Mathematical and Physical Foundation of Extended Gravity (III)

Dark Energy and Dark Matter as Curvature Effects

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Summary

- Dark Energy and Dark matter problems
- Extending General Relativity
- The weak field limit
- Stellar structures and Jeans instability
- Quadrupolar gravitational radiation
- Application to the binary systems
- Testing spiral galaxies
- Testing elliptical galaxies
- Modeling clusters of galaxies
- Cosmography
- Conclusions

Strange Situation in today Physics

- *Astronomy: Excellent Data without Theory!*
- *Quantum Gravity: Excellent Theory without Data!*

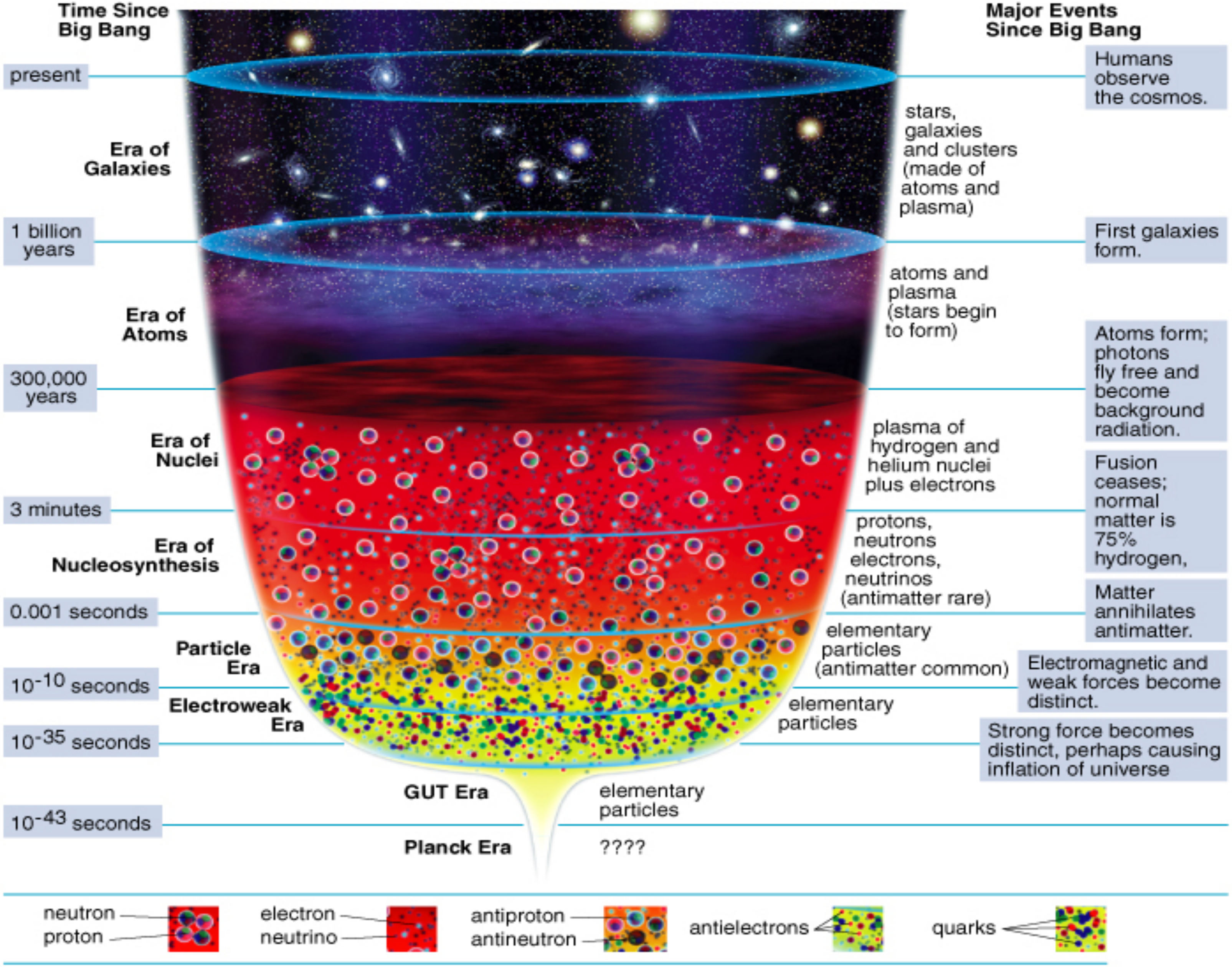
What is in the middle?

Dark Matter & Dark Energy?

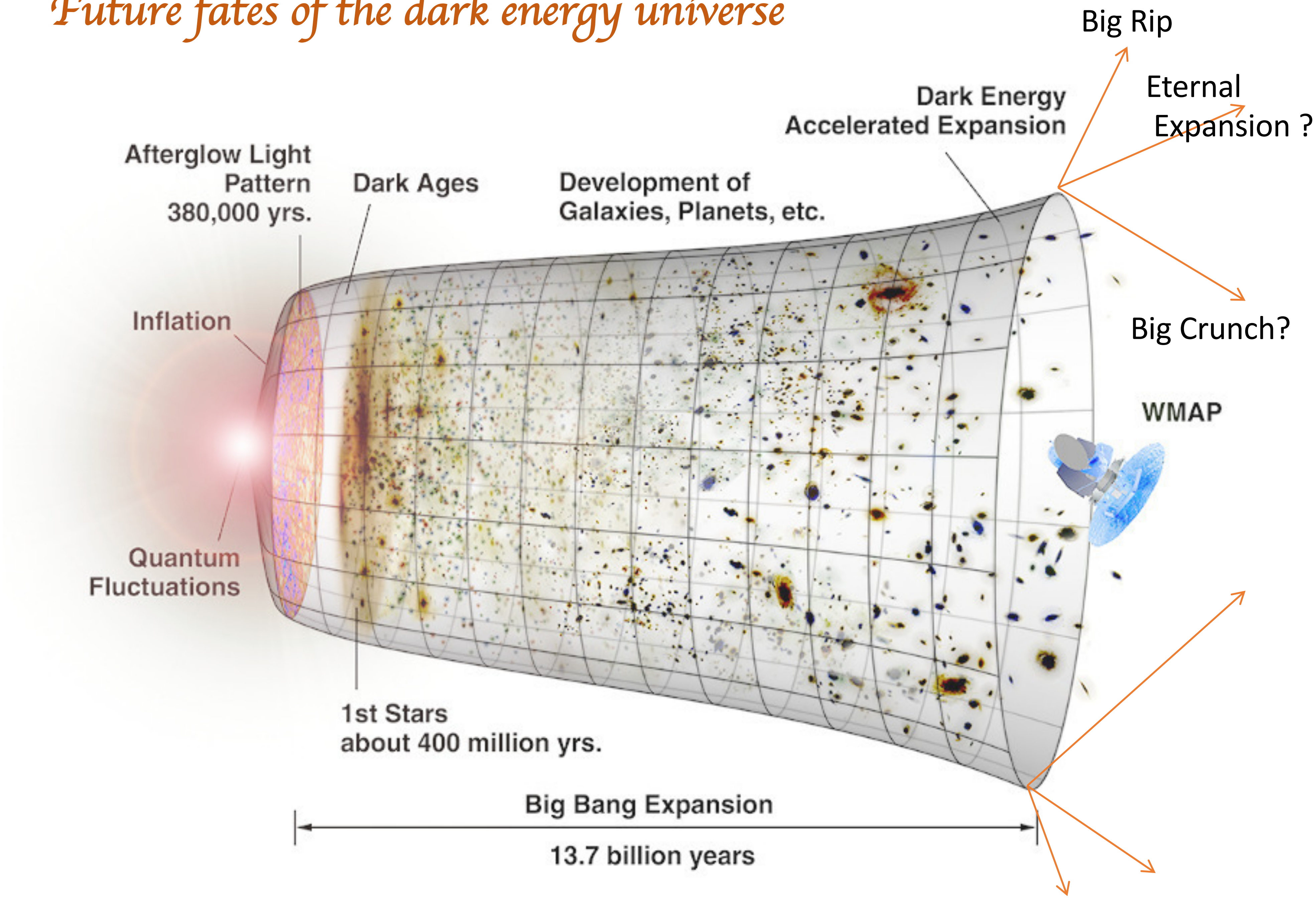
The content of the universe is, up today, absolutely unknown for its largest part. The situation is very "DARK" while the observations are extremely good!

Components of the Universe

The Observed Universe Evolution



Future fates of the dark energy universe



A plethora of theoretical answers!

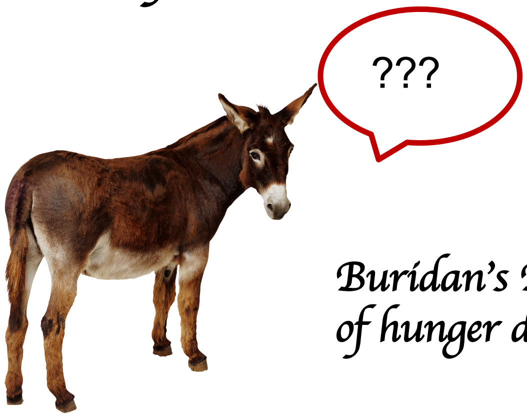
(A tale of unconstrained fantasy)

DARK MATTER

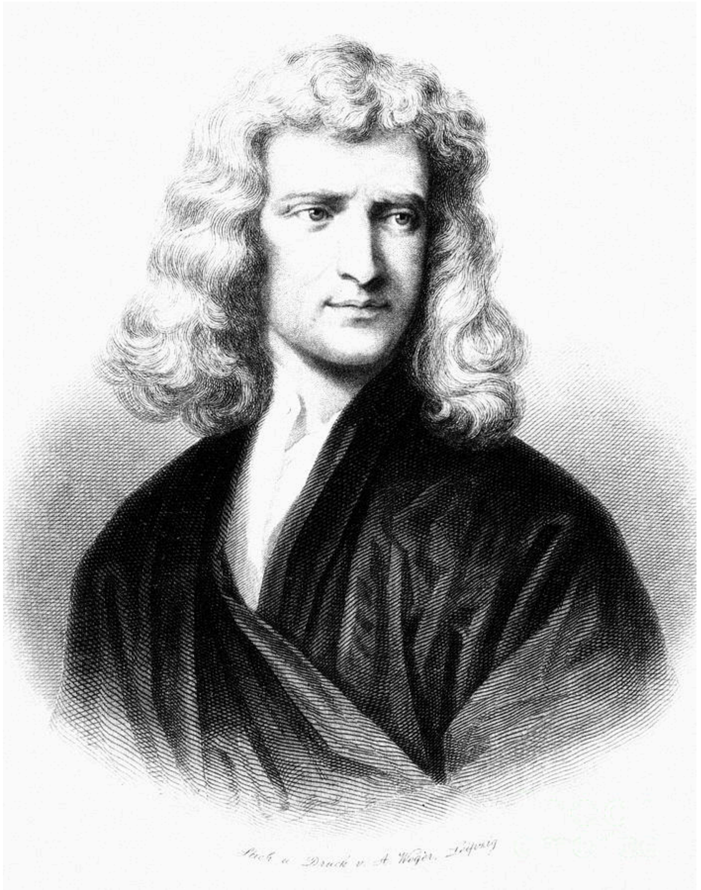
- ✓ Neutrinos
- ✓ WIMPs
- ✓ Wimpzillas,
- ✓ Axions,
- ✓ The "particle forest".....
- ✓ MOND
- ✓ MACHOS
- ✓ Black Holes
- ✓ Neutralinos

DARK ENERGY

- ✓ Cosmological Constant
- ✓ Scalar field Quintessence
- ✓ Phantom fields
- ✓ String-Dilaton scalar field
- ✓ Braneworlds
- ✓ Unified theories



Buridan's Donkey dies
of hunger despite so much food



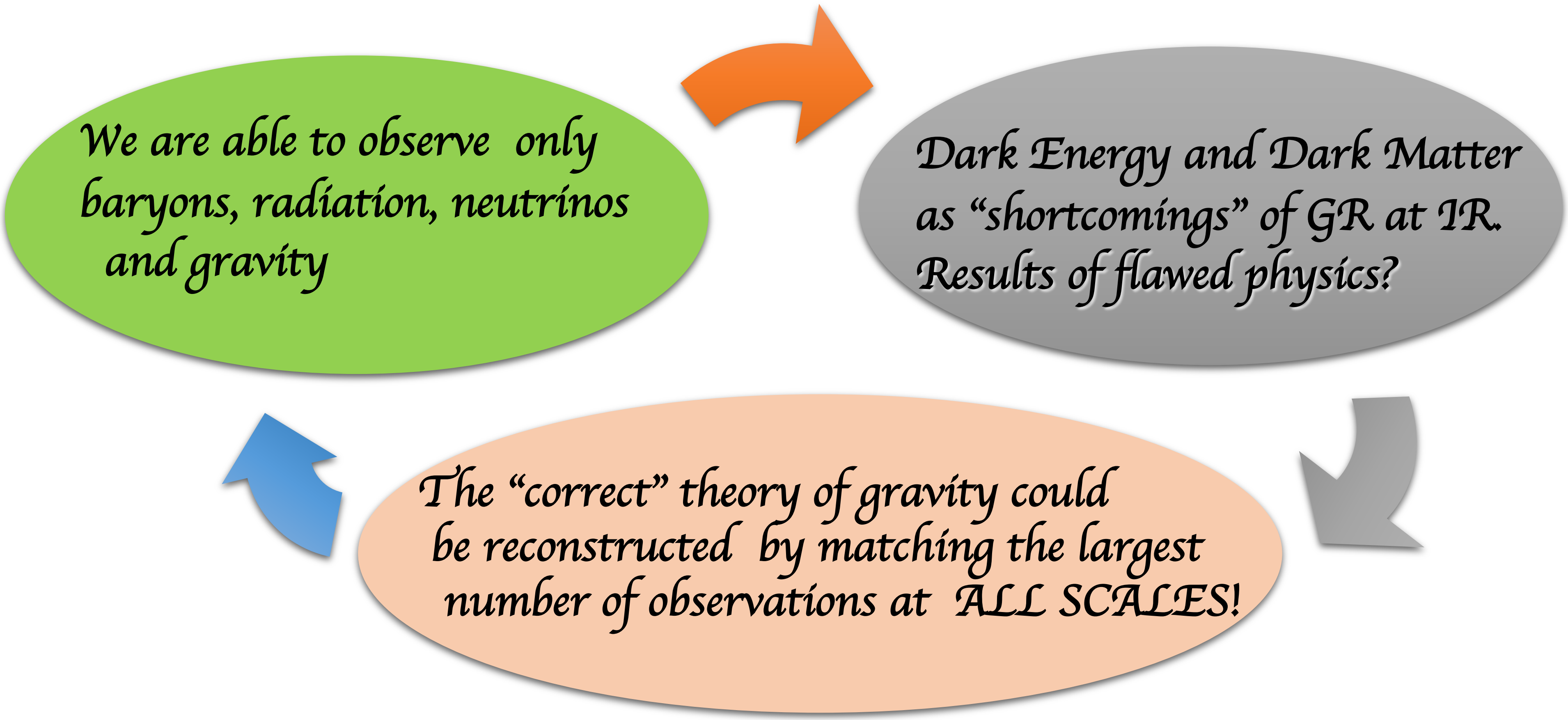
"…there are the ones that invent OCCULT FLUIDS to understand the Laws of Nature. They will come to conclusions, but they now run out into DREAMS and CHIMERAS neglecting the true constitution of things…..
…however there are those that from the simplest observation of Nature, they reproduce New Forces (i.e. New Theories)… "

From the Preface of PRINCIPIA (II Edition) 1687 by Isaac Newton, written by Mr. Roger Cotes

There is a fundamental question:

Are extragalactic observations and cosmology probing the breakdown of General Relativity at large (IR) scales?

