# Cosmological Perturbation Theory and Primordial Gravitational Waves

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- Cosmological Perturbation Theory. Gauge transformations: scalar, vector and tensor modes.
- Generation of gravitational waves during inflation. Second-order tensor modes.
- Linear evolution of gravitational waves. Upper bounds on the gravitational-wave background. Anisotropy and non-Gaussianity of the gravitational-wave background.

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# Cosmological Perturbation Theory

- Cosmological perturbation theory was developed, mostly in the sixties, although there even earlier studies:

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  - R. K. Sachs and A. M. Wolfe, Astrophys. J. 147, 73 (1967)
  - G. B. Field, in Galaxies and the Universe, edited by A. Sandage, M. Sandage, and J. Kristian (University of Chicago Press, Chicago, 1975)
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- In all these cases the definition of perturbation entails a comparison between the physical (perturbed) space-time and an idealized background which is usually taken as the homogeneous and isotropic FLRW model.

- The subsequent classification of modes (scalar, vector and tensor) unavoidably depends on the choice of background.

# The development of cosmological perturbation theory and the gauge problem

- The perturbative approach is a fundamental tool in General Relativity (GR), where exact solutions of Einstein's Equations (EE) are most often too idealized to properly represent the realm of natural phenomena. In other words, exact solutions of EE describe only particular manifolds endowed with symmetries. We can extend our knowledge of the physical universe by considering small deviations from these symmetries: this is the so called "perturbation theory".
- Unfortunately, the invariance of GR under diffeomorphisms (two solutions of EE are physically equivalent if they are diffeomorphic to each other) makes the very definition of perturbations gauge-dependent. Consider diffeomorphisms, $\delta g_{\mu\nu} = \mathcal{L}\, g_{\mu\nu}$, where $\mathcal{L}\, \mathbf{g}_{\mu\nu}$ is the Lie derivative of the metric tensor. The invariance of EE generates redundant degrees of freedom ("*pure gauge modes*"), that must be suppressed because they have no physical meaning. The traditional way to do it is through a *gauge fixing*. Alternatively, one can look for "*gauge-invariant*" perturbations, which are not affected by such transformations (Bardeen 1980). While this approach is easy at first order it becomes very complex at higher-order and some "tricks" have to be adopted.
- A *gauge choice* or *gauge fixing* is an identification of a map between the perturbed (i.e. physical) and the background (idealized) space-times. Generic perturbations are not invariant under a gauge transformation ➔ gauge problem.

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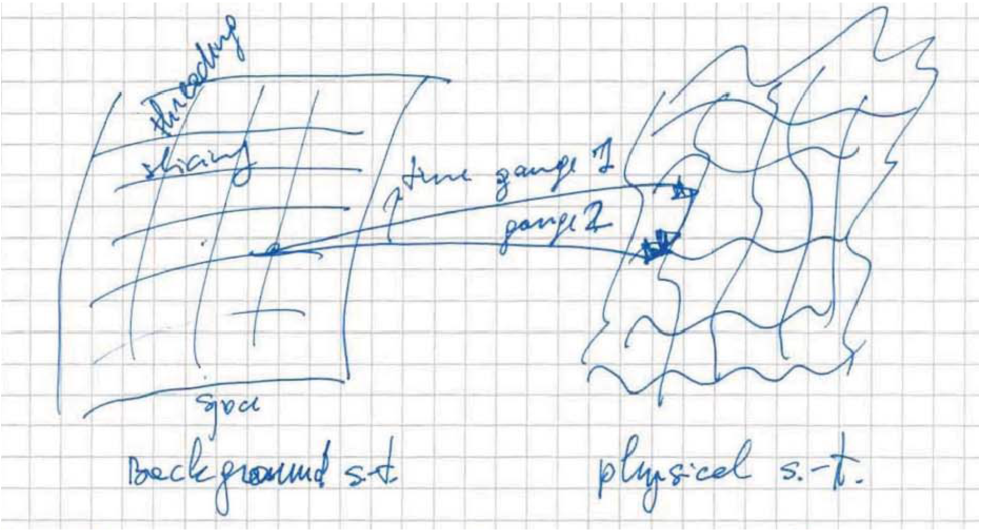
# Physical vs. Backround (FLRW) manifolds

**Bardeen 1980**: "A one-to-one correspondence between points in the background and points in the physical space-time carries these coordinates over into the physical spacetime and defines a choice of gauge. A change in the correspondence, keeping the background coordinates fixed, is called a *gauge transformation*, to be distinguished from a *coordinate transformation* which changes the labelling of points in the background and physical spacetime together.
The perturbation in some quantity is the difference between the value it has at a point in the physical spacetime and the value at the corresponding point in the background spacetime."




A choice of coordinates defines a "*threading*" of space-time into lines (corresponding to fixed spatial coordinates) and "*slicing*" into hypersurfaces (corresponding to fixed times).

"A gauge transformation induces a coordinate transformation in the physical spacetime, but it also changes the point in the background spacetime corresponding to a given point in the physical spacetime. Thus, even if a quantity is a scalar under coordinate transformations, the value of the perturbation in the quantity will not be invariant under gauge transformations if the quantity is non-zero and position dependent in the background."

# Cosmological perturbations. I

- A gravitational perturbation can be written as a small variation of the metric

$$g_{ij} \rightarrow g_{ij} + \delta g_{ij}$$

- where the unperturbed metric represents the background universe, which we will consider to be the FLRW metric for homogeneous and isotropic universes

$$ds^2 = a^2(\tau)[-d\tau^2 + {}^3g_{ij}dx^i dx^j]$$

- where a(τ) is the scale factor, τ is the conformal time and $^3gij$ is the metric tensor for a 3-dimensional space of uniform curvature K, and the choice of the space coordinates is left arbitrary.

# Cosmological perturbations: scalars

The homogeneity and isotropy of the background allow a separation of the time dependence and the spatial one, so without losing any generality we can expand an arbitrary perturbation over spatial spherical harmonics $Q^{(n)}$. Through these functions perturbations can be classified in scalar, vector and tensor quantities, according to how they transform under spatial coordinate transformations in the background spacetime.

A **scalar perturbation** has a spatial dependence derived from scalar harmonics, which are the solutions of the *scalar Helmholtz's equation*:

$$\Delta Q^{(0)} + k^2 Q^{(0)} = 0$$

where $-k^2$ is the eigenvalue of the Laplace-Beltrami operator $\Delta$. Vectors and tensor quantities associated with scalar perturbations can be built from covariant derivatives of $Q^{(0)}$ and the spatial metric tensor; let us define the vector

$$Q_i^{(0)} = -\frac{1}{k} Q_{|i}^{(0)}$$

and the traceless symmetric tensor

$$Q_{ij}^{(0)} = \frac{1}{k^2} Q_{|ij}^{(0)} + \frac{1}{3} {}^{3}g_{ij} Q^{(0)}$$

# Cosmological perturbations: vectors and tensors

A **vector perturbation** is proportional to $Q_i^{(0)}$ , but it describes a divergenceless component that cannot be constructed from scalar harmonics; instead it must be proportional to vector harmonic functions, which are solutions of the *vector Helmholtz's equation*

$$\Delta Q^{(1)i} + k^2 Q^{(1)i} = 0$$

The second rank traceless symmetric tensor associated with the vector harmonics is

$$Q^{(1)ij} = -\frac{1}{2k}(Q^{(1)i|j} + Q^{(1)j|i})$$

In the same way a **tensor perturbation** will be proportional to the solutions of the *tensor Helmholtz's equation*

$$\Delta Q^{(2)ij} + k^2 Q^{(2)ij} = 0$$

# More on tensor perturbations

- Tensor perturbations affect only the traceless part of the metric tensor and the stress-energy tensor:

$$g_{ij} = a^2(\tau)[{}^3g_{ij}(\vec{x}) + 2H_T^{(2)}(\tau)Q_{ij}^{(2)}(\vec{x})]$$

$$T_j^i = P_0[\delta_j^i + \pi_T^{(2)}(\tau)Q_j^{(2)i}] \ .$$

- Note that no density or isotropic pressure perturbation is associated with vector or tensor perturbations.

# Mode independence and mode-mixing

- At the linear level modes are mutually independent (in every possible sense):
  - Different Fourier (or more general eigenmodes of your basis) modes evolve independently.
  - Scalar, vector and tensor modes are mutually independent.

- If you go to order *n* in perturbation theory:
  - Different Fourier modes are coupled (non-linearity → non-Gaussianity)
  - Only scalars, vectors and tensor of the same order *n* remain mutually independent.

