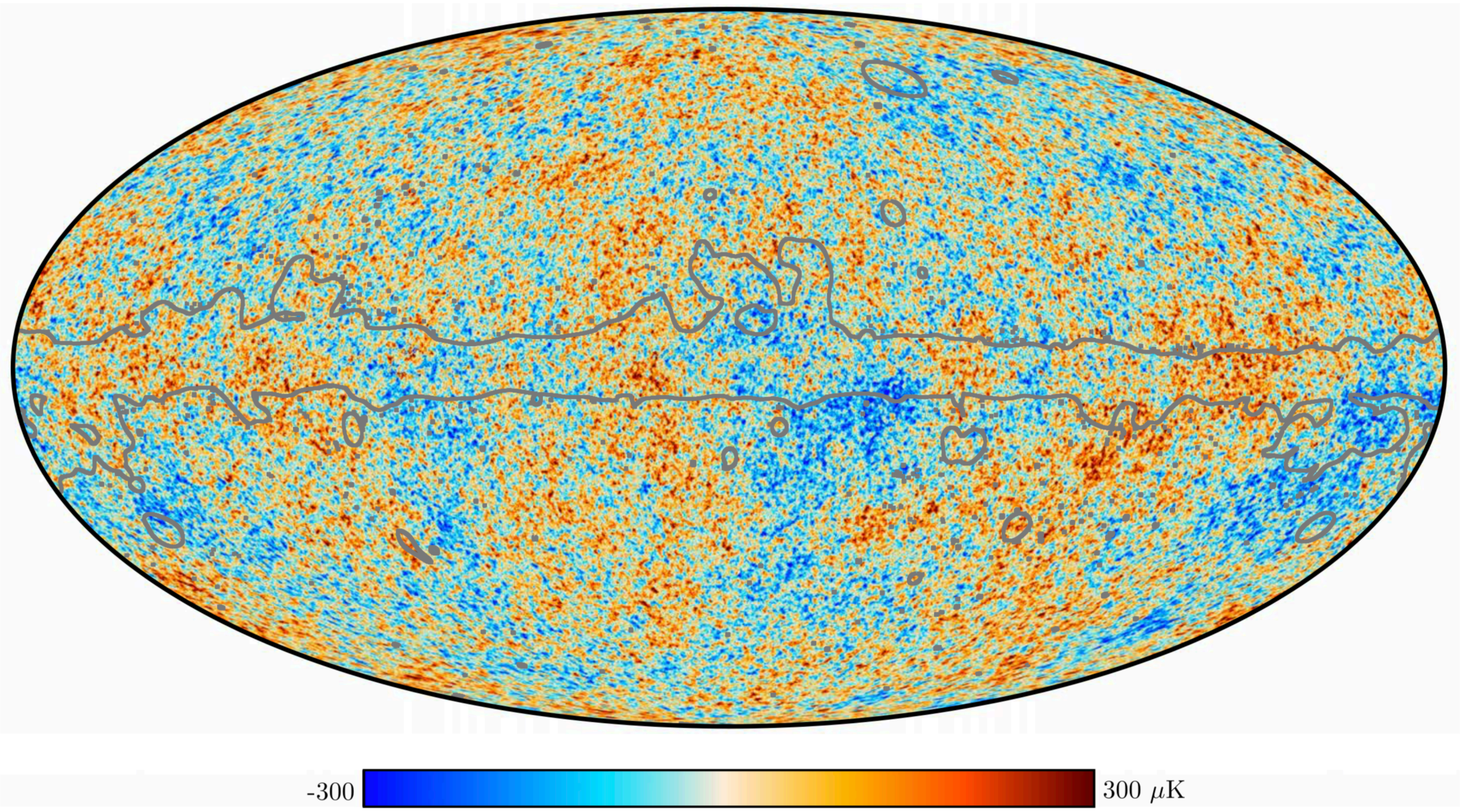
$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{2M_P^2\epsilon}\left(\frac{H_*}{2\pi}\right)^2 \left(\frac{k}{aH_*}\right)^{n_{\mathcal{R}}-1}$$