The problem could be reversed



Accelerating behaviour (DE) and dynamical phenomena (DM) can be dealt as GEOMETRIC EFFECTS

Why extending General Relativity?

- *No final evidence for Dark Energy and Dark Matter at fundamental level (LHC, astroparticle physics, ground based experiments, LUX, XENON, DAMA,…).*
- *The problem can be framed extending GR at infrared scales.*
- *GR does not work at ultraviolet scales (no Quantum Gravity).*
- *Several issues in modern Astrophysics ask for new paradigms.*
- *ETGs as minimal extension of GR considering Quantum Fields in Curved Spaces*
- ***Big issue: Is it possible to find out probes and test-beds for ETGs?***

FROM WHERE?

- *Geodesic motions around compact objects e.g- SgrA**
- *Torsion experiments*
- *Microgravity experiments from atomic physics*
- *Violation of Equivalence Principle*
- ***PROVA REGINA: Further modes from Gravitational Waves!***

Extending General Relativity

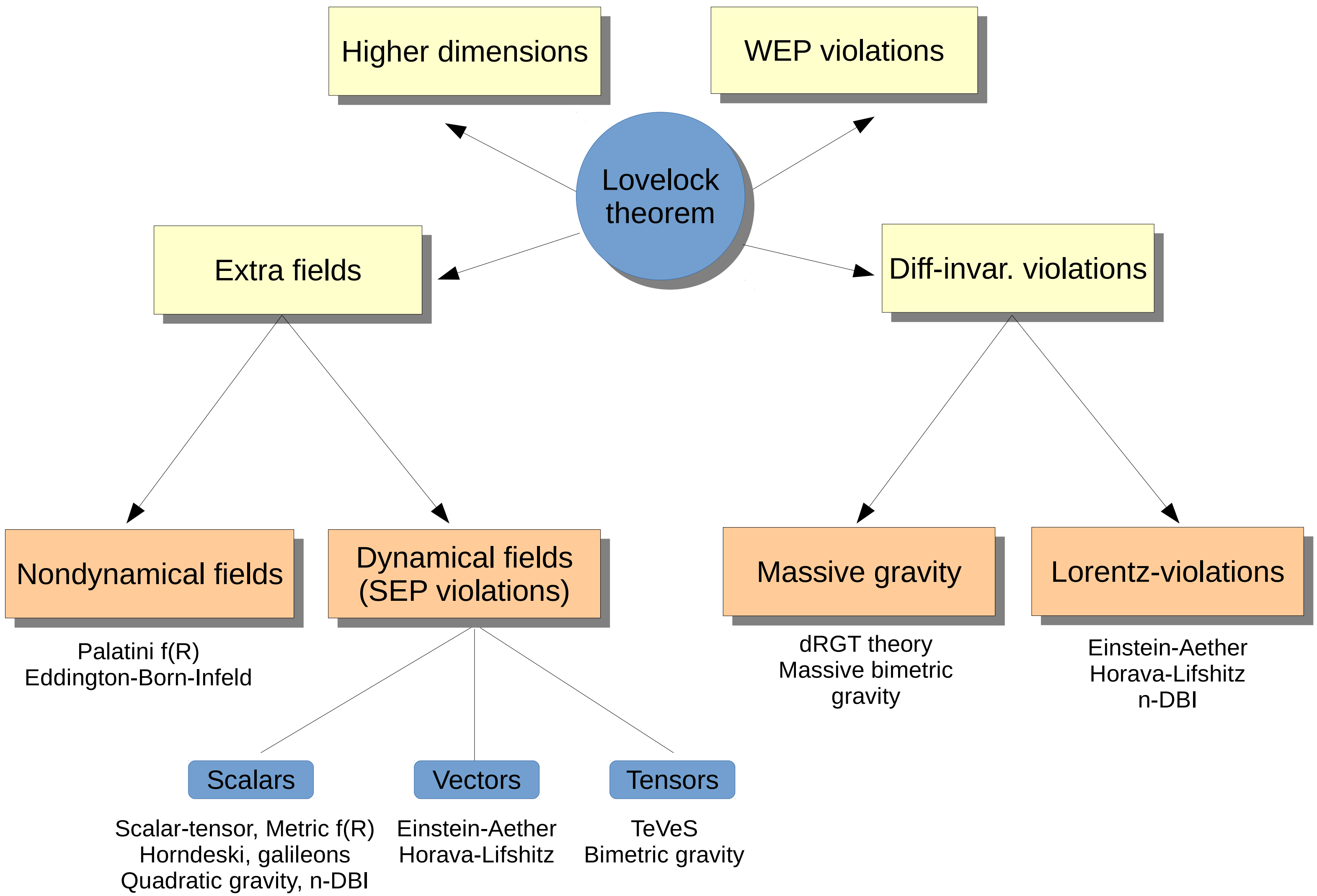
ETGs takes into account two main features in the gravitational action

- Physically motivated scalar fields ;
- Higer order curvature (or torsion) invariants

SCALAR-TENSOR, HIGHER ORDER GRAVITY, GALILEON, HORNDESKI, LOVELOCK, TELEPARALLEL GRAVITY,…..

A. A. Starobinsky, Phys. Lett. B91, 99 (1980).
S. Capozziello, Int. Jou. Mod. Phys. D 11, 483 (2002) .
A. De Felice, S Tsujikawa, Living Rev.Rel. 13 (2010) 3
S. Capozziello, M. De Laurentis, Phys. Rep. 509, 167 (2011).
S. Nojiri, S.D. Odintsov, Phys. Rep. 505, 59 (2011).

Most theories can be reduced to GR +scalar fields by the Lovelock Theorem



E. Berti et al. CQG 32 (2015) 243001

Extending General Relativity

A general class of higher-order-scalar-tensor theories in four dimensions is given by the action

$$\mathcal{S} = \int d^4x \sqrt{-g}\left[F\left(R, \Box R, \Box^2 R, \ldots, \Box^k R, \phi\right) - \frac{\epsilon}{2} g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \mathcal{L}^{(m)}\right]$$

In the metric approach, the field equations are obtained by varying with respect to $g_{\mu\nu}$

$$\begin{aligned} G^{\mu\nu} = \frac{1}{\mathcal{G}}\Bigg[& \kappa T^{\mu\nu} + \frac{1}{2} g^{\mu\nu}(F - \mathcal{G}R) \\ & + \left(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma}\right)\mathcal{G}_{;\lambda\sigma} \\ & + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i}\left(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma}\right)\left(\Box^{j-i}\right)_{;\sigma} \\ & \times \left(\Box^{i-j}\frac{\partial F}{\partial \Box^i R}\right)_{;\lambda} - g^{\mu\nu}g^{\lambda\sigma} \\ & \times \left(\left(\Box^{j-1}R\right)_{;\sigma}\Box^{i-j}\frac{\partial F}{\partial \Box^i R}\right)_{;\lambda}\Bigg], \end{aligned}$$

- $G^{\mu\nu}$ is the Einstein tensor

and

$$\mathcal{G} \equiv \sum_{j=0}^{n}\Box^j\left(\frac{\partial F}{\partial \Box^j R}\right)$$

Extending General Relativity

The simplest extension, f(R) gravity , is achieved assuming in the action

$$\mathcal{F} = f(\mathcal{R}), \quad \varepsilon = 0$$

By varying with respect to $g_{\mu\nu}$, we get

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} = \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)$$

after some manipulations in GR form

$$G_{\mu\nu} = \frac{1}{f'(R)}\left\{\nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) + g_{\mu\nu}\frac{[f(R) - f'(R)R]}{2}\right\}$$

Extending General Relativity

In a perfect-fluid representation

$$G_{\alpha\beta} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\Box f'(R)\right\} + \frac{\kappa T^{(m)}_{\alpha\beta}}{f'(R)} = \underbrace{T^{(\mathrm{curv})}_{\alpha\beta}} + \frac{T^{(m)}_{\alpha\beta}}{f'(R)}$$

In the case of GR, identically vanishes while the standard, minimal coupling is recovered for the matter contribution

it is an effective stress-energy tensor constructed by the extra curvature terms

For f(R) = R, standard GR is restored.

S. Capozziello. C.A. Mantica. L.G. Molinari, IJGMMP 2018

Extending General Relativity

In the same way, one achieves *Scalar-Tensor Gravity*

$$F = F(\phi)R - V(\phi), \qquad \epsilon = -1$$

The variation with respect to $g_{\mu\nu}$ gives the second-order field equations

$$F(\phi)G_{\mu\nu} = F(\phi)\left[R_{\mu\nu} - \frac{1}{2}R_{\mu\nu}\right] = -\frac{1}{2}T^{\phi}_{\mu\nu} - g_{\mu\nu}\Box_g F(\phi) + F(\phi)_{;\mu\nu}$$

The energy-momentum tensor related to the scalar field is

$$T^{\phi}_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{\alpha}_{;} + g_{\mu\nu}V(\phi)$$

The variation with respect to **φ** *provides the Klein–Gordon equation, i.e. the field equation for the scalar field:*

$$\Box_g\phi - RF_{\phi}(\phi) + V_{\phi}(\phi) = 0$$

This last equation is equivalent to the Bianchi contracted identity

Understanding at which scales corrections to General Relativity could work is a crucial point to confirm or rule out any extended/modified model.

The weak field limit

Assuming spherically symmetric metric:

$$ds^2 = g_{\sigma\tau} dx^\sigma dx^\tau$$
$$= g_{00}(x^0, r) dx^{0\,2} - g_{rr}(x^0, r) dr^2 - r^2 d\Omega ,$$

In a Minkowskian background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

Metric entries $\Rightarrow$

$$\begin{cases} g_{tt}(t,r) \simeq 1 + g_{tt}^{(2)}(t,r) + g_{tt}^{(4)}(t,r) , \\ g_{rr}(t,r) \simeq -1 + g_{rr}^{(2)}(t,r) , \\ g_{\theta\theta}(t,r) = -r^2 , \\ g_{\phi\phi}(t,r) = -r^2 \sin^2\theta , \end{cases}$$

The weak field limit

Assuming Taylor expandable f (R) functions with respect to R = R_0:

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R-R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \ldots .$$

In O(2) - order approximation, the field equations in vacuum, results to be

$$\begin{cases} f_1 r R^{(2)} - 2 f_1 g^{(2)}_{tt,r} + 8 f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} + 4 f_2 r R^{(2)} = 0 \,, \\ f_1 r R^{(2)} - 2 f_1 g^{(2)}_{rr,r} + 8 f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} = 0 \,, \\ 2 f_1 g^{(2)}_{rr} - r \\ \times \left[f_1 r R^{(2)} - f_1 g^{(2)}_{tt,r} - f_1 g^{(2)}_{rr,r} + 4 f_2 R^{(2)}_{,r} + 4 f_2 r R^{(2)}_{,rr} \right] = 0 \,, \\ f_1 r R^{(2)} + 6 f_2 \left[2 R^{(2)}_{,r} + r R^{(2)}_{,rr} \right] = 0 \,, \\ 2 g^{(2)}_{rr} + r \left[2 g^{(2)}_{tt,r} - r R^{(2)} + 2 g^{(2)}_{rr,r} + r g^{(2)}_{tt,rr} \right] = 0 \,. \end{cases} \tag{33}$$

The weak field limit

The general solution:

where $\boxed{\xi \doteq \frac{f_1}{6f_2}}$ $\Rightarrow$

$$\begin{cases} g_{tt}^{(2)} = \delta_0 - \dfrac{Y}{f_1 r} - \dfrac{\delta_1(t)e^{-r\sqrt{-\xi}}}{3\xi r} + \dfrac{\delta_2(t)e^{r\sqrt{-\xi}}}{6(-\xi)^{3/2} r} \\ g_{rr}^{(2)} = -\dfrac{Y}{f_1 r} + \dfrac{\delta_1(t)[r\sqrt{-\xi}+1]e^{-r\sqrt{-\xi}}}{3\xi r} \\ -\dfrac{\delta_2(t)[\xi r + \sqrt{-\xi}]e^{r\sqrt{-\xi}}}{6\xi^2 r} \\ R^{(2)} = \dfrac{\delta_1(t)e^{-r\sqrt{-\xi}}}{r} - \dfrac{\delta_2(t)\sqrt{-\xi}e^{r\sqrt{-\xi}}}{2\xi r} \end{cases}$$

For limit $f(R) \to R$, in the case of a point-like source of mass M, we recover the weak field limit of GR

The two arbitrary functions of time $\delta_1(t)$ and $\delta_2(t)$ have respectively the dimensions of length^{-1} and length^{-2} .