# Active vs. Passive view

- There are two approaches to calculate how perturbations change under a gauge transformation. For the ***active*** view we study how perturbations change under a mapping, where the map directly induces the transformation of the perturbed quantities. In the ***passive*** view instead the relation between the two coordinate systems is specified, and we calculate how the perturbations are changed under coordinate transformations.
- In the *active* approach the transformation of the perturbed quantities is evaluated at the *same coordinate point,* whereas in the *passive* approach the transformation is taken at the *same physical point.*

# Active approach

The starting point in the active approach is the exponential map, telling us how a tensor **T** transforms, once the generator of the gauge transformation, $\xi_\mu$, has been specified. The exponential map is **T** → $\widetilde{\mathbf{T}} = e^{\pounds_\xi}\mathbf{T}$.

Expanding the vector $\xi_\mu$ and the exponential map up to 2-nd order we get

$$\xi^\mu \equiv \epsilon\xi_1^\mu + \frac{1}{2}\epsilon^2\xi_2^\mu + O(\epsilon^3). \qquad \exp(\pounds_\xi) = 1 + \epsilon\pounds_{\xi_1} + \frac{1}{2}\epsilon^2\pounds_{\xi_1}^2 + \frac{1}{2}\epsilon^2\pounds_{\xi_2} + \cdots$$

$$\widetilde{\mathbf{T}_0} = \mathbf{T}_0,$$
$$\epsilon\widetilde{\delta\mathbf{T}_1} = \epsilon\delta\mathbf{T}_1 + \epsilon\pounds_{\xi_1}\mathbf{T}_0,$$
$$\epsilon^2\widetilde{\delta\mathbf{T}_2} = \epsilon^2\left(\delta\mathbf{T}_2 + \pounds_{\xi_2}\mathbf{T}_0 + \pounds_{\xi_1}^2\mathbf{T}_0 + 2\pounds_{\xi_1}\delta\mathbf{T}_1\right)$$

Note that the background quantities are not affected by the mapping.

Applying the map to the coordinate functions $x^\mu$ we get a relation for the coordinates of a point q and a point p as

$$x^\mu(q) = x^\mu(p) + \epsilon\xi_1^\mu(p) + \frac{1}{2}\epsilon^2\left(\xi^\mu_{1,\nu}(p)\xi_1{}^\nu(p) + \xi_2^\mu(p)\right)$$

However, in this approach we do not need this eq. to calculate how perturbations change under a gauge transformation, it simply tells us how the coordinates of the points p and q are related.

Credits: Malik & Wands 2009

# Passive approach

In the passive approach we specify the relation between two coordinate systems directly, and then calculate the change in the metric and matter variables when changing from one system to the other. In order to make contact with the active approach, discussed above, we take the last eq. as our starting point. Note, that all quantities in the passive approach are evaluated at the same physical point. This can be rewritten to give a relation between the ‘‘old’’ (untilted) and the ‘‘new’’ (tilded) coordinate systems, evaluated at the same physical point q.

$$\widetilde{x^{\mu}}(q) = x^{\mu}(q) - \epsilon \xi_1^{\mu}(q) + \epsilon^2 \frac{1}{2} \left( \xi_1^{\mu}(q)_{,\nu} \xi_1{}^{\nu}(q) - \xi_2^{\mu}(q) \right)$$

The starting point in the passive approach is to identify an invariant quantity, that allows us to relate quantities to be evaluated in the two coordinate systems. This could be e.g. the energy density, which is a 4-scalar, or the line element, etc...

Credits: Malik & Wands 2009

# Perturbations of the metric tensor – I.

- The components of a perturbed FLRW metric can be written as:

$$g_{00} = -a^2(\tau)\left(1 + 2\sum_{r=1}^{+\infty} \frac{1}{r!}\psi^{(r)}\right),$$

$$g_{0i} = a^2(\tau)\sum_{r=1}^{+\infty} \frac{1}{r!}\omega_i^{(r)},$$

$$g_{ij} = a^2(\tau)\left\{\left[1 - 2\left(\sum_{r=1}^{+\infty} \frac{1}{r!}\phi^{(r)}\right)\right]\delta_{ij} + \sum_{r=1}^{+\infty} \frac{1}{r!}\chi_{ij}^{(r)}\right\}$$

# Perturbations of the metric tensor – II.

The functions $\phi^{(r)}, \hat{\omega}_i^{(r)}, \psi^{(r)}$ and $\hat{\chi}_{ij}^{(r)}$, where $(r) = (1), (2)$, stand for the $r$th–order perturbations of the metric. Notice that such an expansion could a priori include terms of arbitrary order, but for our purposes the first and second–order terms are sufficient. It is standard use to split the perturbations into the so–called scalar, vector and tensor parts according to their transformation properties with respect to the 3-dimensional space with metric $\delta_{ij}$, where scalar parts are related to a scalar potential, vector parts to transverse (divergence–free) vectors and tensor parts to transverse trace–free tensors. Thus in our case

$$\hat{\omega}_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)} ,$$

$$\hat{\chi}_{ij}^{(r)} = D_{ij}\chi^{(r)} + \partial_i \chi_j^{(r)} + \partial_j \chi_i^{(r)} + \chi_{ij}^{(r)} ,$$

where $\omega_i$ and $\chi_i$ are transverse vectors, *i.e.* $\partial^i \omega_i^{(r)} = \partial^i \chi_i^{(r)} = 0$, $\chi_{ij}^{(r)}$ is a symmetric transverse and trace–free tensor, *i.e.* $\partial^i \chi_{ij}^{(r)} = 0$, $\chi^{i(r)}_{\ i} = 0$) and $D_{ij} = \partial_i \partial_j - (1/3)\, \delta_{ij}\, \nabla^2$ is a trace–free operator.

Here and in the following latin indices are raised and lowered using $\delta^{ij}$ and $\delta_{ij}$, respectively.

# Perturbations of the metric tensor (up to second order)

$$
\begin{aligned}
g_{00} &= -a^2(\tau)\left(1 + 2\phi^{(1)} + \phi^{(2)}\right), \\
g_{0i} &= a^2(\tau)\left(\partial_i\omega^{(1)} + \frac{1}{2}\partial_i\omega^{(2)} + \frac{1}{2}\omega_i^{(2)}\right), \\
g_{ij} &= a^2(\tau)\left[\left(1 - 2\psi^{(1)} - \psi^{(2)}\right)\delta_{ij} + D_{ij}\left(\chi^{(1)} + \frac{1}{2}\chi^{(2)}\right) + \chi_{ij}^{(1)}\right. \\
&+ \left.\frac{1}{2}\left(\partial_i\chi_j^{(2)} + \partial_j\chi_i^{(2)} + \chi_{ij}^{(2)}\right)\right].
\end{aligned}
$$

# Einstein tensor – FLRW background

The Einstein tensor in a spatially flat FRW background is given by

$$G^0{}_0 = -\frac{3}{a^2}\left(\frac{a'}{a}\right)^2 ,$$

$$G^i{}_j = -\frac{1}{a^2}\left[2\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right] \delta^i{}_j ,$$

$$G^0{}_i = G^i{}_0 = 0 .$$

Primes denote differentiation w.r.t. conformal time $\tau$.

# Einstein tensor – linear order (up to vector modes)

The first–order perturbations of the Einstein tensor components are

$$\delta^{(1)}G^0{}_0 = \frac{1}{a^2}\left[6\left(\frac{a'}{a}\right)^2\phi^{(1)} + 6\,\frac{a'}{a}\psi^{(1)\prime} + 2\,\frac{a'}{a}\nabla^2\omega^{(1)} - 2\nabla^2\psi^{(1)} - \frac{1}{2}\,\partial_k\partial^i\, D^k_i\chi^{(1)}\right]$$

$$\delta^{(1)}G^i{}_0 = \frac{1}{a^2}\left[4\left(\frac{a'}{a}\right)^2\partial^i\omega^{(1)} - 2\,\frac{a''}{a}\,\partial^i\omega^{(1)} + 2\,\partial^i\,\psi^{(1)\prime} + 2\,\frac{a'}{a}\,\partial^i\phi^{(1)} + \frac{1}{2}\partial_k\, D^{ki}\chi^{(1)\prime}\right],$$

$$\delta^{(1)}G^0{}_i = \frac{1}{a^2}\left(-2\,\frac{a'}{a}\partial_i\phi^{(1)} - 2\,\partial_i\psi^{(1)\prime} - \frac{1}{2}\,\partial_k D^k{}_i\chi^{(1)\prime}\right),$$

$$\begin{aligned}
\delta^{(1)}G^i{}_j = \frac{1}{a^2}\Bigg[&\left(2\,\frac{a'}{a}\phi^{(1)\prime} + 4\,\frac{a''}{a}\phi^{(1)} - 2\left(\frac{a'}{a}\right)^2\phi^{(1)} + \nabla^2\phi^{(1)} + 4\,\frac{a'}{a}\psi^{(1)\prime}\right.\\
&+ \left. 2\,\psi^{(1)\prime\prime} - \nabla^2\psi^{(1)} + 2\,\frac{a'}{a}\nabla^2\omega^{(1)} + \nabla^2\omega^{(1)\prime} - \frac{1}{2}\partial_k\partial^m D^k{}_m\chi^{(1)}\right)\delta^i{}_j\\
&- \partial^i\partial_j\phi^{(1)} + \partial^i\partial_j\psi^{(1)} - 2\,\frac{a'}{a}\partial^i\partial_j\omega^{(1)} - \partial^i\partial_j\omega^{(1)\prime}\\
&+ \frac{a'}{a}D^i{}_j\chi^{(1)\prime} + \frac{1}{2}\,D^i{}_j\chi^{(1)\prime\prime} + \frac{1}{2}\,\partial_k\partial^i\,D^k{}_j\chi^{(1)} + \frac{1}{2}\,\partial_k\partial_j\,D^{ik}\chi^{(1)} - \frac{1}{2}\,\partial_k\partial^k\,D^i{}_j\chi^{(1)}\Bigg] + \frac{a'}{a}\,\chi^i{}_j{}^{(1)\prime}\\
&- \frac{1}{2}\,\nabla^2\chi^i{}_j{}^{(1)} + \frac{1}{2}\,\chi^i{}_j{}^{(1)\prime\prime}
\end{aligned}$$