Tensor (gravity-wave) perturbation power-spectrum

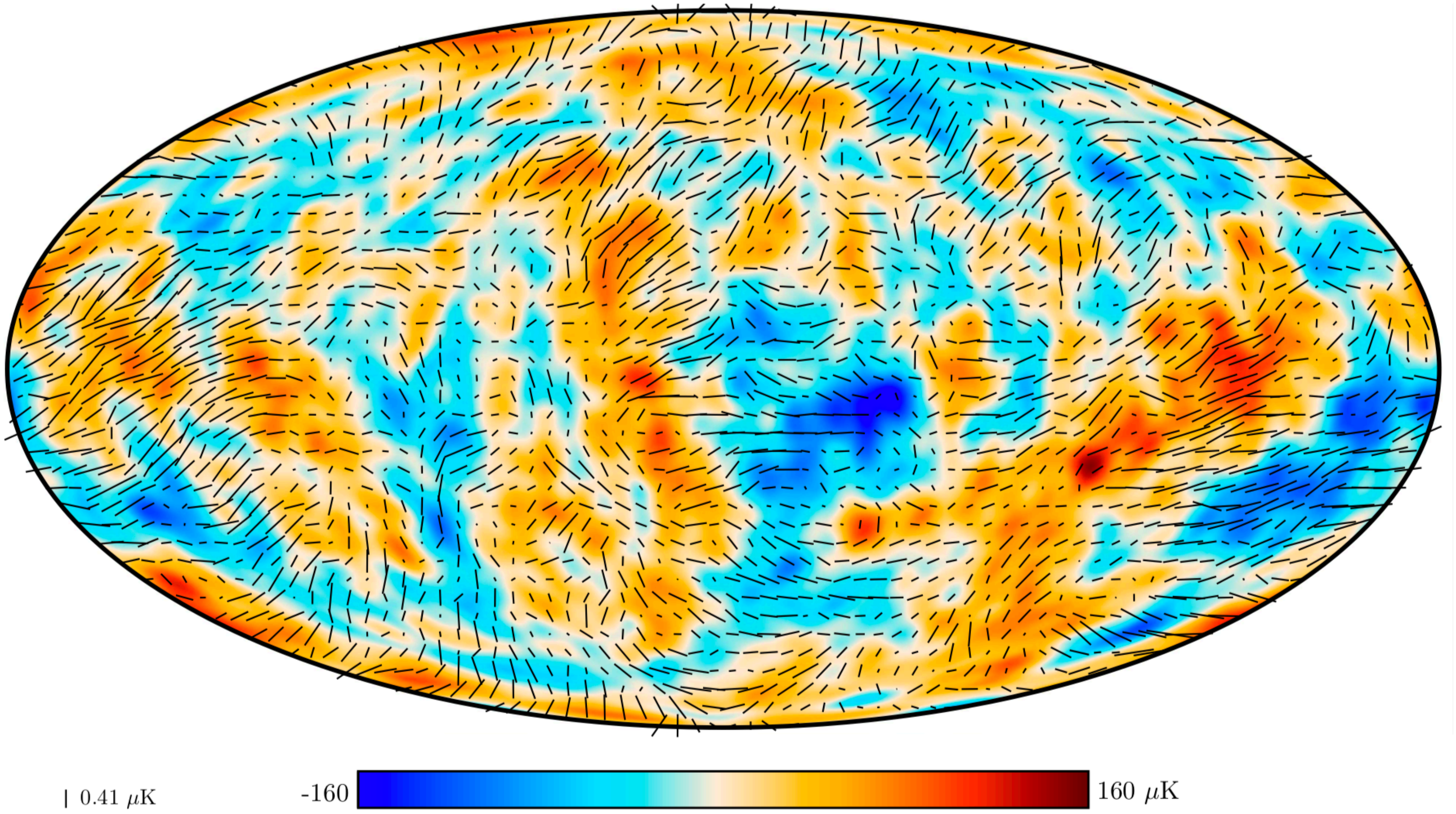
$$\mathcal{P}_T(k) = \frac{k^3}{2\pi^2}\langle h_{ij}^* h^{ij}\rangle = \frac{8}{M_P^2}\left(\frac{H_*}{2\pi}\right)^2 \left(\frac{k}{aH_*}\right)^{n_T}$$

$$\begin{cases} r = \dfrac{\mathcal{P}_T}{\mathcal{P}_{\mathcal{R}}} = 16\epsilon \\ n_T = -2\epsilon \end{cases}$$