The weak field limit

To match at infinity the Minkowskian prescription, one can discard the Yukawa growing mode :

$$\begin{cases} ds^2 = \left[1 - \frac{2GM}{f_1 r} - \frac{\delta_1(t)e^{-r\sqrt{-\xi}}}{3\xi r}\right]dt^2 \\ -\left[1 + \frac{2GM}{f_1 r} - \frac{\delta_1(t)(r\sqrt{-\xi}+1)e^{-r\sqrt{-\xi}}}{3\xi r}\right]dr^2 - r^2 d\Omega, \\ R = \frac{\delta_1(t)e^{-r\sqrt{-\xi}}}{r}. \end{cases}$$

Being $g_{tt} = 1+2\Phi_{grav} = 1+ g(2)_{tt}$, the gravitational potential of $f(R)$ gravity is

$$\boxed{\Phi_{\text{grav}} = -\left(\frac{GM}{f_1 r} + \frac{\delta_1(t)e^{-r\sqrt{-\xi}}}{6\xi r}\right)}$$

The standard Newton potential is recovered only in the particular case $f(R) = R$

The parameters $f_{1,2}$ and the function δ_1 represent the corrections with respect to the standard Newton potential

S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

The weak field limit

The ξ parameter is related to an effective mass

$$m^2 = (3\xi)^{-1} = -\frac{f_1}{3f_2}$$

and can be interpreted also as an effective length L

$$\boxed{\Phi(r) = -\frac{GM}{(1+\delta)r}\left(1+\delta e^{-\frac{r}{L}}\right)}$$

The second term is a modification of the gravity including a scale length

If $\delta = 0$ the Newtonian potential and the standard gravitational coupling are recovered.

Assuming $1+\delta = f_1$, δ is related to $\delta_1(t)$ through

$$\delta_1 = -\frac{6GM}{L^2}\left(\frac{\delta}{1+\delta}\right)$$

Under this assumption, the scale length L could naturally arise and reproduce several phenomena that range from Solar System to cosmological scales depending on the mass and the size of the self-gravitating system

The weak field limit

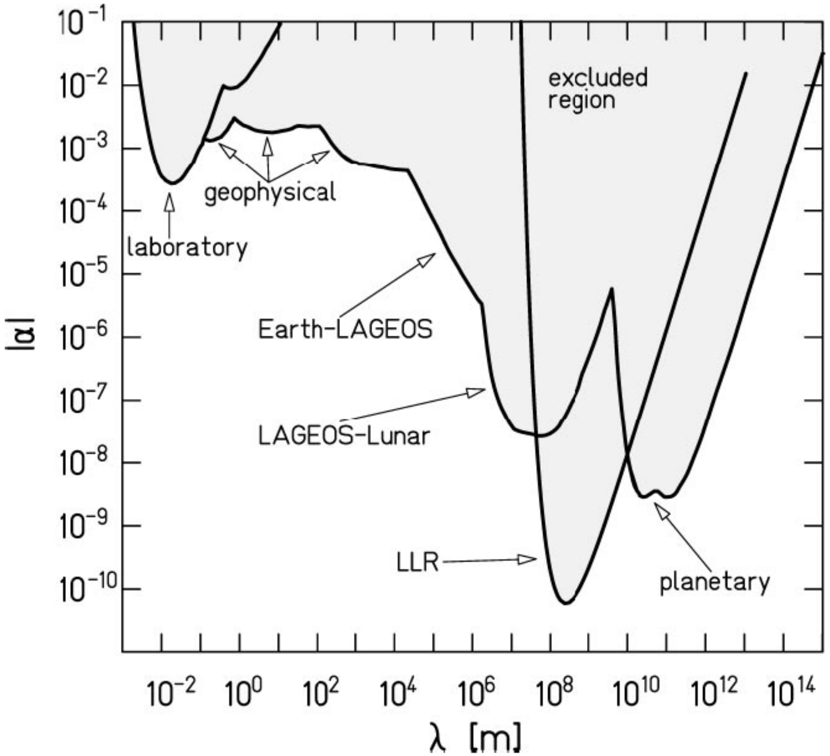
Fifth force

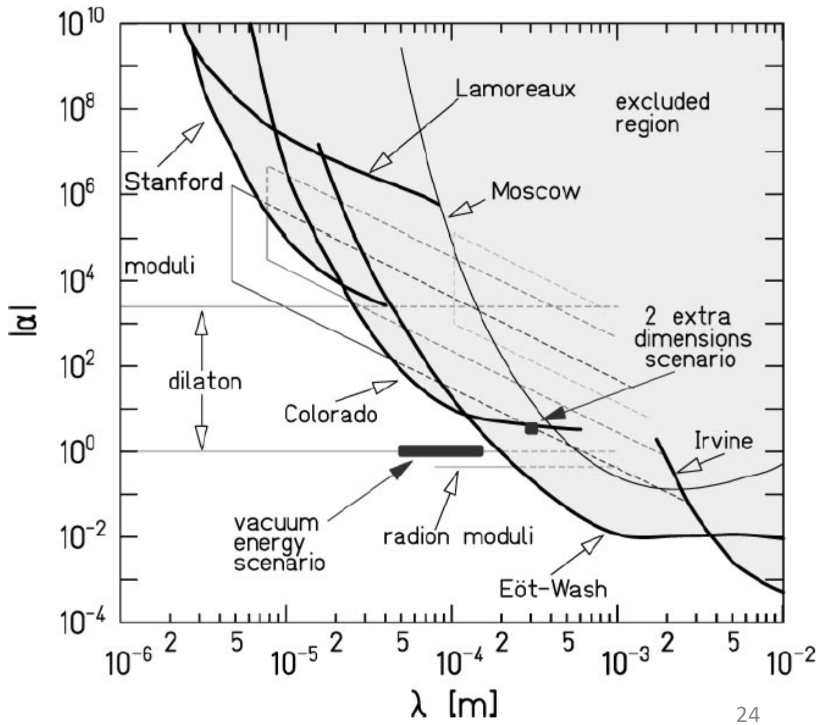
$$V(r) = -G\frac{m_1 m_2}{r}[1 + \alpha e^{-r/\lambda}]$$

α is a dimensionless strength parameter

λ is a length scale or range

Experimental bounds





Stellar structures and Jeans instability

- It is usually assumed that the dynamics of stellar objects is completely determined by the Newton law of gravity
- Considering potential corrections in strong field regimes could be another way to check the viability of Extended Theories of Gravity
- In particular, stellar systems are an ideal laboratory to look for signatures of possible modifications of standard law of gravity
- Some observed stellar systems are incompatible with the standard models of stellar structure: star in instability strips, anomalous neutron stars (e.g. PSRJ 1614-2230), magnetars. e.g. Astashenok, Capozziello, Odintsov JCAP 1312 (2013) 040.

Stellar structures and Jeans instability

Field equations at O (2)-order,

$$R_{tt}^{(2)} - \frac{R^{(2)}}{2} - f''(0) \triangle R^{(2)} = \mathcal{X} T_{tt}^{(0)}$$

$$-3f''(0) \triangle R^{(2)} - R^{(2)} = \mathcal{X} T^{(0)},$$

The energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu - p g_{\mu\nu}.$$

The pressure contribution is negligible in the field equations of Newtonian approximation

$$\triangle \Phi + \frac{R^{(2)}}{2} + f''(0) \triangle R^{(2)} = -\mathcal{X}\rho$$

modified Poisson equation

$$3f''(0) \triangle R^{(2)} + R^{(2)} = -\mathcal{X}\rho,$$

S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

For f''(R) = 0 we have the standard Poisson equation $\triangle \Phi = -4\pi G \rho$

From the Bianchi identity $T^{\mu\nu}{}_{;\mu} = 0 \rightarrow \frac{\partial p}{\partial x^k} = -\frac{1}{2}(p + \epsilon) \frac{\partial \ln g_{tt}}{\partial x^k}.$

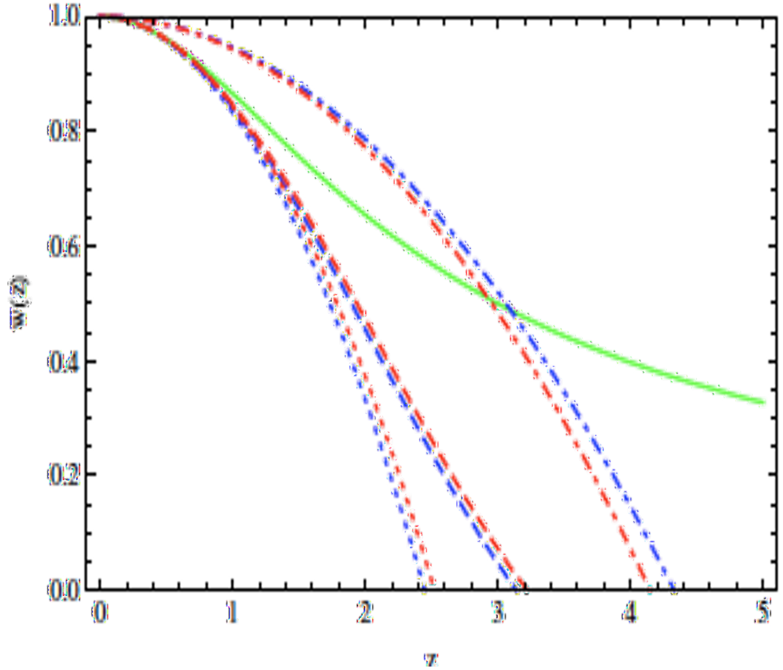
Stellar structures and Jeans instability

Assuming a polytropic equation for matter $p = K^{\gamma} \rho^{\gamma}$

we obtain a Lané-Emden integro-differential equation (standard for f(R) =R)

$$\frac{d^2 w(z)}{dz^2} + \frac{2}{z}\frac{dw(z)}{dz} + w(z)^n = \frac{m\xi_0}{8}\frac{1}{z}\int_0^{\xi/\xi_0} dz'\, z' \left\{ e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|} \right\} w(z')^n$$

Radial profiles for some values of n (polytropic index)



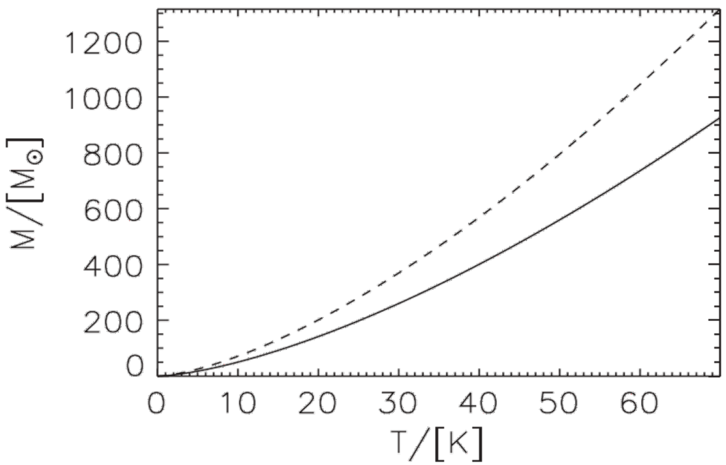
New solutions are physically relevant and could explain exotic systems out of Main Sequence (magnetars, variable stars).

S. Capozziello, M. De Laurentis, A. Stabile, S.D. Odintsov, PRD 83, 064004, (2011)

Stellar structures and Jeans instability

The Jeans mass for various types of interstellar molecular clouds changes

The collapse of an interstellar cloud is affected in a different way with respect to GR



S. Capozziello, M. De Laurentis, I. De Martino, M. Formisano, S.D. Odintsov
Phys.Rev. D85 (2012) 044022

Subject	T (K)	n (10^8 m^{-3})	μ	M_J ($M_\odot$)	$\tilde{M}_J$ ($M_\odot$)
Diffuse hydrogen clouds	50	5.0	1	795.13	559.68
Diffuse molecular clouds	30	50	2	82.63	58.16
Giant molecular clouds	15	1.0	2	206.58	145.41
Bok globules	10	100	2	11.24	7.91

Quadrupolar gravitational radiation

The same approach for gravitational radiation

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R - R_0)^n \simeq f_0 + f_0' R + \frac{1}{2} f_0'' R^2 + \cdots$$

Field equations at the first order of approximation

$$f_0' \left[R^{(1)}_{\mu\nu} - \frac{R^{(1)}}{2}\eta_{\mu\nu} \right] - f_0'' \left[R^{(1)}_{,\mu\nu} - \eta_{\mu\nu} \Box R^{(1)} \right] = \frac{\mathcal{X}}{2} T^{(0)}_{\mu\nu}$$

The Ricci tensor and scalar

$$\begin{cases} R^{(1)}_{\mu\nu} = h^{\sigma}_{(\mu,\nu)\sigma} - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2} h_{,\mu\nu} \\ R^{(1)} = h^{;\sigma\tau}_{\sigma\tau} - \Box h \end{cases}$$

S. Capozziello, M. De Laurentis, Phys. Rep. 509, 167 (2011)

M. De Laurentis, S. Capozziello, Astroparticle Physics 35 , 257 (2011)

Quadrupolar gravitational radiation

In vacuum, with suitable gauge, it is $R^{(1)}_{\mu\nu} = \Box h_{\mu\nu} = 0$

The Landau-Lifshitz gravitational energy momentum tensor is

$$t^\lambda_\alpha = f'\left\{\left[\frac{\partial R}{\partial g_{\rho\sigma,\lambda}} - \frac{1}{\sqrt{-g}}\partial_\xi\left(\sqrt{-g}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}\right)\right]g_{\rho\sigma,\alpha} + \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\right\} - f''R_{,\xi}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\alpha} - \delta^\lambda_\alpha\, f$$

$$t^\lambda_\alpha = f'_0 t^\lambda_{\alpha}{}_{|\mathrm{GR}} + f''_0 t^\lambda_{\alpha}{}_{|f(R)}$$

M. De Laurentis, S. Capozziello, Astroparticle Physics 35 , 257 (2011)

Quadrupolar gravitational radiation

...in term of the h perturbation, it is

$$
t^{\lambda}_{\alpha} \sim f'_0 t^{\lambda}_{\alpha}\big|_{GR} + f''_0 \Big\{ (h^{\rho\sigma}_{,\rho\sigma} - \Box h) \Big[h^{\lambda\xi}_{,\xi\alpha} - h^{,\lambda}_{\alpha} - + \frac{1}{2}\delta^{\lambda}_{\alpha}(h^{\rho\sigma}_{,\rho\sigma} - \Box h) \Big]
$$

$$
- h^{\rho\sigma}_{,\rho\sigma\xi} h^{\lambda\xi}_{,\alpha} + h^{\rho\sigma}_{,\rho\sigma}{}^{,\lambda} h_{,\alpha} + h^{\lambda\xi}_{,\alpha} \Box h_{,\xi} - \Box h^{,\lambda} h_{,\alpha} \Big\}.
$$