# Einstein tensor – Second order (neglecting linear vector and tensor modes)

The second–order perturbed Einstein tensor components are given by

$$
\begin{aligned}
\delta^{(2)}G^0{}_0 \;=\; & \frac{1}{a^2}\Bigg[3\left(\frac{a'}{a}\right)^2\phi^{(2)} + 3\,\frac{a'}{a}\psi^{(2)\prime} - \nabla^2\psi^{(2)} + \frac{a'}{a}\nabla^2\omega^{(2)} - \frac{1}{4}\partial_k\partial_i\, D^{ki}\chi^{(2)} \\
& - \; 12\left(\frac{a'}{a}\right)^2\left(\phi^{(1)}\right)^2 - 12\,\frac{a'}{a}\,\phi^{(1)}\,\psi^{(1)\prime} - 3\,\partial_i\psi^{(1)}\,\partial^i\psi^{(1)} - 8\,\psi^{(1)}\,\nabla^2\psi^{(1)} + 12\,\frac{a'}{a}\psi^{(1)}\,\psi^{(1)\prime} \\
& - \; 3\left(\psi^{(1)\prime}\right)^2 - 4\,\frac{a'}{a}\,\phi^{(1)}\,\nabla^2\omega^{(1)} - 2\,\frac{a'}{a}\partial_k\omega^{(1)}\,\partial^k\phi^{(1)} + 3\left(\frac{a'}{a}\right)^2\partial_k\omega^{(1)}\,\partial^k\omega^{(1)} \\
& + \; \frac{1}{2}\,\partial_i\partial_k\omega^{(1)}\,\partial^i\partial^k\omega^{(1)} - \frac{1}{2}\,\partial_k\partial^k\omega^{(1)}\,\partial_k\partial^k\omega^{(1)} - 2\,\frac{a'}{a}\partial_k\psi^{(1)}\,\partial^k\omega^{(1)} + 4\,\frac{a'}{a}\psi^{(1)}\,\nabla^2\omega^{(1)} \\
& - \; 2\,\partial_k\omega^{(1)}\,\partial^k\psi^{(1)\prime} - 2\psi^{(1)\prime}\nabla^2\omega^{(1)} - 2\,\psi^{(1)}\partial_k\partial^i\,D^k{}_i\chi^{(1)} - \frac{1}{2}\,\partial_i\partial_j\omega^{(1)}\,D^{ij}\chi^{(1)\prime} \\
& + \; \partial_k\partial_i\psi^{(1)}\,D^{ki}\chi^{(1)} - 2\,\frac{a'}{a}\partial_i\partial_k\omega^{(1)}\,D^{ik}\chi^{(1)} - 2\,\frac{a'}{a}\partial_k\omega^{(1)}\,\partial_i\,D^{ik}\chi^{(1)} - \frac{1}{2}\,\partial_k\omega^{(1)}\,\partial^i D^k{}_i\chi^{(1)\prime} \\
& - \; \frac{1}{2}\,\nabla^2\,D_{mk}\chi^{(1)}\,D^{km}\chi^{(1)} + \partial_m\partial^k\,D_{ik}\chi^{(1)}\,D^{im}\chi^{(1)} + \frac{1}{2}\,\partial_k D^{km}\chi^{(1)}\,\partial^i D_{mi}\chi^{(1)} \\
& - \; \frac{3}{8}\,\partial^i D^{km}\chi^{(1)}\,\partial_i D_{km}\chi^{(1)} + \frac{1}{8}\,D^{ik}\chi^{(1)\prime}\,D_{ki}\chi^{(1)\prime} + \frac{a'}{a}D^{ki}\chi^{(1)}\,D_{ik}\chi^{(1)\prime} + \frac{1}{4}\,\partial_k D_{ij}\chi^{(1)}\,\partial^j D^{ik}\chi^{(1)}\Bigg],
\end{aligned}
$$

# Einstein tensor – Second order (neglecting linear vector and tensor modes)

$$
\begin{aligned}
\delta^{(2)}G^{i}{}_{0} \;=\; & \frac{1}{a^2}\Bigg[\frac{a'}{a}\partial^i\phi^{(2)} + \partial^i\psi^{(2)\prime} + \frac{1}{4}\partial_k D^{ki}\chi^{(2)\prime} + \frac{1}{4}\nabla^2\chi^{i(2)\prime} - \frac{1}{4}\nabla^2\omega^{i(2)} \\
& - \frac{a''}{a}\partial^i\omega^{(2)} - \frac{a''}{a}\omega^{i(2)} + 2\left(\frac{a'}{a}\right)^2\partial^i\omega^{(2)} + 2\left(\frac{a'}{a}\right)^2\omega^{i(2)} - 4\frac{a'}{a}\phi^{(1)}\partial^i\phi^{(1)} + 4\frac{a'}{a}\psi^{(1)}\partial^i\phi^{(1)} \\
& - 2\psi^{(1)\prime}\partial^i\phi^{(1)} + 4\psi^{(1)\prime}\partial^i\psi^{(1)} + 8\psi^{(1)}\partial^i\psi^{(1)\prime} - \partial^i\phi^{(1)}\nabla^2\omega^{(1)} - \partial^k\omega^{(1)}\partial^i\partial_k\phi^{(1)} \\
& + \nabla^2\phi^{(1)}\partial^i\omega^{(1)} + \partial^i\partial_k\omega^{(1)}\partial^k\phi^{(1)} + 4\frac{a''}{a}\phi^{(1)}\partial^i\omega^{(1)} - 8\left(\frac{a'}{a}\right)^2\phi^{(1)}\partial^i\omega^{(1)} \\
& + 2\frac{a'}{a}\phi^{(1)\prime}\partial^i\omega^{(1)} + \nabla^2\omega^{(1)\prime}\partial^i\omega^{(1)} - \partial^k\omega^{(1)}\partial^i\partial_k\omega^{(1)\prime} + 2\psi^{(1)\prime\prime}\partial^i\omega^{(1)} \\
& + 8\left(\frac{a'}{a}\right)^2\psi^{(1)}\partial^i\omega^{(1)} - 4\frac{a''}{a}\psi^{(1)}\partial^i\omega^{(1)} - 2\frac{a'}{a}\psi^{(1)\prime}\partial^i\omega^{(1)} - \frac{1}{2}\partial^k\phi^{(1)}D^i{}_k\chi^{(1)\prime} \\
& - 2\frac{a'}{a}\partial_k\phi^{(1)}D^{ki}\chi^{(1)} - \frac{1}{2}\partial_k\psi^{(1)}D^{ki}\chi^{(1)\prime} + 2\psi^{(1)}\partial_k D^{ki}\chi^{(1)\prime} + \psi^{(1)\prime}\partial_k D^{ki}\chi^{(1)} \\
& - \partial_k\psi^{(1)\prime}D^{ki}\chi^{(1)} + \frac{1}{2}\partial^k\omega^{(1)}D^i{}_k\chi^{(1)\prime\prime} + \frac{a'}{a}\partial^k\omega^{(1)}D^i{}_k\chi^{(1)\prime} - 4\left(\frac{a'}{a}\right)^2\partial_k\omega^{(1)}D^{ik}\chi^{(1)} \\
& + 2\frac{a''}{a}\partial_k\omega^{(1)}D^{ik}\chi^{(1)} - \frac{1}{2}\partial_k D^{km}\chi^{(1)}D^i{}_m\chi^{(1)\prime} - \frac{1}{2}\partial_k D^i{}_m\chi^{(1)\prime}D^{km}\chi^{(1)} \\
& + \frac{1}{4}\partial^i D_{mk}\chi^{(1)}D^{km}\chi^{(1)\prime} + \frac{1}{2}\partial^i D_{mk}\chi^{(1)\prime}D^{km}\chi^{(1)} - \frac{1}{2}D^{ik}\chi^{(1)}\partial_m D^m_k\chi^{(1)\prime}\Bigg],
\end{aligned}
$$