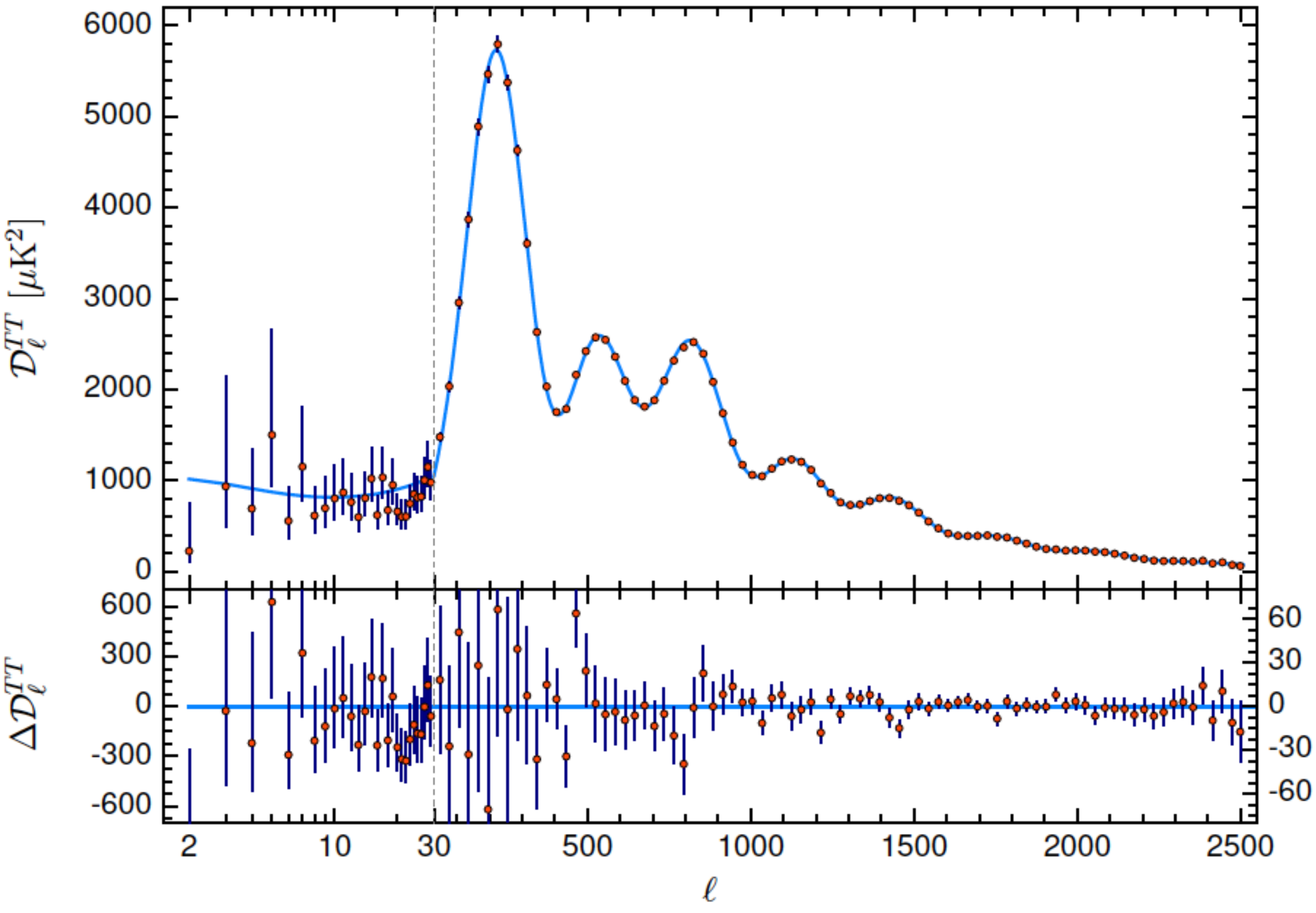
PLANCK 2018: TEMPERATURE ANISOTROPIES



Planck 2018: Polarization Anisotropies

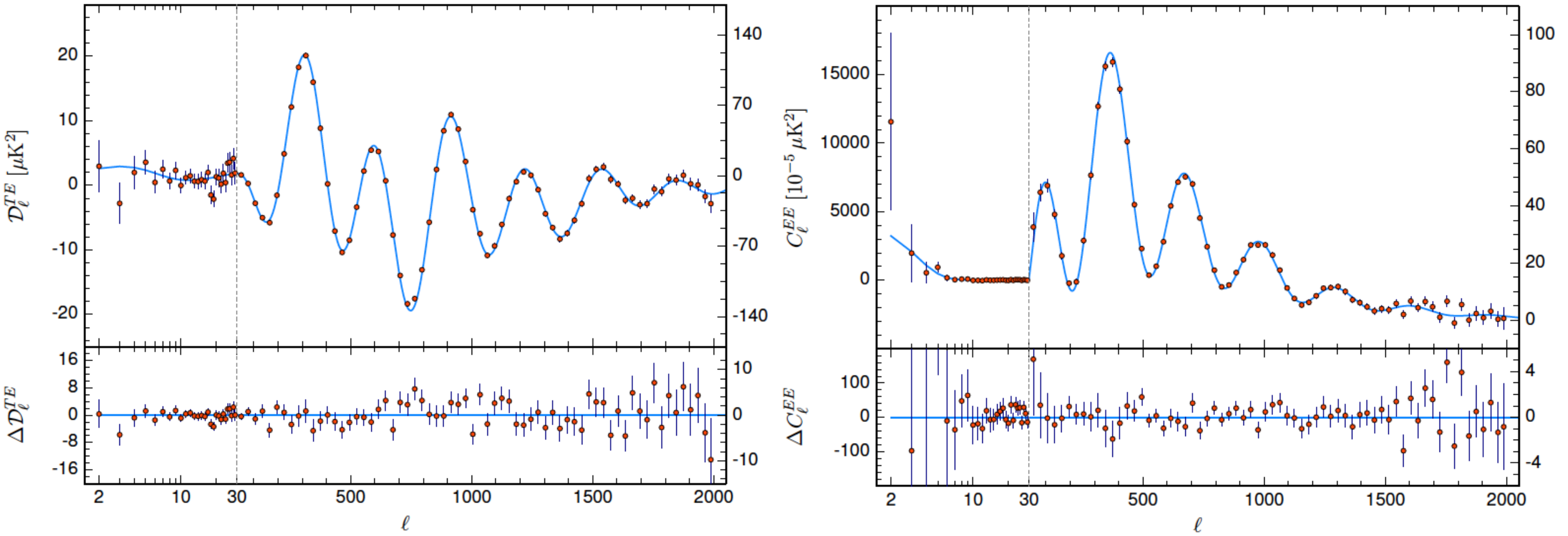


Planck 2018 TT-spectrum



and ... including polarization

Planck 2018



Baseline LCDM results 2018

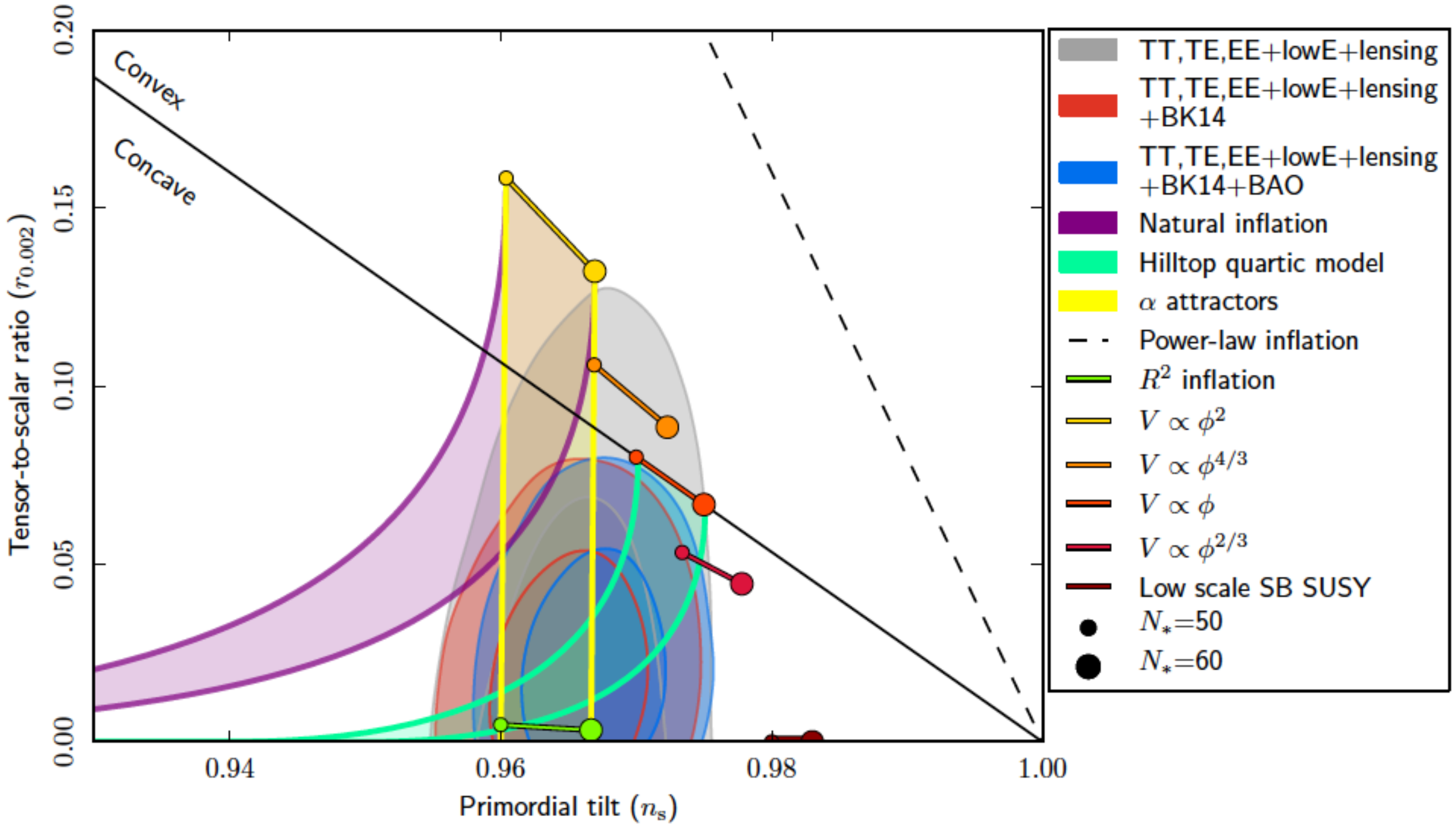
(Planck legacy: Temperature+polarization+CMB lensing)

	Mean	σ	[%]
$\Omega_b h^2$ Baryon density	0.02237	0.00015	0.7
$\Omega_c h^2$ DM density	0.1200	0.0012	1
100θ Acoustic scale	1.04092	0.00031	0.03
τ Reion. Optical depth	0.0544	0.0073	13
$\ln(A_s 10^{10})$ Power Spectrum amplitude	3.044	0.014	0.7
n_s Scalar spectral index	0.9649	0.0042	0.4
H_0 Hubble	67.36	0.54	0.8
Ω_m Matter density	0.3153	0.0073	2.3
σ_8 Matter perturbation amplitude	0.8111	0.0060	0.7

credits: S. Galli

- Most parameters determined at (sub-) percent level!
- Best determined parameter is the angular scale of sound horizon θ to 0.03%.
- τ lower and tighter due to HFI data at large scales.
- n_s 8σ away from scale invariance (even in extended models, always >3σ)
- Best (indirect) **0.8%** determination of the Hubble constant to date.

Planck 2018 constraints on inflation models



Marginalized joint 68% and 95% CL regions for n_s and $r_{0.002}$ from Planck in combination with other datasets, vs. theoretical prediction of selected inflation models.