In the weak field limit, the source $h_{\mu\nu}$ is written as function of time t' = t – r, and plane wave approximation

the energy momentum tensor assumes the form:

$$
\boxed{t^{\lambda}_{\alpha} = \underbrace{f'_0 k^{\lambda} k_{\alpha} \left(\dot{h}^{\rho\sigma} \dot{h}_{\rho\sigma} \right)}_{GR} - \underbrace{\frac{1}{2} f''_0 \delta^{\lambda}_{\alpha} \left(k_{\rho} k_{\sigma} \ddot{h}^{\rho\sigma} \right)^2}_{f(R)}}
$$

M. De Laurentis, S. Capozziello, Astroparticle Physics 35 , 257 (2011)
De Laurentis M., De Martino I., 2013, MNRAS., doi:10.1093/mnras/stt216

Quadrupolar gravitational radiation

The average energy flux dE/dt away from the systems and the momenta of the mass-energy distribution

$$\underbrace{\left\langle \frac{dE}{dt} \right\rangle}_{(total)} = \frac{G}{60} \left\langle \underbrace{f_0' \left(\dddot{Q}^{ij} \dddot{Q}_{ij} \right)}_{GR} - \underbrace{f_0'' \left(\ddddot{Q}^{ij} \ddddot{Q}_{ij} \right)}_{f(R)} \right\rangle$$

for $f''_0 = 0$ and $f'_0 = 4/3$

$$\underbrace{\left\langle \frac{dE}{dt} \right\rangle}_{(GR)} = \frac{G}{45} \left\langle \dddot{Q}^{ij} \dddot{Q}_{ij} \right\rangle$$

In terms of massive mode

$$\underbrace{\left\langle \frac{dE}{dt} \right\rangle}_{(total)} = \frac{G f_0'}{60} \left\langle \left(\dddot{Q}^{ij} \dddot{Q}_{ij} \right) - \frac{1}{m^2} \left(\ddddot{Q}^{ij} \ddddot{Q}_{ij} \right) \right\rangle$$

This could represent a signature to investigate such theories in the GW strong-field regime.

Application to the binary systems

Assuming Keplerian motion and the orbit in the (x; y)-plane
the quadrupole matrix is

$$Q_{ij} = \mu r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix}_{ij}$$

the time average of the radiated power

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{T}\int_0^T dt \frac{dE(\psi)}{dt} = \frac{1}{T}\int_0^{2\pi} \frac{d\psi}{\dot\psi}\frac{dE(\psi)}{dt}$$ where $$\dot\psi = \left(\frac{Gm_c}{a^3}\right)^{\frac{1}{2}} (1-\epsilon^2)^{-\frac{3}{2}} (1+\epsilon\cos\psi)^2$$

The time derivative of the orbital period

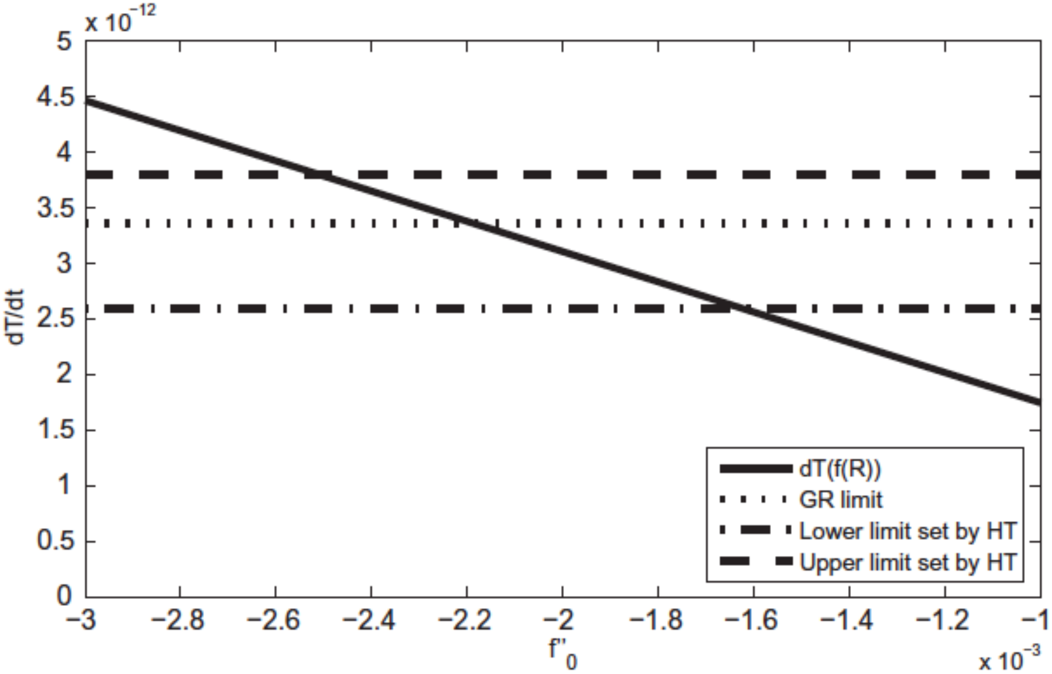
$$\boxed{\begin{aligned} \dot T_b = & -\frac{3}{20}\left(\frac{T}{2\pi}\right)^{-\frac{5}{3}} \frac{\mu G^{\frac{5}{3}}(m_c+m_p)^{\frac{2}{3}}}{c^5(1-n^2)^{\frac{7}{2}}} \times \\ & \times \left[f'_0\,(37\epsilon^4 + 292\epsilon^2 + 96) - \frac{f''_0 \pi^2 T^{-1}}{2(1+n^2)^3} \times \right. \\ & \left. \times\ (891\epsilon^8 + 28016\epsilon^6 + 82736\epsilon^4 + 43520\epsilon^2 + 3072) \right] \end{aligned}}$$

$f(R)$ can be constrained comparing with data

Application to the binary systems: The PSR 1913 + 16 case

Using the values for the specific example of PSR 1913 + 16 to numerically evaluate the above equations

PSR 1913 + 16	Chacteristic features
Pulsar mass	$m = 1.39M_{\odot}$
Companion mass	$M = 1.44M_{\odot}$
Inclination angle	$\sin i = 0.81$
Orbit semimajor axis	$a = 8.67 \times 10^{10}$ cm
Eccentricity	$\epsilon = 0.617155$
Gravitational constant	$G = 6.67 \times 10^{-8}$ dyn cm^2 g^{-2}
Speed of light	$c = 2.99 \times 10^{10}$ cm s^{-1}



Orbital decay rate for PSR 1913 + 16 in f(R)-gravity. Upper limit set by Taylor et al. in dashed line. GR limit 3.36×10^{-12} in dotted line and the lower limit set by Taylor et al. in dashdot line. Solid line is $dT_{f(R)}$

A class of f(R) agrees with data!

Extended Theories of Gravity can impact on DM properties at galactic scales

Testing spiral galaxies

Yukawa-like corrections are a general feature in the framework of $f(R)$-gravity

The potential

$$\boxed{\Phi(r) = -\frac{GM}{(1+\delta)r}\left(1+\delta e^{-\frac{r}{L}}\right)}$$

is the starting point for the computation of the rotation curve of an extended system.

R.H. Sanders, Astron. Astrophys. 136 (1984) L21
A. Stabile and S. Capozziello, Phys. Rev. D 87 (2013) 064002

Testing spiral galaxies

Using cylindrical coordinates (R,θ, z) and the corresponding dimensionless variables (η,θ,ζ) (with ζ = z/rs), the total force then reads:

$$F(\mathbf{r}) = \frac{G\rho_0 r_s}{1+\delta}\int_0^\infty \eta' d\eta' \int_{-\infty}^{\infty} d\zeta' \int_0^\pi f_r(\Delta)\tilde{\rho}(\eta',\theta',\zeta')d\theta'$$

with ˜ρ = ρ/ρo, ρo a reference density, we have

$$\Delta = \left[\eta^2 + \eta'^2 - 2\eta\eta'\cos(\theta-\theta') + (\zeta-\zeta')^2\right]^{1/2}$$

Circular velocity

$$v_c^2(R) = \frac{G\rho_0 R_d^2 \eta}{1+\delta}\int_0^\infty \eta' d\eta' \int_{-\infty}^{\infty} \tilde{\rho}(\eta',\zeta')d\zeta' \int_0^\pi f_r(\Delta_0)d\theta'$$

S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

Testing spiral galaxies

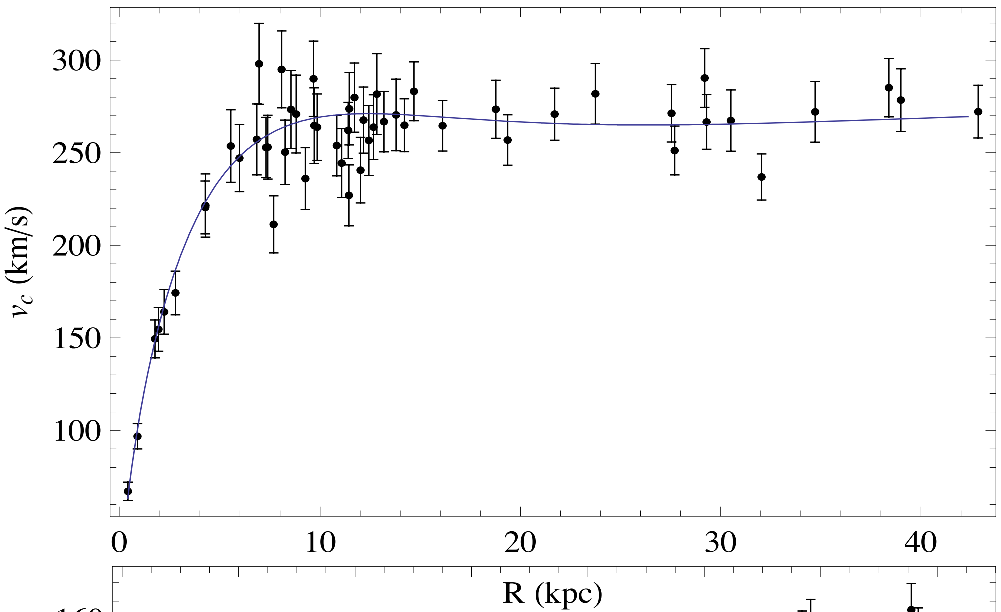
The total rotation curve is:

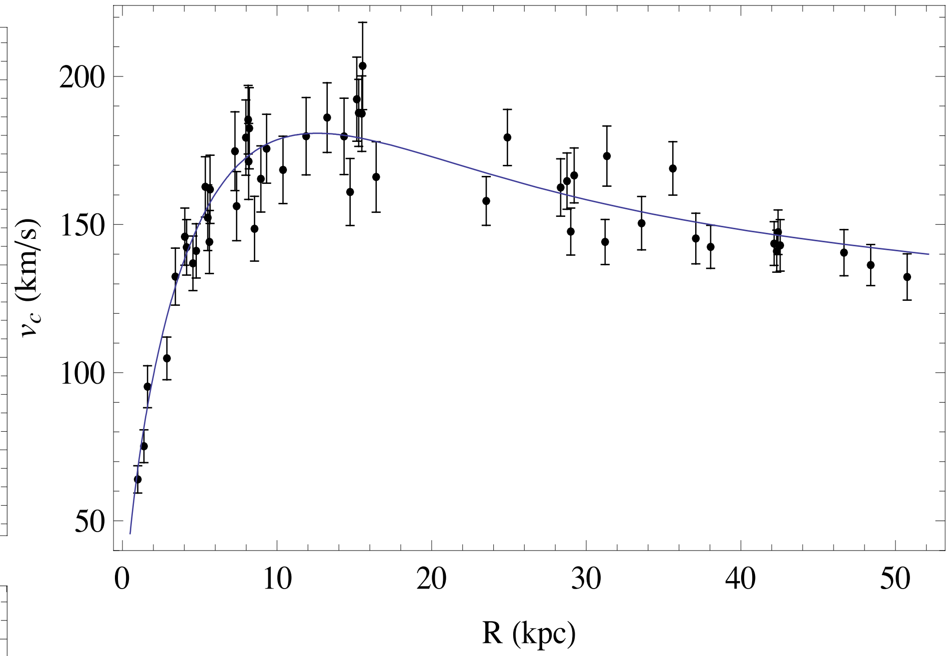
$$v_c^2(R, M_d, \mathbf{p}_i) = v_{dN}^2(R, M_d) + v_{hN}^2(R, \mathbf{p}_i) + v_{dY}^2(R, M_d) + v_{hY}^2(R, \mathbf{p}_i)$$

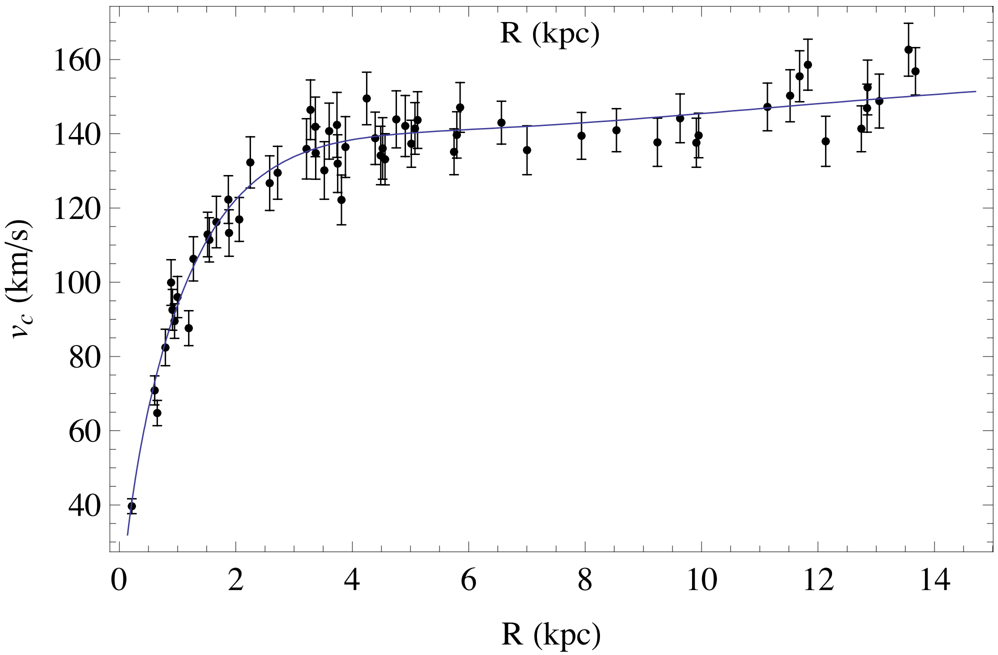
M_d *is the disc mass,* d *and* h *denote disc and halo related quantities, while* N *and* Y *refer to the Newtonian and Yukawa-like contributions*

One may model a spiral galaxy as the sum of a thick disc and a spherical halo without DM contribution.