# Einstein tensor – Second order (neglecting linear vector and tensor modes)

$$
\begin{aligned}
\delta^{(2)}G^{0}{}_{i} \;=\; & \frac{1}{a^2}\Bigg[-\frac{a'}{a}\partial_i\phi^{(2)} - \partial_i\psi^{(2)\prime} - \frac{1}{4}\partial_k\, D^k{}_i\chi^{(2)\prime} - \frac{1}{4}\nabla^2\chi_i^{(2)\prime} + \frac{1}{4}\nabla^2\omega_i^{(2)} \\
& + 8\frac{a'}{a}\phi^{(1)}\partial_i\phi^{(1)} + 4\,\phi^{(1)}\,\partial_i\psi^{(1)\prime} + 2\,\psi^{(1)\prime}\partial_i\phi^{(1)} - 4\,\psi^{(1)\prime}\partial_i\psi^{(1)} - 4\,\psi^{(1)}\,\partial_i\psi^{(1)\prime} \\
& + \partial_i\phi^{(1)}\,\nabla^2\omega^{(1)} - \partial_i\partial_k\omega^{(1)}\,\partial^k\phi^{(1)} \\
& - 2\,\frac{a'}{a}\,\partial^k\omega^{(1)}\,\partial_i\partial_k\omega^{(1)} + \nabla^2\psi^{(1)}\,\partial_i\omega^{(1)} + \partial^k\omega^{(1)}\,\partial_i\partial_k\psi^{(1)} \\
& + \frac{1}{2}\,\partial^k\phi^{(1)}\,D_{ik}\chi^{(1)\prime} - \psi^{(1)}\,\partial_k D^k{}_i\chi^{(1)\prime} + \frac{1}{2}\,\partial_k\psi^{(1)}\,D^k{}_i\chi^{(1)\prime} - \psi^{(1)\prime}\partial_k D^k{}_i\chi^{(1)} \\
& - \partial_k\psi^{(1)\prime}D^k{}_i\chi^{(1)} + \frac{1}{2}\,\partial_k\,\omega^{(1)}\,\partial^k\,\partial^m\,D_{im}\,\chi^{(1)} \\
& + \frac{1}{2}\partial^k\omega^{(1)}\,\partial_m\partial_i\,D^m_k\chi^{(1)} + \phi^{(1)}\partial^k\,D_{ki}\chi^{(1)\prime} \\
& - \frac{1}{2}\,\partial^m\omega^{(1)}\,\partial_k\partial^k\,D_{im}\chi^{(1)} + \frac{1}{2}\,\partial_k D^{km}\chi^{(1)}\,D_{im}\chi^{(1)\prime} + \frac{1}{2}\,\partial_k D_{im}\chi^{(1)\prime}\,D^{km}\chi^{(1)} \\
& - \frac{1}{4}\partial_i D_{mk}\chi^{(1)}\,D^{km}\chi^{(1)\prime} - \frac{1}{2}\,\partial_i D_{mk}\chi^{(1)\prime}\,D^{km}\chi^{(1)}\Bigg],
\end{aligned}
$$

# Einstein tensor – Second order (neglecting linear vector and tensor modes)

$$
\begin{aligned}
\delta^{(2)}G^{d\,i}_{\ \ j} = \frac{1}{a^2}\Bigg[ & \frac{1}{2}\nabla^2\phi^{(2)} + \frac{a'}{a}\phi^{(2)\prime} + 2\,\frac{a''}{a}\,\phi^{(2)} - \left(\frac{a'}{a}\right)^2\phi^{(2)} - \frac{1}{2}\nabla^2\psi^{(2)} + \psi^{(2)\prime\prime} \\
& + 2\,\frac{a'}{a}\,\psi^{(2)\prime} + \frac{a'}{a}\,\nabla^2\omega^{(2)} + \frac{1}{2}\nabla^2\omega^{(2)\prime} - \frac{1}{4}\,\partial_k\partial_i\,D^{ki}\chi^{(2)} + 4\left(\frac{a'}{a}\right)^2\left(\phi^{(1)}\right)^2 \\
& - 8\,\frac{a''}{a}\left(\phi^{(1)}\right)^2 - 8\,\frac{a'}{a}\,\phi^{(1)}\,\phi^{(1)\prime} - \partial_k\phi^{(1)}\,\partial^k\phi^{(1)} - 2\phi^{(1)}\,\nabla^2\phi^{(1)} - 4\,\phi^{(1)}\,\psi^{(1)\prime\prime} \\
& - 2\,\phi^{(1)\prime}\psi^{(1)\prime} - 8\,\frac{a'}{a}\,\phi^{(1)}\,\psi^{(1)\prime} - 2\,\partial_k\psi^{(1)}\,\partial^k\psi^{(1)} - 4\,\psi^{(1)}\,\nabla^2\psi^{(1)} + \left(\psi^{(1)\prime}\right)^2 \\
& + 8\,\frac{a'}{a}\,\psi^{(1)}\,\psi^{(1)\prime} + 4\,\psi^{(1)}\,\psi^{(1)\prime\prime} + 2\,\psi^{(1)}\,\nabla^2\phi^{(1)} - \phi^{(1)\prime}\nabla^2\omega^{(1)} \\
& - 2\,\phi^{(1)}\,\nabla^2\omega^{(1)\prime} - 2\,\frac{a'}{a}\,\partial_k\omega^{(1)}\,\partial^k\phi^{(1)} - 4\,\frac{a'}{a}\,\phi^{(1)}\,\nabla^2\omega^{(1)} + 2\frac{a''}{a}\partial_k\omega^{(1)}\,\partial^k\omega^{(1)} \\
& - \left(\frac{a'}{a}\right)^2\partial_k\omega^{(1)}\,\partial^k\omega^{(1)} + 2\,\frac{a'}{a}\partial_k\omega^{(1)}\,\partial^k\omega^{(1)\prime} - \frac{1}{2}\,\nabla^2\omega^{(1)}\,\nabla^2\omega^{(1)} \\
& + \frac{1}{2}\,\partial^m\partial^k\omega^{(1)}\,\partial_m\partial_k\omega^{(1)} + 4\,\frac{a'}{a}\,\psi^{(1)}\,\nabla^2\omega^{(1)} + 2\,\psi^{(1)}\,\nabla^2\omega^{(1)\prime} \\
& - 2\,\partial_k\omega^{(1)}\,\partial^k\psi^{(1)\prime} - \psi^{(1)\prime}\nabla^2\omega^{(1)} - \partial_k\partial_m\phi^{(1)}\,D^{km}\chi^{(1)} - \partial_k\phi^{(1)}\,\partial_m D^{mk}\chi^{(1)} \\
& - \partial_k\psi^{(1)}\,\partial_m D^{mk}\chi^{(1)} - \partial_k\omega^{(1)}\,\partial^i D^k_{\ i}\chi^{(1)\prime} - \partial_k\omega^{(1)\prime}\,\partial_m D^{mk}\chi^{(1)} \\
& - \partial_k\partial_m\omega^{(1)\prime}\,D^{km}\chi^{(1)} - 2\,\frac{a'}{a}\partial^k\omega^{(1)}\,\partial_m D^m_k\chi^{(1)} - 2\,\frac{a'}{a}\partial_m\partial^k\omega^{(1)}\,D^m_k\chi^{(1)} \\
& + \partial_k\partial^l\,D_{ml}\chi^{(1)}\,D^{km}\chi^{(1)} - \frac{1}{2}\,\nabla^2\,D_{ml}\chi^{(1)}\,D^{ml}\chi^{(1)} - \frac{1}{2}\,\partial_k\,\partial_m\,\omega^{(1)}D^{km}\chi^{(1)\prime} \\
& + \frac{1}{2}\,\partial_k D_{km}\chi^{(1)}\,\partial^l D^{ml}\chi^{(1)} - \frac{3}{8}\,\partial^l D_{km}\chi^{(1)}\,\partial_l D^{km}\chi^{(1)} - 2\,\psi^{(1)}\,\partial_k\,\partial_m\,D^{km}\chi^{(1)} \\
& + \frac{1}{2}\,D^{mk}\chi^{(1)}\,D_{mk}\chi^{(1)\prime\prime} + \frac{3}{8}\,D^{mk}\chi^{(1)\prime}\,D_{mk}\chi^{(1)\prime} + \frac{a'}{a}\,D^{mk}\chi^{(1)}\,D_{km}\chi^{(1)\prime} \\
& + \frac{1}{4}\partial^l D^{km}\chi^{(1)}\,\partial_m D_{kl}\chi^{(1)}\Bigg]\delta^i_{\ j}
\end{aligned}
$$