Best fit: Starobinsky model

- A. Starobinski in 1980 proposed a model for the Early Universe originally motivated by conformal (trace) anomaly. This corresponds to the Lagrangian (Jordan frame)

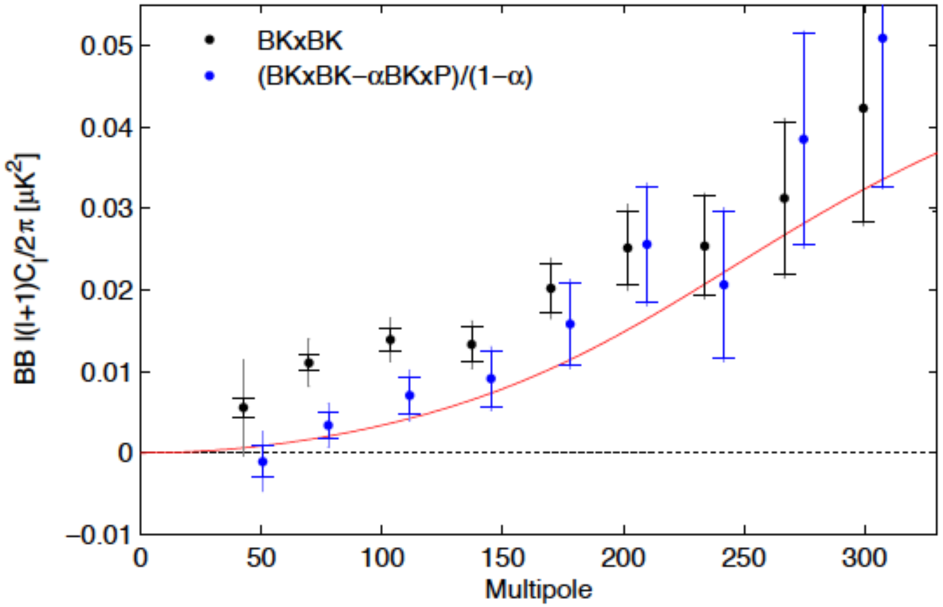
$$L = R + R^2/6M^2$$

- The corresponding action in the Einstein frame leads to a plateau + an exponential branch
- Question: is there a way to distinguish Starobinski's model from Higgs inflation?
- Hint: look at disformal ("Lifshitz scaling") anomalies (Celoria & Matarrese, in preparation; see also Celoria, Matarrese & Pilo 2018: symmetry of continuous media with constant equation of state).

“B2KP” constraints

- Joint Analysis of BICEP2/Keck Array and Planck Data (2015):
  <<... The final result is expressed as a likelihood curve for r, and yields an upper limit $r_{0.05} < 0.12$ at 95% confidence. Marginalizing over dust and r, lensing B-modes are detected at 7.0 σ significance.>>

BB spectrum of the BICEP2/Keck maps before and after subtraction of the dust contribution, estimated from the cross-spectrum with *Planck* 353 GHz. The error bars are standard deviations of simulations, which, in the latter case, have been scaled and combined in the same way.
The inner error bars are from lensed-CDM+noise sims, while the outer error bars are from lensed-CDM+noise+dust sims.



Consequences for high energy physics

For values of $r \approx 10^{-1}$, inflation probes the GUT scale, i.e. high-energy scales never achievable in laboratories