Testing spiral galaxies







Perfect congruence between theoretical curves and data!

V.F. Cardone, S. Capozziello MNRAS 414 (2011)1301

A big issue: can Elliptical and Spiral Galaxies be addreessed with the same type of DM?

Testing elliptical galaxies

- *The modified potential can be tested also for elliptical galaxies checking whether it is able to provide a reasonable match with their kinematics.*

- *Such self-gravitating systems are very different with respect to spirals so addressing both classes of objects under the same standard could be a fundamental step versus DM*

- *One may construct equilibrium models based on the solution of the radial Jeans equation to interpret the kinematics of planetary nebulae*

- *We use the inner long slit data and the extended planetary nebulae kinematics for three galaxies within DM halo framework*
- *(see Napolitano, Capozziello, Capaccioli, Romanowski ApJ 748 (2012) 87).*

NGC 3379 , (DL +09) , NGC 4494 N +09 , NGC 4374 (N + 11).

Testing elliptical galaxies

Circular velocity as a function of the potential parameters L and **δ** *for NGC 4494 and NGC 4374.*

From a theoretical point of view, **δ** *is a free parameter that can assume positive and negative values. Comparing results for spirals and ellipticals, it is clear that the morphology of these two classes of systems strictly depends on the sign and the value of* **δ**.

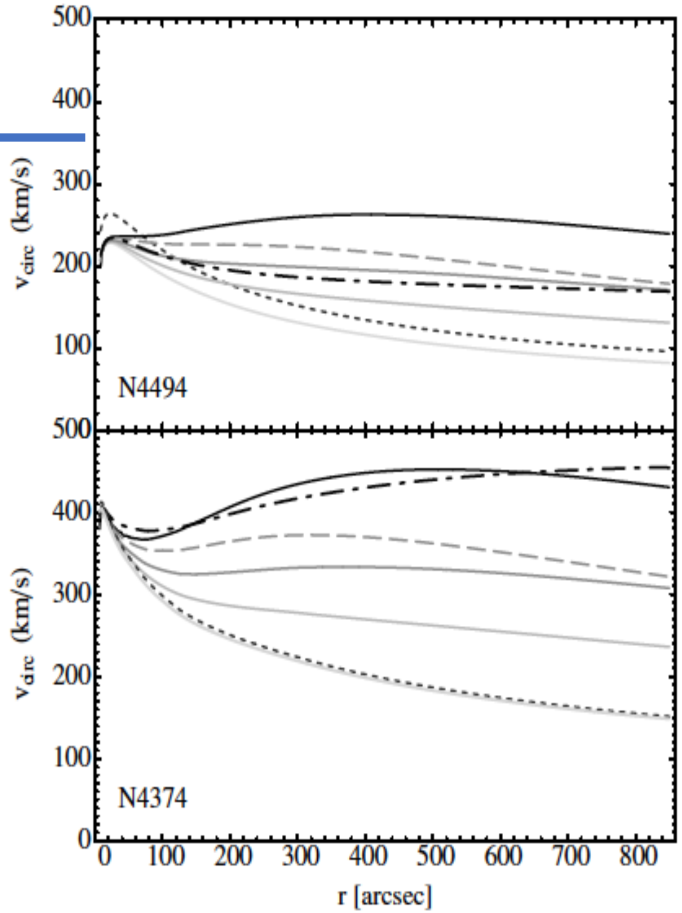


Figure 6 Circular velocity produced by the modified potential for the two galaxies N4494 (top) and N4374 (bottom). In both cases the $M/\mathscr{L}_*$ has been fixed to some fiducial value (as expected from stellar population models and Kroupa 2001 IMF): $M/\mathscr{L}_* = 4.3\Upsilon_{\odot,B}$ for NGC 4494 and $M/\mathscr{L}_* = 5.5\Upsilon_{\odot,V}$ for NGC 4374. The potential parameters adopted are: $L = 250''$ and δ=0, -0.65, -0.8, -0.9 (lighter to darker solid lines) and $L = 180''$ and δ=-0.8 (dashed lines). The dotted line is a case with positive coefficient of the Yukawa-like term and $L = 5000''$ which illustrates that positive δ cannot produce flat circular velocity curves. Finally some reference Navarro-Frenk-White (NFW) models are shown as dot-dashed lines [108].

Testing elliptical galaxies

The match of the model curves with data is remarkably good and it is comparable with models obtained with DM modeling (gray lines)

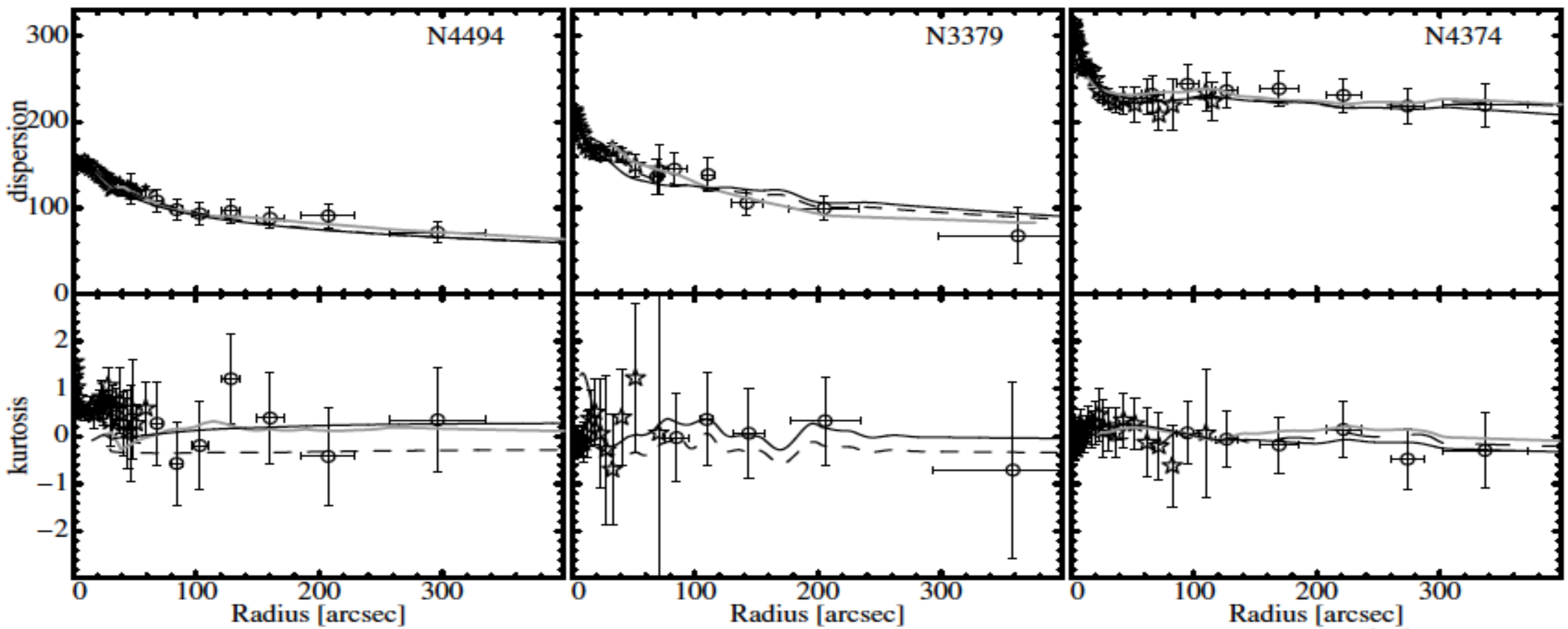


Figure 7 Dispersion in kms (top) and kurtosis (bottom) fit of the galaxy sample for the different $f(R)$ parameter sets: the anisotropic solution (solid lines) is compared with the isotropic case (dashed line – for NGC 4374 and NGC 4494 this is almost indistinguishable from the anisotropic case). From the left, NGC 4494, NGC 3379 and NGC 4374 are shown with DM models as gray lines from N+09, DL+09 (no kurtosis is provided), and N+11 respectively [108].

Testing elliptical galaxies

The marginalized confidence contours of the main two potential parameters for the three considered elliptical galaxies

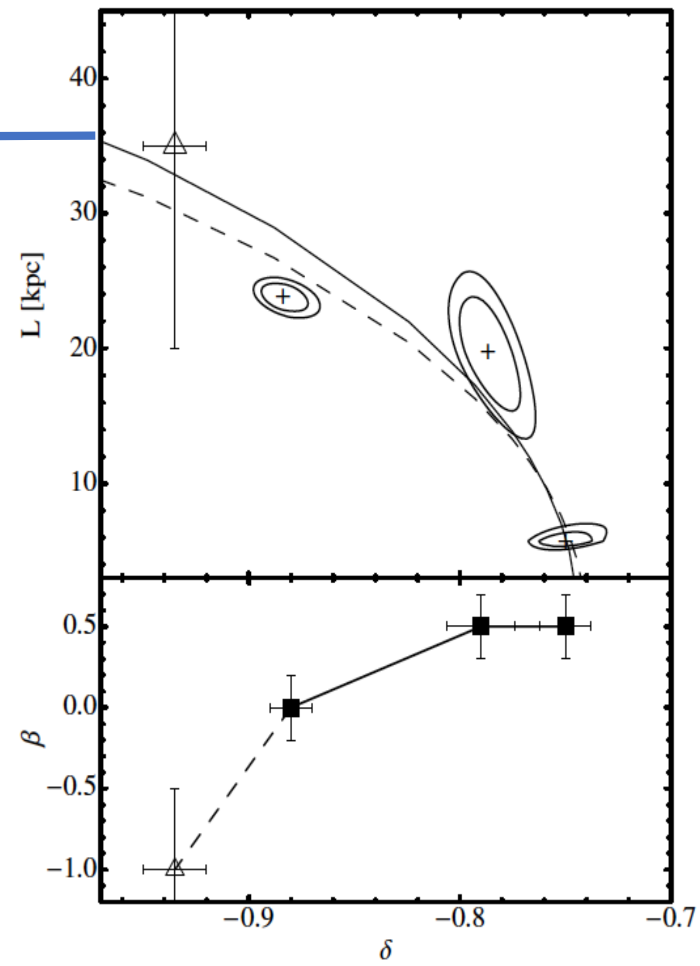


Figure 8 Top: 1- and 2-σ confidence levels in the $\delta - L$ space marginalized over $M/\mathscr{L}_\star$ and β (see also Table 4). Spiral galaxy results from [105] are shown as empty triangle with error bars. Solid (dashed) curve shows the tentative best-fit to the data including (excluding) the spiral galaxies and assuming a $L \propto \sqrt{\delta/(1+\delta)}$. Bottom: the anisotropy and the δ parameters turn out to be correlated for the elliptical sample (full squares). This correlation seems to include also the spiral sample cumulatively shown as the empty triangle (here we have assumed $\beta = -1.0 \pm 0.5$ as a fiducial value for spiral galaxies to draw a semi-quantitative trend across galaxy types) [108].

The results can have interesting implications on the capability of the theory of making predictions on the internal structure of the gravitating systems after their spherical collapse. However, this possibility has to be confirmed on larger galaxy samples

Modeling clusters of galaxies

Modeling clusters of galaxies

- A fundamental issue is related to clusters and superclusters of galaxies.
- These systems rule the large scale structure and are the intermediate step between galaxies and cosmology.
- As galaxies, they appear DM dominated but the distribution of DM component seems clustered and organized in a very different way with respect to galaxies. It seems that DM clustering is governed by the scale and also its fundamental nature could depend on the scale
- The goal is to reconstruct the cluster mass profiles without DM adopting the above strategy where DM effects are figured out by corrections to the Newton potential

Modeling clusters of galaxies

Standard Cluster Model: spherical mass distribution in hydrostatic equilibrium

- Boltzmann equation:

$$-\frac{d\Phi}{dr} = \frac{kT(r)}{\mu m_p r}\left[\frac{d\ln\rho_{gas}(r)}{d\ln r} + \frac{d\ln T(r)}{d\ln r}\right]$$

- Newton classical approach:

$$\begin{cases} \phi(r) = -\frac{GM}{r} \\ \rho_{cl,EC}(r) = \rho_{dark} + \rho_{gas}(r) + \rho_{gal}(r) + \rho_{CDgal}(r) \end{cases}$$

- f(R) approach:

$$\begin{cases} \phi(r) = -\frac{3GM}{4a_1 r}\left(1 + \frac{1}{3}e^{-\frac{r}{L}}\right) \\ \rho_{cl,EC}(r) = \rho_{gas}(r) + \rho_{gal}(r) + \rho_{CDgal}(r) \end{cases}$$

- Rearranging the Boltzmann equation:

$$\phi_N(r) = -\frac{3GM}{4a_1 r}$$

$$\phi_C(r) = -\frac{GM}{4a_1}\frac{e^{-\frac{r}{L}}}{r}$$

$$\begin{cases} M_{bar,th}(r) = \frac{4a_1}{3}\left[-\frac{kT(r)}{\mu m_p G}r\left(\frac{d\ln\rho_{gas}(r)}{d\ln r} + \frac{d\ln T(r)}{d\ln r}\right)\right] - \frac{4a_1}{3G}r^2\frac{\mathrm{d}\Phi_C}{\mathrm{d}r}(r) \\ M_{bar,obs}(r) = M_{gas}(r) + M_{gal}(r) + M_{CDgal}(r) \end{cases}$$