$$
\begin{aligned}
\delta^{(2)}G^{nd\,i}_{\ \ \ j} = \frac{1}{a^2}\Bigg[ & -\frac{1}{2}\,\partial^i\partial_j\phi^{(2)} + \frac{1}{2}\,\partial^i\partial_j\psi^{(2)} - \frac{a'}{a}\partial^i\partial_j\omega^{(2)} - \frac{1}{2}\,\partial^i\partial_j\omega^{(2)\prime} \qquad (6 \\
& - \frac{1}{2}\frac{a'}{a}\left(\partial^i\,\omega^{(2)}_j + \partial_j\,\omega^{i(2)}\right) - \frac{1}{4}\left(\partial^i\,\omega^{(2)\prime}_j + \partial_j\,\omega^{i(2)\prime}\right) \\
& + \frac{1}{2}\frac{a'}{a}\left(D^i_{\ j}\chi^{(2)\prime} + \partial^i\chi^{(2)\prime}_j + \partial_j\chi^{i(2)\prime} + \chi^{i(2)\prime}_{\ j}\right) + \frac{1}{2}\partial_k\partial^i\,D^k_{\ j}\chi^{(2)} \\
& - \frac{1}{4}\nabla^2 D^i_{\ j}\chi^{(2)} - \frac{1}{4}\nabla^2\chi^{i(2)}_{\ j} + \frac{1}{4}\left(D^i_{\ j}\chi^{(2)\prime\prime} + \partial^i\chi^{(2)\prime\prime}_j + \partial_j\chi^{i(2)\prime\prime} + \chi^{i(2)\prime\prime}_{\ j}\right) \\
& + \partial^i\phi^{(1)}\,\partial_j\phi^{(1)} + 2\,\phi^{(1)}\,\partial^i\partial_j\phi^{(1)} - 2\,\psi^{(1)}\partial^i\partial_j\phi^{(1)} - \partial_j\phi^{(1)}\,\partial^i\psi^{(1)} - \partial^i\phi^{(1)}\,\partial_j\psi^{(1)} \\
& + 3\,\partial^i\psi^{(1)}\,\partial_j\psi^{(1)} + 4\,\psi^{(1)}\,\partial^i\partial_j\psi^{(1)} + 2\,\frac{a'}{a}\,\partial^i\omega^{(1)}\,\partial_j\phi^{(1)} + 4\frac{a'}{a}\phi^{(1)}\,\partial^i\partial_j\omega^{(1)} \\
& + \phi^{(1)\prime}\,\partial^i\partial_j\omega^{(1)} + 2\,\phi^{(1)}\,\partial^i\partial_j\omega^{(1)\prime} + \nabla^2\omega^{(1)}\,\partial^i\partial_j\omega^{(1)} - \partial_j\partial^k\omega^{(1)}\,\partial^i\partial_k\omega^{(1)} \\
& - 2\,\frac{a'}{a}\partial^i\psi^{(1)}\,\partial_j\omega^{(1)} - 2\,\frac{a'}{a}\partial^i\omega^{(1)}\,\partial_j\psi^{(1)} - \partial^i\psi^{(1)\prime}\,\partial_j\omega^{(1)} + \partial_j\psi^{(1)\prime}\,\partial^i\omega^{(1)} \\
& - \partial^i\psi^{(1)}\,\partial_j\omega^{(1)\prime} - \partial_j\psi^{(1)}\,\partial^i\omega^{(1)\prime} - 2\,\psi^{(1)}\,\partial^i\partial_j\omega^{(1)\prime} + \psi^{(1)\prime}\,\partial^i\partial_j\omega^{(1)} \\
& - 4\,\frac{a'}{a}\,\psi^{(1)}\,\partial^i\partial_j\omega^{(1)} - 2\,\frac{a'}{a}\phi^{(1)}\,D^i_{\ j}\chi^{(1)\prime} - \frac{1}{2}\phi^{(1)\prime}\,D^i_{\ j}\chi^{(1)\prime} - \phi^{(1)}\,D^i_{\ j}\chi^{(1)\prime\prime} \\
& + \frac{1}{2}\partial_k\phi^{(1)}\,\partial^i D^k_{\ j}\chi^{(1)} + \frac{1}{2}\partial_k\phi^{(1)}\,\partial_j D^{ki}\chi^{(1)} - \frac{1}{2}\partial_k\phi^{(1)}\,\partial^k D^i_{\ j}\chi^{(1)} + \partial_j\partial_k\phi^{(1)}\,D^{ki}\chi^{(1)} \\
& + \frac{1}{2}\psi^{(1)\prime}\,D^i_{\ j}\chi^{(1)\prime} + \psi^{(1)\prime\prime}D^i_{\ j}\chi^{(1)} + 2\,\frac{a'}{a}\psi^{(1)\prime}\,D^i_{\ j}\chi^{(1)} + \frac{1}{2}\partial_k\psi^{(1)}\,\partial^i D^k_{\ j}\chi^{(1)} \\
& + 2\,\frac{a'}{a}\,\psi^{(1)}\,D^i_{\ j}\chi^{(1)\prime} + \psi^{(1)}\,D^i_{\ j}\chi^{(1)\prime\prime} + \frac{1}{2}\partial_k\psi^{(1)}\,\partial_j D^{ki}\chi^{(1)} - \frac{3}{2}\,\partial_k\psi^{(1)}\,\partial^k D^i_{\ j}\chi^{(1)} \\
& + 2\,\psi^{(1)}\,\partial_k\partial^i D^k_{\ j}\chi^{(1)} + 2\,\psi^{(1)}\,\partial_k\partial_j D^{ki}\chi^{(1)} - 2\,\psi^{(1)}\,\partial_k\partial^k D^i_{\ j}\chi^{(1)} - \nabla^2\psi^{(1)}\,D^i_{\ j}\chi^{(1)} \\
& + \partial^i\psi^{(1)}\,\partial_k D^k_{\ j}\chi^{(1)} + \partial_j\psi^{(1)}\,\partial_k D^{ki}\chi^{(1)} + \partial_k\partial^i\psi^{(1)}\,D^k_{\ j}\chi^{(1)} + \frac{1}{2}\,\partial^i\omega^{(1)}\,\partial_k D^k_{\ j}\chi^{(1)\prime} \\
& + \frac{1}{2}\,\partial_k\partial^i\omega^{(1)}\,D^k_{\ j}\chi^{(1)\prime} + \frac{1}{2}\,\partial_k\partial_j\omega^{(1)}\,D^{ki}\chi^{(1)\prime} - \frac{1}{2}\,\partial_k\partial^k\omega^{(1)}\,D^i_j\chi^{(1)\prime} + \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial^i D_{kj}\chi^{(1)\prime} \\
& + \frac{1}{2}\,\partial^k\omega^{(1)}\,\partial_j D^i_{\ k}\chi^{(1)\prime} - \partial^k\omega^{(1)}\,\partial_k D^i_{\ j}\chi^{(1)\prime} + \frac{1}{2}\,\partial^k\omega^{(1)\prime}\,\partial^i D_{kj}\chi^{(1)} + \frac{1}{2}\,\partial^k\omega^{(1)\prime}\,\partial_j D^i_{\ k}\chi^{(1)} \\
& - \frac{1}{2}\,\partial^k\omega^{(1)\prime}\,\partial_k D^i_{\ j}\chi^{(1)} + \partial_k\partial_j\omega^{(1)\prime}\,D^{ik}\chi^{(1)} + \frac{a'}{a}\partial^k\omega^{(1)}\,\partial^i D_{kj}\chi^{(1)} + \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_j D^i_{\ k}\chi^{(1)} \\
& - \frac{a'}{a}\partial^k\omega^{(1)}\,\partial_k D^i_{\ j}\chi^{(1)} + 2\,\frac{a'}{a}\,\partial_k\partial_j\omega^{(1)}\,D^{ik}\chi^{(1)} - \frac{1}{2}\,D^{ki}\chi^{(1)\prime}\,D_{kj}\chi^{(1)\prime} \\
& - \frac{1}{2}\partial^i D_{mj}\chi^{(1)}\,\partial_k D^{km}\chi^{(1)} - \frac{1}{2}\partial_j D^i_{\ m}\chi^{(1)}\,\partial_k D^{km}\chi^{(1)} + \frac{1}{2}\partial_m D^i_{\ j}\chi^{(1)}\,\partial_k D^{km}\chi^{(1)} \\
& - \frac{1}{2}\partial_k\partial^i D_{mj}\chi^{(1)}\,D^{km}\chi^{(1)} - \frac{1}{2}\partial_k\partial_j D^i_{\ m}\chi^{(1)}\,D^{km}\chi^{(1)} + \frac{1}{2}\partial_k\partial_m D^i_{\ j}\chi^{(1)}\,D^{km}\chi^{(1)} \\
& + \frac{1}{2}D^{km}\chi^{(1)}\,\partial^i\partial_j D_{km}\chi^{(1)} + \frac{1}{4}\partial^i D^{mk}\chi^{(1)}\,\partial_j D_{mk}\chi^{(1)} \\
& - \frac{a'}{a}\,D_{kj}\chi^{(1)\prime}D^{ik}\chi^{(1)} - \frac{1}{2}\,D_{kj}\chi^{(1)\prime\prime}D^{ki}\chi^{(1)} - \partial_m\partial_k\,D^m_j\chi^{(1)}\,D^{ki}\chi^{(1)} \\
& + \frac{1}{2}\,\partial_m\partial^m\,D_{kj}\chi^{(1)}\,D^{ki}\chi^{(1)} + \frac{1}{2}\,\partial_m D^{ik}\chi^{(1)}\partial^m D_{kj}\chi^{(1)} - \frac{1}{2}\,\partial_m D^{ik}\chi^{(1)}\,\partial_k D^m_{\ j}\,\chi^{(1)}\Bigg],
\end{aligned}
$$