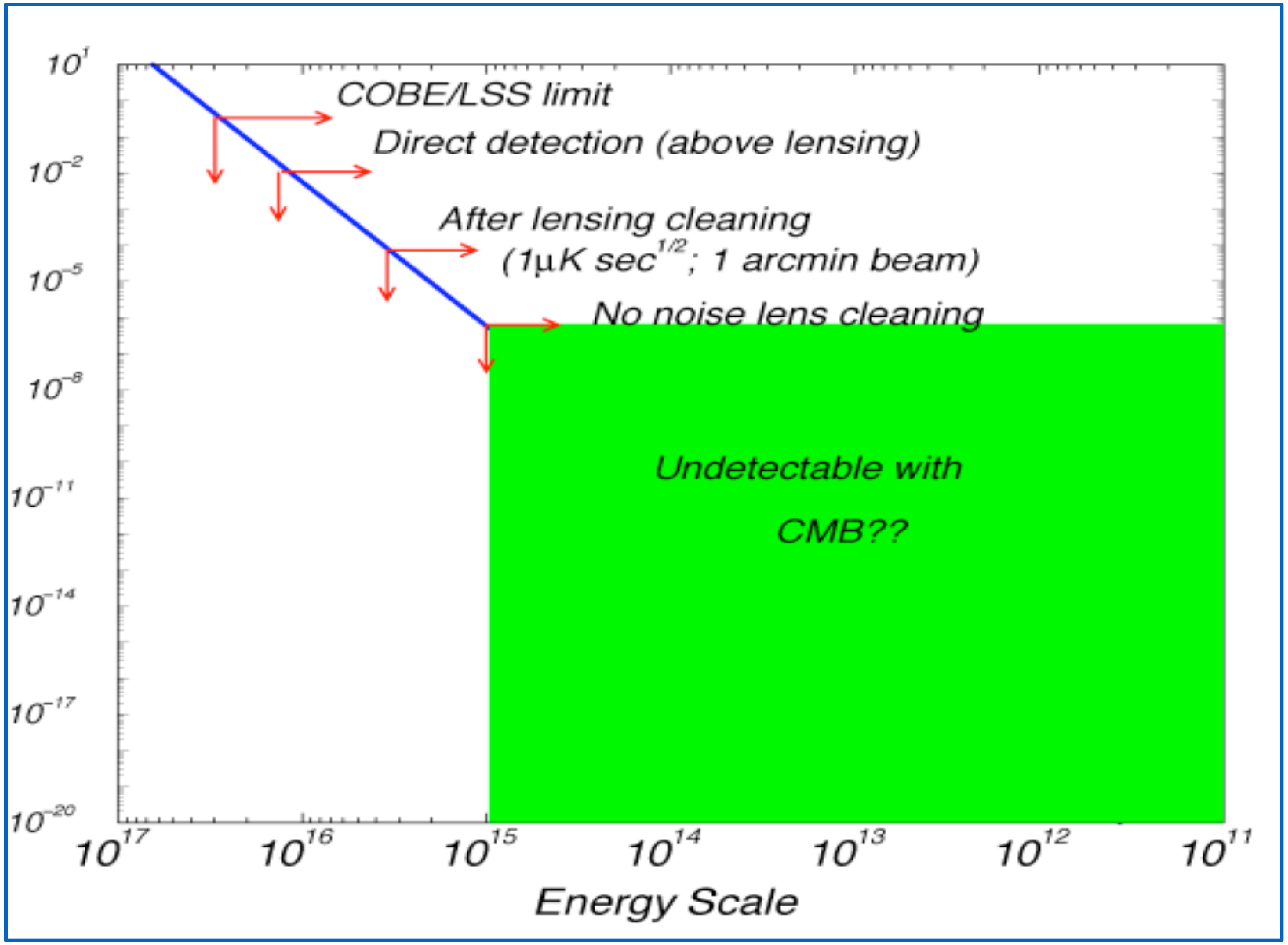
$$V^{1/4} = 1.94 \times 10^{16} \left(\frac{r}{0.12}\right)^{1/4} \text{GeV}$$

The many observational confirmations of inflation predictions (may) provide evidence of physics beyond the Standard Model of particle physics

Who is the inflaton??

This question has become more and more pressing.

Tensor-to-scalar ratio



Cooray 2004

Standard inflation is still alive ... and in very good shape!

Standard inflation i.e.

- single scalar field
- canonical kinetic term
- slow-roll dynamics
- Bunch-Davies initiual vacuum state
- standard Einstein gravity

which predicts $O(10^{-2})$ primordial NG signal, still consistent with data.

Testable predictions of inflation

Cosmological aspects

- Critical density Universe
- Almost scale-invariant and nearly Gaussian, adiabatic density fluctuations
- Almost scale-invariant stochastic background of relic gravitational waves

Particle physics aspects

- Nature of the inflaton
- Inflation energy scale

Final remarks on Part II

- Inflation is a very successful theory (a “paradigm”?), which solves the main internal contradiction of the standard “Hot Big Bang” theory and provides a physical mechanism for the generation of the seeds out of which CMB anisotropies and polarization have originated.

- The prediction of a stochastic gravitational-wave background appears to be ubiquitous: it is a consequence of the lack of Weyl invariance of the gravitational-wave action.

- Detecting cosmological GWs is a very hard task. We have to search for indirect consequences, such as effects on CMB anisotropies and polarization.

- The quantum-to-classical transition of cosmological GW has to be properly understood.