Modeling clusters of galaxies

Fitting mass Profile with data:

- *Sample: 12 clusters from Chandra (Vikhlinin 2005, 2006)*
- *Temperature profile from spectroscopy*
- *Gas density: modified beta-model*

$$n_p n_e = n_0^2 \cdot \frac{(r/r_c)^{-\alpha}}{(1+r^2/r_c^2)^{3\beta-\alpha/2}} \cdot \frac{1}{(1+r^\gamma/r_s^\gamma)^{\epsilon/\gamma}} + \frac{n_{02}^2}{(1+r^2/r_{c2}^2)^{3\beta_2}}$$

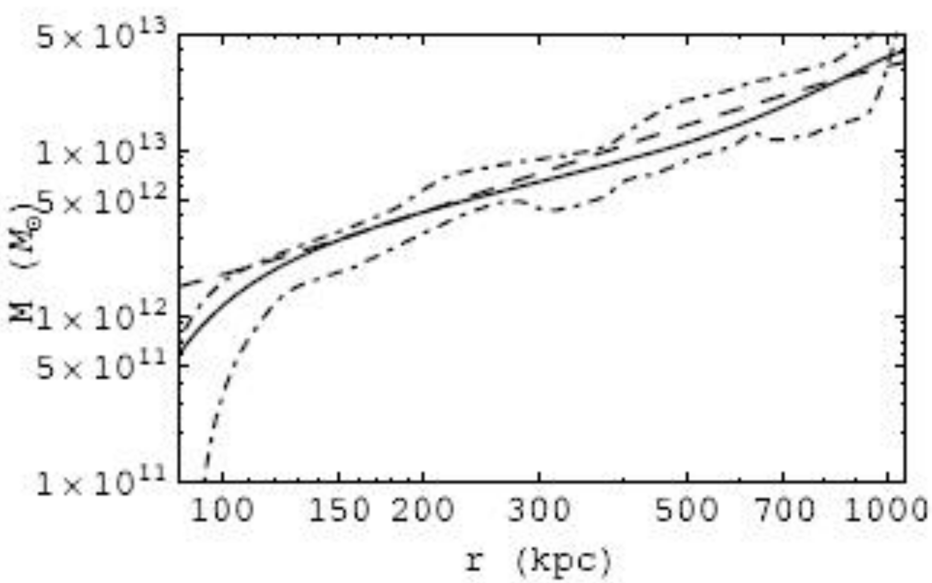
- *Galaxy density:*

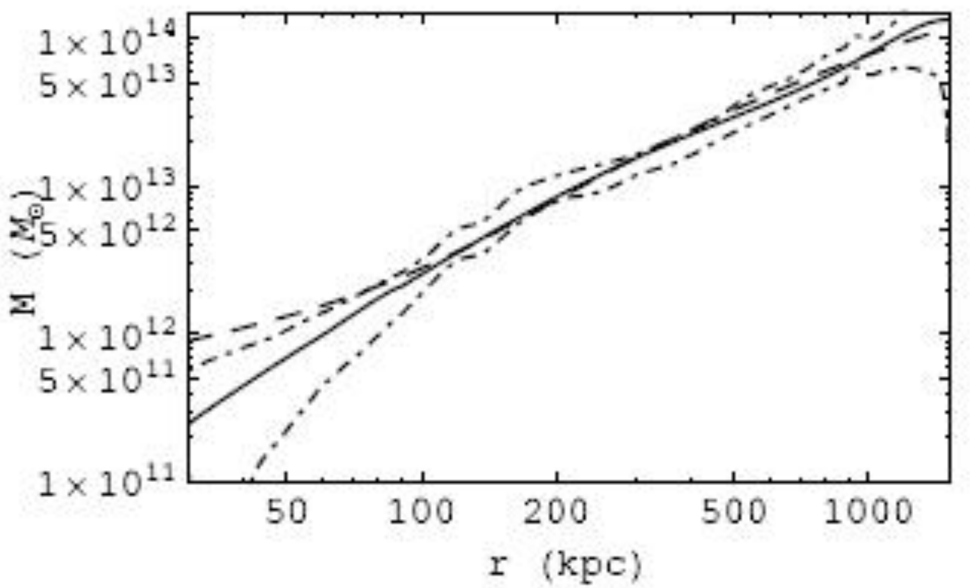
$$\rho_{gal}(r) = \begin{cases} \rho_{gal,1} \cdot \left[1+\left(\frac{r}{R_c}\right)^2\right]^{-\frac{3}{2}} & r < R_c \\ \rho_{gal,2} \cdot \left[1+\left(\frac{r}{R_c}\right)^2\right]^{-\frac{2.6}{2}} & r > R_c \end{cases} \qquad \rho_{CDgal} = \frac{\rho_{0,J}}{\left(\frac{r}{r_c}\right)^2 \left(1+\frac{r}{r_c}\right)^2}$$

Table 1. Column 1: Cluster name. Column2: Richness. Column 2: cluster total mass. Column 3: gas mass. Column 4: galaxy mass. Column 5: cD-galaxy mass. All mass values are estimated at $r = r_{max}$. Column 6: ratio of total galaxy mass to gas mass. Column 7: minimum radius. Column 8: maximum radius.

name	R	$M_{cl,N}$ ($M_\odot$)	M_{gas} ($M_\odot$)	M_{gal} ($M_\odot$)	M_{cDgal} ($M_\odot$)	$\frac{gal}{gas}$	r_{min} (kpc)	r_{max} (kpc)
A133	0	$4.35874 \cdot 10^{14}$	$2.73866 \cdot 10^{13}$	$5.20269 \cdot 10^{12}$	$1.10568 \cdot 10^{12}$	0.23	86	1060
A262	0	$4.45081 \cdot 10^{13}$	$2.76659 \cdot 10^{12}$	$1.71305 \cdot 10^{11}$	$5.16382 \cdot 10^{12}$	0.25	61	316
A383	2	$2.79785 \cdot 10^{14}$	$2.82467 \cdot 10^{13}$	$5.88048 \cdot 10^{12}$	$1.09217 \cdot 10^{12}$	0.25	52	751
A478	2	$8.51832 \cdot 10^{14}$	$1.05583 \cdot 10^{14}$	$2.15567 \cdot 10^{13}$	$1.67513 \cdot 10^{12}$	0.22	59	1580
A907	1	$4.87657 \cdot 10^{14}$	$6.38070 \cdot 10^{13}$	$1.34129 \cdot 10^{13}$	$1.66533 \cdot 10^{12}$	0.24	563	1226
A1413	3	$1.09598 \cdot 10^{15}$	$9.32466 \cdot 10^{13}$	$2.30728 \cdot 10^{13}$	$1.67345 \cdot 10^{12}$	0.26	57	1506
A1795	2	$1.24313 \cdot 10^{14}$	$1.00530 \cdot 10^{13}$	$4.23211 \cdot 10^{12}$	$1.93957 \cdot 10^{12}$	0.11	79	1151
A1991	1	$1.24313 \cdot 10^{14}$	$1.00530 \cdot 10^{13}$	$1.24608 \cdot 10^{12}$	$1.08241 \cdot 10^{12}$	0.23	55	618
A2029	2	$8.92392 \cdot 10^{14}$	$1.24129 \cdot 10^{14}$	$3.21543 \cdot 10^{13}$	$1.11921 \cdot 10^{12}$	0.27	62	1771
A2390	1	$2.09710 \cdot 10^{15}$	$2.15726 \cdot 10^{14}$	$4.91580 \cdot 10^{13}$	$1.12141 \cdot 10^{12}$	0.23	83	1984
MKW4	-	$4.69503 \cdot 10^{13}$	$2.83207 \cdot 10^{12}$	$1.71153 \cdot 10^{11}$	$5.29855 \cdot 10^{11}$	0.25	60	434
RXJ1159	-	$8.97997 \cdot 10^{13}$	$4.33256 \cdot 10^{12}$	$7.34414 \cdot 10^{11}$	$5.38799 \cdot 10^{11}$	0.29	64	568

Modeling clusters of galaxies





- *Differences between theoretical and observed fit less than 5%*
- *Typical scale in [100; 150] kpc range where is a turning-point:*
  - *Break in the hydrostatic equilibrium*
  - *Limits in the expansion series of f(R):* $R - R_0 << \frac{a_1}{a_2}$ *in the range [19;200] kpc*

  Proper gravitational scale (as for galaxies, see Capozziello et al MNRAS 2007)

  - *Similar issues in Metric-Skew-Tensor-Gravity (Brownstein, 2006): we have better and more detailed approach*

Modeling clusters of galaxies

Results

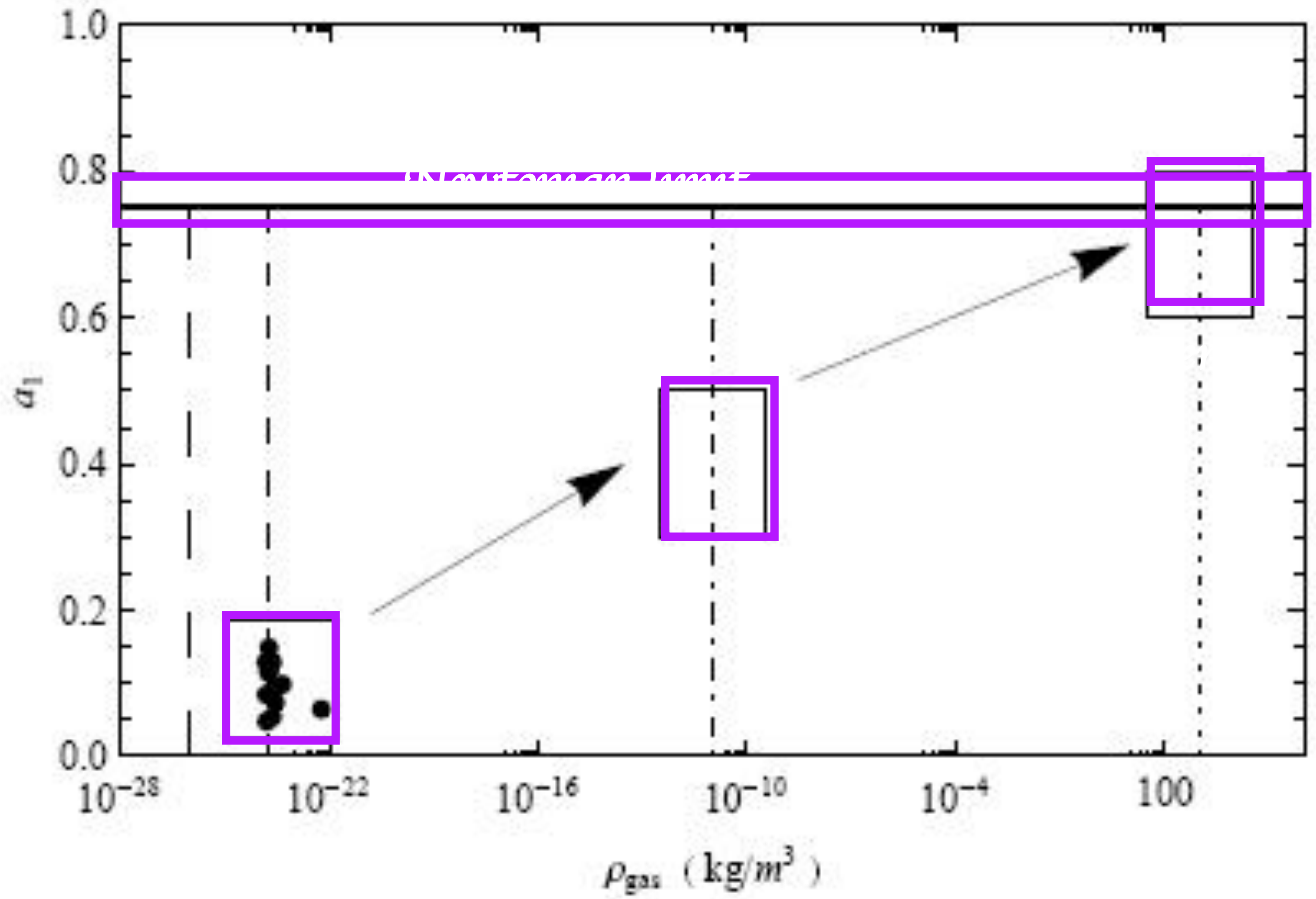
name	a_1	$[a_1 - 1\sigma,\ a_1 + 1\sigma]$	a_2 (kpc^2)	$[a_2 - 1\sigma,\ a_2 + 1\sigma]$ (kpc^2)	L (kpc)	$[L - 1\sigma,\ L + 1\sigma]$ (kpc)
A133	0.085	[0.078, 0.091]	$-4.98 \cdot 10^3$	$[-2.38 \cdot 10^4,\ -1.38 \cdot 10^3]$	591.78	[323.34, 1259.50]
A262	0.065	[0.061, 0.071]	-10.63	$[-57.65,\ -3.17]$	31.40	[17.28, 71.10]
A383	0.099	[0.093, 0.108]	$-9.01 \cdot 10^2$	$[-4.10 \cdot 10^3,\ -3.14 \cdot 10^2]$	234.13	[142.10, 478.06]
A478	0.117	[0.114, 0.122]	$-4.61 \cdot 10^3$	$[-1.01 \cdot 10^4,\ -2.51 \cdot 10^3]$	484.83	[363.29, 707.73]
A907	0.129	[0.125, 0.136]	$-5.77 \cdot 10^3$	$[-1.54 \cdot 10^4,\ -2.83 \cdot 10^3]$	517.30	[368.84, 825.00]
A1413	0.115	[0.110, 0.119]	$-9.45 \cdot 10^4$	$[-4.26 \cdot 10^5,\ -3.46 \cdot 10^4]$	2224.57	[1365.40, 4681.21]
A1795	0.093	[0.084, 0.103]	$-1.54 \cdot 10^3$	$[-1.01 \cdot 10^4,\ -2.49 \cdot 10^2]$	315.44	[133.31, 769.17]
A1991	0.074	[0.072, 0.081]	-50.69	$[-3.42 \cdot 10^2,\ -13]$	64.00	[32.63, 159.40]
A2029	0.129	[0.123, 0.134]	$-2.10 \cdot 10^4$	$[-7.95 \cdot 10^4,\ -8.44 \cdot 10^3]$	988.85	[637.71, 1890.07]
A2390	0.149	[0.146, 0.152]	$-1.40 \cdot 10^6$	$[-5.71 \cdot 10^6,\ -4.46 \cdot 10^5]$	7490.80	[4245.74, 15715.60]
MKW4	0.054	[0.049, 0.060]	-23.63	$[-1.15 \cdot 10^2,\ -8.13]$	51.31	[30.44, 110.68]
RXJ1159	0.048	[0.047, 0.052]	-18.33	$[-1.35 \cdot 10^2,\ -4.18]$	47.72	[22.86, 125.96]