# Gauge transformations – I. (see Matarrese, Mollerach & Bruni 1998, Appendix A).

- Gauge choices for perturbations entail the comparison of the tensor field representing a certain physical and/or geometrical quantity in the perturbed  spacetime with the tensor field representing the same quantity in the background spacetime. Gauge transformations entail the comparison of tensors at different points in the background spacetime. A smallness parameter $\lambda$ is involved, so that these comparisons are always carried out at the required order of accuracy in $\lambda$ , using Taylor expansions. The comparison of tensors is meaningful only when we consider them at the same point. Therefore, if we want to compare a tensor field T at points p and q, we need to define a transport law from q  to p. This gives us two tensors at p, T, itself, and the transported one, which can now be directly compared.

The simplest transport law we need to consider is the Lie dragging by a vector field, which allows us to compare $T$ with its pullback $\widetilde{T}(\lambda)$ (the new tensor defined by this transport). To fix ideas, let us first consider, on a manifold $\mathcal{M}$, the comparison of tensors at first order in $\lambda$ (which we shall define shortly). Suppose a coordinate system $x^{\mu}$ has been given on (an open set of) $\mathcal{M}$, together with a vector field $\xi$. From $dx^{\mu}/d\lambda = \xi^{\mu}$, $\xi$ generates on $\mathcal{M}$ a congruence of curves $x^{\mu}(\lambda)$: thus $\lambda$ is the parameter along the congruence. Given a point $p$, this will always lie on one of these curves, and we can always take $p$ to correspond to $\lambda = 0$ on this. The coordinates of a second point $q$ at a parameter distance $\lambda$ from $p$ on the same curve will be given by

$$\tilde{x}^{\mu}(\lambda) = x^{\mu} + \lambda \xi^{\mu} + \cdots, \tag{A1}$$

where the $x^{\mu}$ are the coordinates of $p$ and the $\tilde{x}^{\mu}$ are those of $q$, approximated here at first order in $\lambda$. Equation (A1) is usually called an "infinitesimal point transformation," or an "active coordinate transformation"

. At the same time, we may think that a new coordinate system $y^{\mu}$ has been introduced on $\mathcal{M}$, *defined* in such a way that the $y$ coordinates of the point $q$ coincide with the $x$ coordinates of the point $p$; using Eq. (A1) it then follows from this definition that

$$\begin{aligned} y^{\mu}(q) &:= x^{\mu}(p) = x^{\mu}(q) - \lambda \xi^{\mu}(x(p)) + \cdots \\ &\simeq x^{\mu}(q) - \lambda \xi^{\mu}(x(q)) + \cdots . \end{aligned} \tag{A2}$$

In practice, we have in this way defined at every point a "passive coordinate transformation" (i.e., just an ordinary relabeling of point's names), which at first order reads

$$y^{\mu}(\lambda) = x^{\mu} - \lambda \xi^{\mu} + \cdots . \tag{A3}$$

Suppose now that a tensor field has been given on $\mathcal{M}$; e.g., to fix ideas, consider the vector field $Z$ with components $Z^{\mu}$ in the $x$-coordinate system. In the same way that we defined a new coordinate system $y^{\mu}$ once a relation between points was assigned through Eq. (A1) by the action of $\xi$, so we can now define a new vector field $\widetilde{Z}$, with components $\widetilde{Z}^{\mu}$ in the $x$ coordinates, such that these components at the coordinate point $x^{\mu}(p)$ are equal to the components $Z'^{\mu}$ the old vector $Z$ has in the $y$ coordinates at the coordinate point $y(q)$:

$$\widetilde{Z}^{\mu}(x(p)) := Z'^{\mu}(y(q)) = \left[ \frac{\partial y^{\mu}}{\partial x^{\nu}} \right]_{x(q)} Z^{\nu}(x(q)). \tag{A4}$$

The last equality in this equation is just the ordinary (passive) transformation between the components of $Z$ in the two coordinate systems: we need it in order to relate $\widetilde{Z}$ and $Z$ in a single system (the $x$ frame here), thus eventually obtaining a covariant relation. Indeed, substitution of Eq. (A3) into Eq. (A4) and a first-order expansion in $\lambda$ about $x(p)$ in the right-hand side (RHS) gives

$$\widetilde{Z}^{\mu}(\lambda) = Z^{\mu} + \lambda \pounds_{\xi} Z^{\mu} + \cdots , \tag{A5}$$

$$\pounds_{\xi} Z^{\mu} := Z^{\mu}_{,\nu} \xi^{\nu} - \xi^{\mu}_{,\nu} Z^{\nu}, \tag{A6}$$

where, given that the point $p$ is arbitrary, the dependence of all terms on $x(p)$ has been omitted. The vector field $\tilde{Z}$ is called the pullback of $Z$, because it is defined by dragging $Z$ back from $q$ to $p$, an operation that gives at $p$ a new vector with components $\tilde{Z}^{\mu}$, given by Eq. (A4). In the particular case of the transformation (A3) this is the Lie dragging. Now, having at the same point two vectors, these can be directly compared: at first order, $\tilde{Z}(\lambda)$ and $Z$ are related by Eqs. (A5), (A6). In fact, in the limit $\lambda \to 0$, it is this comparison that allows us to define the Lie derivative, with components (A6); Eq. (A15) below generalizes this to a generic tensor $T$.

Although the story so far is a textbook one (cf. [29,35–37]), recalling it in some detail allows us to easily extend it to higher order. First, one has to realize that Eq. (A1) is just the first-order Taylor expansion about $x(p)$ of the solution of the ordinary differential equation $\mathrm{d}x^{\mu}/\mathrm{d}\lambda = \xi^{\mu}$ defining the congruence $x^{\mu}(\lambda)$ associated with $\xi$. The exact solution of this equation is the Taylor series (

$$x^{\mu}(q) = x^{\mu}(p) + \lambda \xi^{\mu}(x(p)) + \frac{\lambda^2}{2} \xi^{\mu}{}_{,\nu} \xi^{\nu}(x(p)) + \cdots, \tag{A7}$$

on using $dx^{\mu}/d\lambda = \xi^{\mu}$, $\mathrm{d}^2 x^{\mu}/\mathrm{d}\lambda^2 = \xi^{\mu}_{,\nu}\xi^{\nu}$, etc. In practice, since $p$ and $q$ are arbitrary, we may simply write

$$\tilde{x}^{\mu}(\lambda) = x^{\mu} + \lambda \xi^{\mu} + \frac{\lambda^2}{2} \xi^{\mu}{}_{,\nu} \xi^{\nu} + \cdots, \tag{A8}$$

$$= \exp[\lambda \pounds_{\xi}] x^{\mu}. \tag{A9}$$

The latter exponential notation is useful, in that it allows us to see the coordinate functions $\tilde{x}^{\mu}$ as the pullbacks of the functions $x^{\mu}$ given by the exponential pullback operator $\exp[\lambda \pounds_{\xi}]$. Furthermore, it is clearly seen by $\exp[(\lambda_1 + \lambda_2)\pounds_{\xi}] = \exp[\lambda_1 \pounds_{\xi}]\exp[\lambda_2 \pounds_{\xi}]$ that the point transformations (A8) form a one-parameter group of transformations. Using again the definition $y^{\mu}(q) := x^{\mu}(p)$ for the $y$ coordinates, we get, from Eq. (A7),

$$y^{\mu}(\lambda) = x^{\mu} - \lambda \xi^{\mu} + \frac{\lambda^2}{2} \xi^{\mu}{}_{,\nu} \xi^{\nu} + \cdots, \tag{A10}$$

on expanding all terms about $x(q)$, eventually omitting again the $x(q)$ dependence, since $q$ is arbitrary. Finally, using Eq. (A10) into Eq. (A4) and expanding all the terms about $x(p)$, we get the $x$ components $\tilde{Z}^{\mu}(\lambda)$ of the pullback $\tilde{Z}(\lambda)$, which reads [39]

$$\tilde{Z}^{\mu}(\lambda) = [\exp[\lambda \pounds_{\xi}] Z]^{\mu} \tag{A11}$$

$$= Z^{\mu} + \lambda \pounds_{\xi} Z^{\mu} + \frac{\lambda^2}{2} \pounds_{\xi}^2 Z^{\mu} + \cdots. \tag{A12}$$