Modeling clusters of galaxies

Results: expectations

- *First derivative, a_1 : very well constrained* → *It scales with the system size*

- *Newtonian limit:* $$\phi(r) = -\frac{3GM}{4a_1 r}\left(1 + \frac{1}{3}e^{-\frac{r}{L}}\right)$$ → $a_1 \rightarrow 3/4$



Modeling clusters of galaxies

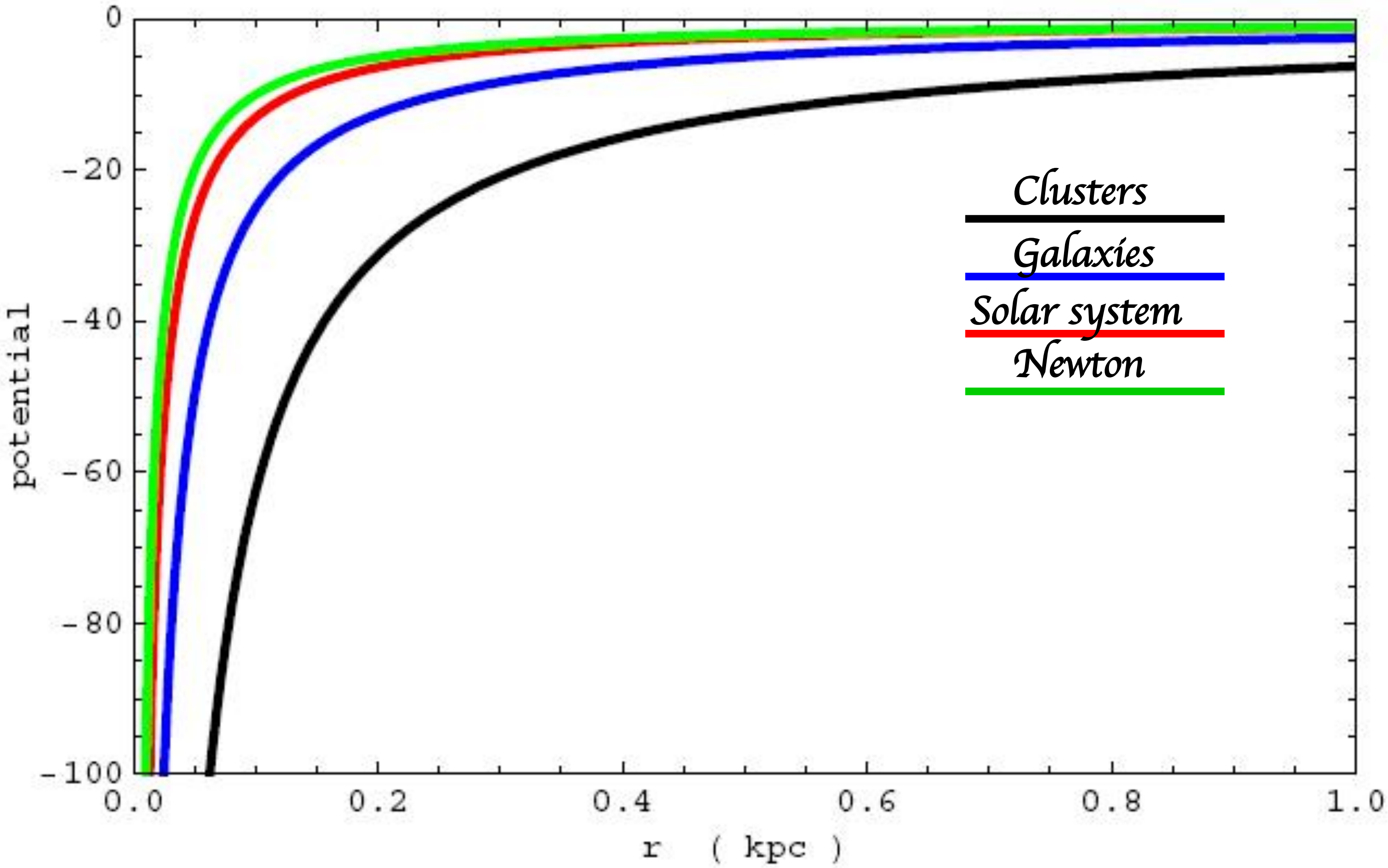
Point like potential:

Cluster of Galaxies: $a_1 = 0.16$ - $L = 1000$ kpc

Galaxies: $a_1 = 0.4$ - $L = 100$ kpc

Solar System: $a_1 = 0.75$ - $L = 1$ kpc

Newton Limit: $a_1 = 0.75$ - $L = 0$ kpc



Modeling clusters of galaxies

- *Gravitational length:* $L \equiv L(a_1, a_2) = \left(-\frac{6a_2}{a_1}\right)^{1/2}$

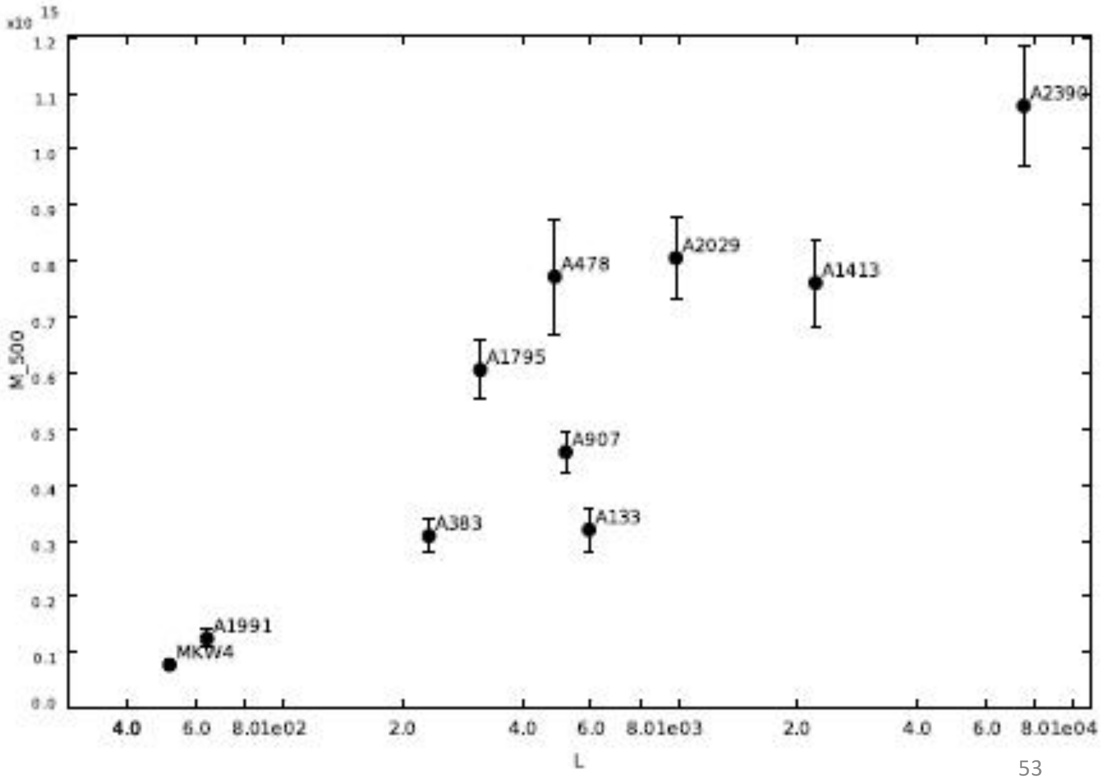
Strong characterization of Gravitational potential

- *Mean length:*

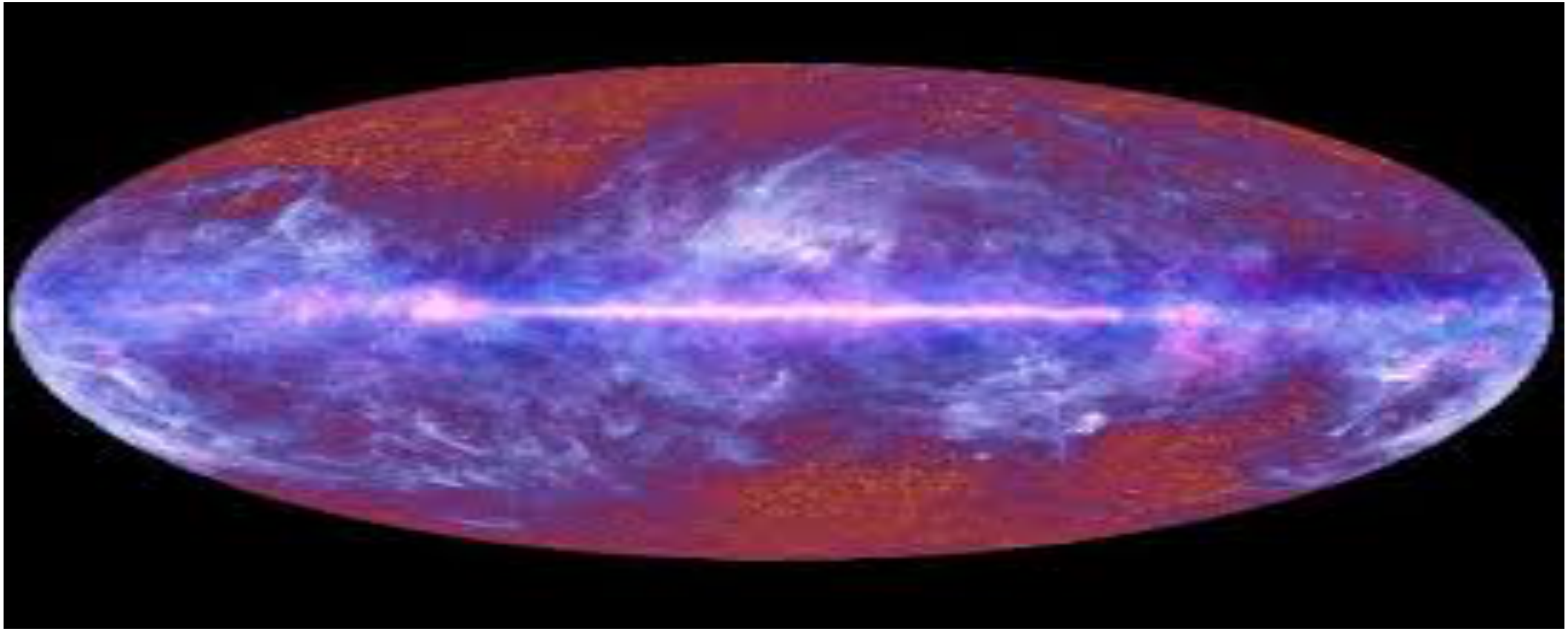
$$< L >_\rho \quad = \quad 318 \text{ kpc} \qquad < a_2 >_\rho = -3.40 \cdot 10^4$$
$$< L >_M \quad = \quad 2738 \text{ kpc} \qquad < a_2 >_M = -4.15 \cdot 10^5$$

- *Strongly related to virial mass (the same for gas mass):*

- *Strongly related to average temperature:*



Cosmology is the main arena where Dark Side can be confronted with ETGs



Cosmography

GR based models vs $f(R)$ gravity

Agreement with Data...

How can we discriminate?

- No a priori dynamical model = Model Independent Approach;
- Friedmann - Robertson – Walker metric;
- Expansion series of the scale factor with respect to cosmic time:

$$\frac{a(t)}{a(t_0)} = 1+H_0(t-t_0)-\frac{q_0}{2}H_0^2(t-t_0)^2+\frac{j_0}{3!}H_0^3(t-t_0)^3+\frac{s_0}{4!}H_0^4(t-t_0)^4+\frac{l_0}{5!}H_0^5(t-t_0)^5+O[(t-t_0)^6$$

$$q(t) = -\frac{1}{a}\frac{d^2a}{dt^2}\frac{1}{H^2} \qquad j(t) = \frac{1}{a}\frac{d^3a}{dt^3}\frac{1}{H^3} \qquad s(t) = \frac{1}{a}\frac{d^4a}{dt^4}\frac{1}{H^4} \qquad l(t) = \frac{1}{a}\frac{d^5a}{dt^5}\frac{1}{H^5}$$

Deceleration Jerk Snap Lerk

Expansion up to fifth order :

- error on $d_L(z)$ less than 10% up to $z = 1$
- error on $\mu(z)$ less than 3% up to $z = 2$

Cosmography with f(R)-gravity

- *Definitions:* $H(t) = \frac{1}{a}\frac{da}{dt}, \; q(t) = -\frac{1}{a}\frac{d^2a}{dt^2}\frac{1}{H^2}, \; j(t) = \frac{1}{a}\frac{d^3a}{dt^3}\frac{1}{H^3}, \; s(t) = \frac{1}{a}\frac{d^4a}{dt^4}\frac{1}{H^4}, \; l(t) = \frac{1}{a}\frac{d^5a}{dt^5}\frac{1}{H^5}$