Equation (A4) is readily generalized to more general tensors than $Z$: we simply have to add to the RHS of Eq. (A4) the right number of transformation matrices. Thus, the pullback $\tilde{T}$ of a tensor field $T$ of type $(p,q)$ is defined by having $x$ components given by

$$
\begin{aligned}
&\tilde{T}^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}(x(p)) \\
&\quad := T'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}(y(q)) \\
&\quad = \left[\frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial y^{\mu_p}}{\partial x^{\alpha_p}}\frac{\partial x^{\beta_1}}{\partial y^{\nu_1}}\cdots\frac{\partial x^{\beta_q}}{\partial y^{\nu_q}}\right]_{x(q)} \\
&\qquad \times T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}(x(q)). \qquad \text{(A13)}
\end{aligned}
$$

Using Eq. (A10) as above then gives, omitting indices for brevity,

$$\tilde{T}(\lambda)=T+\lambda\pounds_\xi T+\frac{\lambda^2}{2}\pounds_\xi^2 T+\cdots. \qquad \text{(A14)}$$

To summarize, each of the diffeomorphisms forming a one-parameter group, as mathematicians call the transformations generated by a vector field $\xi$ and represented in coordinates by Eq. (A9), gives rise to a new field, the pullback $\tilde{T}(\lambda)$, from any given tensor field $T$ and for any given value of $\lambda$. Thus $\tilde{T}(\lambda)$ and $T$ may be compared at every point, which allows one to define the Lie derivative along $\xi$ as the limit $\lambda\to 0$ of the difference $\tilde{T}(\lambda)-T$:

$$\pounds_\xi T:=\left[\frac{d}{d\lambda}\right]_{\lambda=0}\tilde{T}(\lambda)=\lim_{\lambda\to 0}\frac{1}{\lambda}[\tilde{T}(\lambda)-T]. \qquad \text{(A15)}$$

At higher order we have

$$\pounds_\xi^k T:=\left[\frac{d^k}{d\lambda^k}\right]_{\lambda=0}\tilde{T}(\lambda). \qquad \text{(A16)}$$

On the other hand, the relation at each point between any tensor field $T$ and its pullback $\tilde{T}(\lambda)$ is expressed at the required order of accuracy by the Taylor expansion (A14).

In order to proceed, considering more general point transformations than Eq. (A8) and more general Taylor expansions than Eq. (A14), some general remarks are in order. First, it should be noticed that the definition $y^\mu(q):=x^\mu(p)$ for the $y$-coordinate system is completely general, given a first coordinate system (the $x$ frame here) and any suitable association between pairs of points (more precisely, any diffeomorphism), of which the one-parameter group of transformations (A8) is a particular example. Second, the same generality is present in the definition of the pullback, Eq. (A13), which is also independent from the specific type of transformation chosen.

we have shown that the action of any given one-parameter family of transformations can be represented by the successive action of one-parameter groups, in a fashion that, to order $\lambda^2$, reminds us the motion of the knight on the chessboard:

$$\tilde{x}^{\mu}(\lambda)=x^{\mu}+\lambda\,\xi^{\mu}_{(1)}+\frac{\lambda^2}{2}(\xi^{\mu}_{(1),\nu}\xi^{\nu}_{(1)}+\xi^{\mu}_{(2)})+\cdots. \tag{A17}$$

A vector field $\xi_{(k)}$ is associated with the $k$th one-parameter group of transformations, with parameter $\lambda_k$ (we denote $\lambda_1 =\lambda$). Similarly to the knight, the action of the transformation (A17) first moves from point $p$ (with coordinates $x^{\mu}$) by an amount $\lambda$ along the integral curve of $\xi_{(1)}$ [i.e., according to Eq. (A8)]; then, it moves along the integral curve of $\xi_{(2)}$ by an amount $\lambda_2=\lambda^2/2$. At each $k$th higher order, a new vector field $\xi_{(k)}$ is involved, generating a motion by $\lambda_k=\lambda^k/k!$. Thus, the action of a one-parameter family of transformations is approximated, at order $k$, by a ‘‘knight transformation’’ of order $k$ , of which Eq. (A17) is the second-order example.

Given the ‘‘knight transformation’’ (A17), we can now use it to define the $y$ coordinates, which will be given by

$$\begin{aligned} y^{\mu}(q):=x^{\mu}(p)=x^{\mu}(q)-\lambda\,\xi^{\mu}_{(1)}(x(p)) \\ -\frac{\lambda^2}{2}[\xi^{\mu}_{(1),\nu}(x(p))\xi^{\nu}_{(1)}(x(p))+\xi^{\mu}_{(2)}(x(p))] \\ +\cdots. \end{aligned} \tag{A18}$$

Expanding the various quantities on the RHS around $q$, and omitting the $x(q)$ dependence, Eq. (A18) becomes, finally,

$$y^{\mu}(\lambda)=x^{\mu}-\lambda\,\xi^{\mu}_{(1)}+\frac{\lambda^2}{2}(\xi^{\mu}_{(1),\nu}\xi^{\nu}_{(1)}-\xi^{\mu}_{(2)})+\cdots. \tag{A19}$$

Using again the case of the vector field $Z$ as our paradigmatic example, we can now derive the pullback $\tilde{Z}(\lambda)$ generated by a one-parameter family of transformations. Substituting Eq. (A19) into Eq. (A4), and expanding again every term about $x(p)$, we obtain the $x$ components $\tilde{Z}^{\mu}(\lambda)$ of $\tilde{Z}(\lambda)$, which (after properly collecting terms) at second order read

$$\tilde{Z}^{\mu}(\lambda)=Z^{\mu}+\lambda\,£_{\xi_{(1)}}Z^{\mu}+\frac{\lambda^2}{2}(£^2_{\xi_{(1)}}+£_{\xi_{(2)}})Z^{\mu}+\cdots. \tag{A20}$$

For a generic tensor $T$, again omitting indices for brevity, use of Eq. (A19) in Eq. (A13) obviously gives

$$\tilde{T}=T+\lambda\,£_{\xi_{(1)}}T+\frac{\lambda^2}{2}(£^2_{\xi_{(1)}}+£_{\xi_{(2)}})T+\cdots. \tag{A21}$$

# Lie derivatives

The Lie derivative of any tensor $T$ of type $(p,q)$ (a tensor with $p$ contravariant and $q$ covariant indices, which we omit here and in the following) is also a tensor of the same type $(p,q)$. For a scalar $f$, a contravariant vector $Z$ and a covariant tensor $T$ of rank 2, the expressions of the Lie derivative along $\xi$ are, respectively,

$$£_{\xi} f = f_{,\mu} \xi^{\mu}, \tag{1}$$

$$£_{\xi} Z^{\mu} = Z^{\mu}_{,\nu} \xi^{\nu} - \xi^{\mu}_{,\nu} Z^{\nu}, \tag{2}$$

$$£_{\xi} T_{\mu\nu} = T_{\mu\nu,\sigma} \xi^{\sigma} + \xi^{\sigma}_{,\mu} T_{\sigma\nu} + \xi^{\sigma}_{,\nu} T_{\mu\sigma} . \tag{3}$$

Expressions for any other tensor can easily be derived from these. A second or higher Lie derivative is easily defined from these formulas; e.g., for a vector we have $£^2_{\xi} Z = £_{\xi}(£_{\xi} Z)$: since one clearly sees from Eq. (2) that $£_{\xi} Z$ is itself a contravariant vector, one needs only to apply Eq. (2) two times to obtain the components of $£^2_{\xi} Z$. Similarly, one derives expressions for the second Lie derivative of any tensor.

# Gauge tranformations in practice

- The gauge tranformation is determined by the vectors $\xi^{\mu}_{(r)}$
- Splitting their time and space parts we can write

$$\xi^0_{(r)} = \alpha^{(r)}$$

and

$$\xi^i_{(r)} = \partial^i \beta^{(r)} + d^{(r)i},$$

with $\partial_i d^{(r)i} = 0$.

- Here $\alpha^{(r)}$ and $\beta^{(r)}$ are scalars, whereas $d^{(r)}$ is a vector.
- No tensor modes are involved → **linear tensor modes (GW) are gauge-invariant**

# Gauge fixing

As we have seen, the vector $\xi^\mu$ generating the gauge transformation involves two scalars $(\alpha, \beta)$ and one divergence-free vector $d^i$. This holds at any order in perturbation theory. Hence the various gauges are defined by suitable choices of ~~these~~ two scalars and one vector.

Let us consider here some popular gauge choices.

# Poisson gauge

1) Poisson gauge

The Poisson gauge is defined by the choice

$$w^{\parallel} = 0$$

$$\chi^{\parallel} = 0$$

$$\chi_i^{\perp} = 0$$

This generalizes the so-called longitudinal or (conformal) Newtonian gauge in which vector and tensor perturbations are not considered (not that this is not a gauge-choice but a dynamical statement).

# Synchronous (and time-orthogonal) gauge

## 2) Synchronous gauge

This is defined by the choice

$$\psi = 0$$

If we also take $\omega^{\parallel} = \omega_i^{\perp} = 0$ this is called synchronous and time-orthogonal gauge. In this gauge the proper time for observers at fixed spatial coordinates coincides with cosmic time in the FRW background, i.e.

$$dt = a(\tau)\, d\tau .$$

It could be easily seen that the synchronous gauge is plagued by residual gauge freedom (somehow similar to the Gribov ambiguity in electrodynamics)

# Other popular gauges

- Spatially flat gauge

The spatially flat or uniform curvature gauge is identified by the condition that one selects spatial hypersurfaces on which the induced 3-metric of spatial hypersurfaces is left unperturbed by scalar or vector perturbations, which requires

$$\varphi = \chi^{\parallel} = \chi_i^{\perp} = 0$$

- Uniform density gauge

This gauge is defined by the condition

$$\delta\rho = 0$$

which leaves freedom in one scalar and one vector perturbation.

# Comoving gauge

3) Comoving gauge

The comoving gauge is defined by the condition that the 3-velocity of the fluid vanishes i.e.

$$v^i = 0 \Rightarrow v^\parallel = v_i^T = 0$$

If we also require orthogonality of the constant-$\tau$ hypersurfaces to the 4-velocity, ($T^0{}_i = 0$) this gives

$$v^\parallel + w^\parallel = 0$$

(zero momentum). Notice that we cannot require simultaneously $v_i^T = 0$ & $w_i^T = 0$ as a gauge condition (but it can be a dynamical requirement).

**Important remark**: if and only if pressure gradients vanish (as e.g. in the case of pressureless matter + cosmological constant) we can simultaneously fix our gauge to be synchronous, time-orthogonal and comoving.

# Second-order tensor modes are not gauge-invariant

- Second order perturbations of the metric read:

$$\delta^2 \tilde{g}_{\mu\nu} = \delta^2 g_{\mu\nu} + 2\pounds_{\xi_{(1)}} \delta g_{\mu\nu} + \pounds^2_{\xi_{(1)}} g^{(0)}_{\mu\nu} + \pounds_{\xi_{(2)}} g^{(0)}_{\mu\nu}$$

- If we focus only on the spatial traceless part of the metric we find

$$\begin{aligned}\tilde{\chi}^{(2)}_{ij} = {} & \chi^{(2)}_{ij} + 2\left(\chi^{(1)\prime}_{ij} + 2\frac{a'}{a}\chi^{(1)}_{ij}\right)\alpha_{(1)} + 2\chi^{(1)}_{ij,k}\xi^k_{(1)} + 2\left(-4\phi_{(1)} + \alpha_{(1)}\partial_0 + \xi^k_{(1)}\partial_k + 4\frac{a'}{a}\alpha_{(1)}\right)(d^{(1)}_{(i,j)} + \mathrm{D}_{ij}\beta_{(1)}) \\ & + 2\left[(2\omega^{(1)}_{(i} - \alpha^{(1)}_{,(i} + \xi^{(1)\prime}_{(i})\alpha^{(1)}_{,j)} - \frac{1}{3}\delta_{ij}(2\omega^k_{(1)} - \alpha^{,k}_{(1)} + \xi^{k\prime}_{(1)})\alpha^{(1)}_{,k}\right] \\ & + 2\left[(2\chi^{(1)}_{(i|k|} + \xi^{(1)}_{k,(i} + \xi^{(1)}_{(i,|k|})\xi^{(1)k}_{,j)} - \frac{1}{3}\delta_{ij}(2\chi^{(1)}_{lk} + \xi^{(1)}_{k,l} + \xi^{(1)}_{l,k})\xi^{k,l}_{(1)}\right] + 2(d^{(2)}_{(i,j)} + \mathrm{D}_{ij}\beta_{(2)}).\end{aligned}$$

which clearly involves several divergence-less contributions in the transformation

# General recipe for gauge-invariace

- It is possible to establish a condition for gauge-invariance to a given perturbative order n even without knowledge of the gauge transformation rules holding at that order. Let's focus here on gauge invariance up to second order only. The most natural definition of gauge invariance is that a tensor T is gauge invariant to order n if and only if $\delta^k \widetilde{T} = \delta^k T$ , for every k≤ n. Let is define $\delta^0 T := T_0,\ \delta T := \delta^1 T$
- Thus, a tensor T is gauge invariant to second order if $\delta^2 \widetilde{T} = \delta^2 T \ and \ \delta \widetilde{T} = \delta T.$
- Let's now consider the infinitesimal point transformation

$$\widetilde{x}^\mu(\lambda) = x^\mu + \lambda \xi^\mu_{(1)} + \frac{\lambda^2}{2}(\xi^\mu_{(1),\nu}\xi^\nu_{(1)} + \xi^\mu_{(2)}) + \mathcal{O}(\lambda^3) \quad \text{and} \quad \widetilde{T}(\lambda) = T(\lambda) + \lambda \pounds_{\xi_{(1)}} T + \frac{\lambda^2}{2}(\pounds^2_{\xi_{(1)}} + \pounds_{\xi_{(2)}})T + \mathcal{O}(\lambda^3)$$

- Hence in two arbitrarily chosen gauges we have

$$T(\lambda) = T_0 + \lambda\,\delta T + \frac{\lambda^2}{2}\delta^2 T + \mathcal{O}(\lambda^3), \qquad \delta\widetilde{T} = \delta T + \pounds_{\xi_{(1)}} T_0,$$

and

$$\widetilde{T}(\lambda) = T_0 + \lambda\,\delta \widetilde{T} + \frac{\lambda^2}{2}\delta^2 \widetilde{T} + \mathcal{O}(\lambda^3), \qquad \delta^2\widetilde{T} = \delta^2 T + 2\pounds_{\xi_{(1)}}\delta T + \pounds^2_{\xi_{(1)}} T_0 + \pounds_{\xi_{(2)}} T_0$$

- This condition implies $\pounds_\xi T_0 = 0 \ \text{and} \ \pounds_\xi \delta T = 0$ for every vector field $\xi^\mu$ . Therefore, apart from trivial cases—i.e., constant scalars and combinations of Kronecker deltas with constant coefficients—gauge invariance to second order requires that $T_0 = 0 \ \text{and} \ \delta T = 0$ in any gauge. This condition generalizes to second order the standard results for first-order gauge invariance and is easily extended to order n.

# A practical recipe to build "gauge-invariant" quantities

Malik & Wands 1998, 2004 noticed the following:

- A gauge-invariant theory of linear perturbations about FRW metric was proposed by Bardeen (1980), but no such gauge-invariant formalism has been developed for non-linear cosmological perturbations. According to the *Stewart-Walker lemma* (1974) in fact, "any truly gauge-independent perturbation must be constant in the background spacetime". This apparently limits our ability to make a gauge-invariant study of quantities that evolve in the background spacetime, such as, e.g., density perturbations in the expanding Universe.
- One can however construct gauge-invariant definitions of unambiguous – i.e. physically defined – perturbations, which – in spite of not being automatically gauge-independent (i.e. with no gauge dependence, such as perturbations about a constant scalar field) – are in general gauge-dependent (such as the curvature perturbation) **but can have a gauge-invariant definition once their gauge-dependence is fixed** (such as the curvature perturbation on uniform-density hypersurfaces). This technique, however may lead to some troubles in the case of transformations which rescale the background (e.g. Weinberg 2003).
- This technique should be used with extra-care when starting from the synchronous gauge, which is affected by residual gauge ambiguities.

# Tenacious myths of higher-order perturbation theory

→ Handle with care the traditional statement: “Linear theory always yields the leading contribution on large scales”

→Rather say: “Linear theory yields the leading contribution of quantities like the power-spectrum on large scales, but higher-order statistics get their leading contribution on large scales by higher-order perturbation theory. E.g. the leading-order large-scale contribution to the bispectrum is coming from second-order perturbation theory. Moreover, quantities like e.g. the ratio of the skewness to the variance squared do not converge to zero on large scales (but the ratio of the skewness to the 3/2 power of the variance does vanish on large scales).”

→As a general rule: the answer depends on the considered observable!

# Final remarks on Part I

- Cosmological perturbations are defined as deviations from the background FLRW model.
- A gauge issue affects most perturbation modes. Exact gauge-invariance almost impossible to implement beyond linear order.
- Mode-mode coupling (hence non-Gaussianity) is a generic feature, as soon as one goes beyond linear theory, and it also implies that scalar and vector modes feed tensor modes, scalar and tensor modes feed vector modes, tensor and vector modes feed scalar modes. This has profound consequences for cosmological observables (Tomita 1967; Matarrese et al. 1998; …).