- *Derivatives of H(t):*

$$\dot{H} = -H^2(1+q)$$

$$\ddot{H} = H^3(j+3q+2)$$

$$d^3H/dt^3 = H^4\left[s-4j-3q(q+4)-6\right]$$

$$d^4H/dt^4 = H^5\left[l-5s+10(q+2)j+30(q+2)q+24\right]$$

- *Derivatives of scalar curvature:*

$$R = -6(\dot{H}+2H^2)$$

$$R_0 = -6H_0^2(1-q_0)$$

$$\dot{R}_0 = -6H_0^3(j_0-q_0-2)$$

$$\ddot{R}_0 = -6H_0^4\left(s_0+q_0^2+8q_0+6\right)$$

$$d^3R_0/dt^3 = -6H_0^5\left[l_0-s_0+2(q_0+4)j_0-6(3q_0+8)q_0-24\right]$$

Cosmography with f(R) gravity

- *1st Friedmann eq. :*

$$H_0^2 = \frac{H_0^2\Omega_M}{f'(R_0)} + \frac{f(R_0) - R_0 f'(R_0) - 6H_0\dot{R}_0 f''(R_0)}{6f'(R_0)},$$

- *2nd Friedmann eq. :*

$$-\dot{H}_0 = \frac{3H_0^2\Omega_M}{2f'(R_0)} + \frac{\dot{R}_0^2 f'''(R_0) + (\ddot{R}_0 - H_0\dot{R}_0)f''(R_0)}{2f'(R_0)}.$$

- *Derivative of 2nd Friedmann eq. :*

$$\begin{aligned}\ddot{H} &= \frac{\dot{R}^2 f'''(R) + \left(\ddot{R} - H\dot{R}\right) f''(R) + 3H_0^2\Omega_M a^{-3}}{2\left[\dot{R}f''(R)\right]^{-1}\left[f'(R)\right]^2} - \frac{\dot{R}^3 f^{(iv)}(R) + \left(3\dot{R}\ddot{R} - H\dot{R}^2\right) f'''(R)}{2f'(R)} \\ &- \frac{\left(d^3R/dt^3 - H\ddot{R} + \dot{H}\dot{R}\right) f''(R) - 9H_0^2\Omega_M H a^{-3}}{2f'(R)}\end{aligned}$$

- *Constraint from gravitational constant:*

$$H^2 = \frac{8\pi G}{3f'(R)}[\rho_m + \rho_{\text{curv}} f'(R)] \quad \Rightarrow \quad G_{\text{eff}}(z=0) = G \to f'(R_0) = 1.$$

Cosmography with f(R) gravity

- *Final solutions:*

$$\frac{f(R_0)}{6H_0^2} = -\frac{\mathcal{P}_0(q_0, j_0, s_0, l_0)\Omega_M + \mathcal{Q}_0(q_0, j_0, s_0, l_0)}{\mathcal{R}(q_0, j_0, s_0, l_0)}$$

$$f'(R_0) = 1$$

$$\frac{f''(R_0)}{(6H_0^2)^{-1}} = -\frac{\mathcal{P}_2(q_0, j_0, s_0)\Omega_M + \mathcal{Q}_2(q_0, j_0, s_0)}{\mathcal{R}(q_0, j_0, s_0, l_0)}$$

$$\frac{f'''(R_0)}{(6H_0^2)^{-2}} = -\frac{\mathcal{P}_3(q_0, j_0, s_0, l_0)\Omega_M + \mathcal{Q}_3(q_0, j_0, s_0, l_0)}{(j_0 - q_0 - 2)\mathcal{R}(q_0, j_0, s_0, l_0)}$$

- *Taylor expansion f(R) in series of R up to third order (higher not necessary)*
- *Linear equations in f(R) and derivatives*
- *Ω_M is model dependent:*

$$\Omega_M = 0.041$$
$$\Omega_M = 0.250.$$

f(R) gravity and CPL model

"Precision cosmology" → *Values of cosmographic parameters?*

Cosmographic parameters → *Dark energy parameters = equivalent f(R)*

CPL approach:
(Chevallier, Polarski, Linder)

$$w = w_0 + w_a(1-a) = w_0 + w_a z(1+z)^{-1}$$

Cosmographic parameters:

$$q_0 = \frac{1}{2} + \frac{3}{2}(1-\Omega_M)w_0$$

$$j_0 = 1 + \frac{3}{2}(1-\Omega_M)\left[3w_0(1+w_0) + w_a\right]$$

$$\begin{aligned} s_0 \;=\; & -\frac{7}{2} - \frac{33}{4}(1-\Omega_M)w_a - \frac{9}{4}(1-\Omega_M)\left[9 + (7-\Omega_M)w_a\right]w_0 + \\ & - \frac{9}{4}(1-\Omega_M)(16-3\Omega_M)w_0^2 - \frac{27}{4}(1-\Omega_M)(3-\Omega_M)w_0^3 \end{aligned}$$

$$\begin{aligned} l_0 \;=\; & \frac{35}{2} + \frac{1-\Omega_M}{4}\left[213 + (7-\Omega_M)w_a\right]w_a + \frac{(1-\Omega_M)}{4}\left[489 + 9(82-21\Omega_M)w_a\right]w_0 + \\ & + \frac{9}{2}(1-\Omega_M)\left[67 - 21\Omega_M + \frac{3}{2}(23-11\Omega_M)w_a\right]w_0^2 + \frac{27}{4}(1-\Omega_M)(47-24\Omega_M)w_0^3 + \\ & + \frac{81}{2}(1-\Omega_M)(3-2\Omega_M)w_0^4 \end{aligned}$$

CPL Cosmography and f(R): the ΛCDM Model

ΛCDM model: $(w_0, w_a) = (-1, 0)$

$$q_0 = \frac{1}{2} - \frac{3}{2}\Omega_M; \quad j_0 = 1; \quad s_0 = 1 - \frac{9}{2}\Omega_M; \quad l_0 = 1 + 3\Omega_M + \frac{27}{2}\Omega_M^2$$

$$f(R_0) = R_0 + 2\Lambda, \qquad f''(R_0) = f'''(R_0) = 0,$$

ΛCDM fits well many data → *cosmographic values strictly depend on Ω_M*

$$q_0 = q_0^\Lambda \times (1 + \varepsilon_q), \qquad j_0 = j_0^\Lambda \times (1 + \varepsilon_j),$$
$$s_0 = s_0^\Lambda \times (1 + \varepsilon_s), \qquad l_0 = l_0^\Lambda \times (1 + \varepsilon_l),$$

$$\eta_{20} = f''(R_0)/f(R_0) \times H_0^4$$

$$\eta_{30} = f'''(R_0)/f(R_0) \times H_0^6$$

$$\begin{cases}
\eta_{20} = \dfrac{64 - 6\Omega_M(9\Omega_M + 8)}{[3(9\Omega_M + 74)\Omega_M - 556]\,\Omega_M^2 + 16} \times \dfrac{\varepsilon}{27} \\[2ex]
\eta_{30} = \dfrac{6\,[(81\Omega_M - 110)\Omega_M + 40]\,\Omega_M + 16}{[3(9\Omega_M + 74)\Omega_M - 556]\,\Omega_M^2 + 16} \times \dfrac{\varepsilon}{243\Omega_M^2} \\[2ex]
\begin{cases} \eta_{20} \simeq 0.15 \times \varepsilon & \text{for } \Omega_M = 0.041 \\ \eta_{20} \simeq -0.12 \times \varepsilon & \text{for } \Omega_M = 0.250 \end{cases} \\[2ex]
\begin{cases} \eta_{30} \simeq 4 \times \varepsilon & \text{for } \Omega_M = 0.041 \\ \eta_{30} \simeq -0.18 \times \varepsilon & \text{for } \Omega_M = 0.250 \end{cases}
\end{cases}$$

Constraining f(R) models by Cosmography

- Procedure:
  1. Estimate ($q(0), j(0), s(0), l(0)$) observationally
  2. Compute $f(R_0), f'(R_0), f''(R_0), f'''(R_0)$
  3. Solve for f(R) parameters from derivatives
  4. Constrain f(R) models

- e.g. Double Power-Law: $\boxed{f(R) = R(1 + \alpha R^n + \beta R^{-m})}$

$$\begin{cases} f(R_0) = R_0(1 + \alpha R_0^n + \beta R_0^{-m}) \\ f'(R_0) = 1 + \alpha(n+1)R_0^n - \beta(m-1)R_0^{-m} \\ f''(R_0) = \alpha n(n+1)R_0^{n-1} + \beta m(m-1)R_0^{-(1+m)} \\ f'''(R_0) = \alpha n(n+1)(n-1)R_0^{n-2} \\ \qquad\qquad - \beta m(m+1)(m-1)R_0^{-(2+m)}. \end{cases}$$

$$\begin{cases} \alpha = \frac{1-m}{n+m}\left(1 - \frac{\phi_0}{R_0}\right)R_0^{-n} \\ \beta = -\frac{1+n}{n+m}\left(1 - \frac{\phi_0}{R_0}\right)R_0^{m}, \end{cases}$$

$$\begin{cases} \alpha = \frac{\phi_2 R_0^{1-n}[1+m+(\phi_3/\phi_2)R_0]}{n(n+1)(n+m)} \\ \beta = \frac{\phi_2 R_0^{1+n}[1-n+(\phi_3/\phi_2)R_0]}{m(1-m)(n+m)}. \end{cases}$$

$$\begin{cases} \frac{n(n+1)(1-m)(1-\phi_0/R_0)}{\phi_2 R_0[1+m+(\phi_3/\phi_2)R_0]} = 1 \\ \frac{m(n+1)(m-1)(1-\phi_0/R_0)}{\phi_2 R_0[1-n+(\phi_3/\phi_2)R_0]} = 1. \end{cases}$$

$$\begin{cases} m = -[1 - n + (\phi_3/\phi_2)R_0] \\ n = \frac{1}{2}\left[1 + \frac{\phi_3}{\phi_2}R_0 \pm \frac{\sqrt{\mathcal{N}(\phi_0, \phi_2, \phi_3)}}{\phi_2 R_0(1 + \phi_0/R_0)}\right] \end{cases}$$

Constraining f(R) models by Cosmography

- *Cosmographic parameters from SNeIa: What we have to expect from data*

$$q_0 = -0.90 \pm 0.65, \qquad j_0 = 2.7 \pm 6.7,$$
$$s_0 = 36.5 \pm 52.9, \qquad l_0 = 142.7 \pm 320.$$

- *Fisher information matrix method:*

$$F_{ij} = \left\langle \frac{\partial^2 L}{\partial \theta_i \partial \theta_j} \right\rangle$$

- *FM ingredients :*

$$\begin{cases} \chi^2(H_0, \mathbf{p}) = \sum_{n=1}^{\mathcal{N}_{SNeIa}} \left[\frac{\mu_{obs}(z_i) - \mu_{th}(z_n, H_0, \mathbf{p})}{\sigma_i(z_i)} \right]^2 \\ d_L(z) = \mathcal{D}_L^0 z + \mathcal{D}_L^1 \ z^2 + \mathcal{D}_L^2 \ z^3 + \mathcal{D}_L^3 \ z^4 + \mathcal{D}_L^4 \ z^5 \\ \sigma(z) = \sqrt{\sigma_{\rm sys}^2 + \left(\frac{z}{z_{\max}}\right)^2 \sigma_m^2} \end{cases}$$

- *Estimating error on g:*

$$\sigma_g^2 = \left| \frac{\partial g}{\partial \Omega_M} \right|^2 \sigma_M^2 + \sum_{i=1}^{i=4} \left| \frac{\partial g}{\partial p_i} \right|^2 \sigma_{p_i}^2 + \sum_{i \neq j} 2 \frac{\partial g}{\partial p_i} \frac{\partial g}{\partial p_j} C_{ij}$$

- Survey: Davis (2007)

$\sigma_M/\Omega_M = 10\%$; $\sigma_{sys} = 0.15$

$N_{SNeIa} = 2000$; $\sigma_m = 0.33$

$z_{max} = 1.7$

$\sigma_1 = 0.38$

$\sigma_2 = 5.4$

$\sigma_3 = 28.1$

$\sigma_4 = 74.0$

$\sigma_{20} = 0.04$

$\sigma_{30} = 0.04$

- Snap like survey:

$\sigma_M/\Omega_M = 1\%$; $\sigma_{sys} = 0.15$

$N_{SNeIa} = 2000$; $\sigma_m = 0.02$

$z_{max} = 1.7$

$\sigma_1 = 0.08$

$\sigma_2 = 1.0$

$\sigma_3 = 4.8$

$\sigma_4 = 13.7$

$\sigma_{20} = 0.007$

$\sigma_{30} = 0.008$

- Ideal PanSTARRS survey:

$\sigma_M/\Omega_M = 0.1\%$; $\sigma_{sys} = 0.15$

$N_{SNeIa} = 60000$; $\sigma_m = 0.02$

$z_{max} = 1.7$

$\sigma_1 = 0.02$

$\sigma_2 = 0.2$

$\sigma_3 = 0.9$

$\sigma_4 = 2.7$

$\sigma_{20} = 0.0015$

$\sigma_{30} = 0.0016$

SC et al. Phys. Rev. D 78 (2008) 063504

Conclusions (DM)

- In principle, corrections to the Newtonian potential can affect gravity at any scale.
- Exotic stellar structures could be compatible with ETGs
- Orbital period of binary systems are in agreement with f(R)-gravity
- Search for EXPERIMENTUM CRUCIS via GWs
- Rotation curves of galaxies can be naturally reproduced, without huge amounts of DM
- The baryonic Tully- Fisher relation has a natural explanation in the framework of f(R) theories.
- Haloes of elliptical galaxies are reproduced by the same mechanism..
- Good evidences also for galaxy clusters
- In this perspective, gravity is not a scale-invariant interaction.

Conclusions (DE)

- Dark Side of the Universe can be accounted by ETGs without exotic fluids but only by geometry
- Following Starobinsky, R can be considered a "geometric" scalar field.
- Comfortable results are obtained by matching the theory with data (SNeIa, BAO, CMBR…PLANCK does not exclude f(R) gravity).
- Generic quintessential and DE models can be easily "mimicked" by $f(R)$
- Main role of Cosmography.
- A comprehensive cosmological model from early to late epochs should be achieved by $f(R)$ see e.g. S. Nojiri, S.D. Odintsov, Phys. Rep. 505, 59 (2011).

Perspectives and Open Issues

DE & DM as curvature effects

- Matching DE models
- Jordan Frame and Einstein Frame
- Systematic studies of rotation curves for galaxies
- Galaxy cluster dynamics (virial theorem, SZE, etc.)
- Luminosity profiles of galaxies .
- Faber-Jackson & Tully-Fisher, Bullet Cluster

Weak Fields, GW, Further results

- Systematic studies of PPN formalism
- Relativistic Experimental Tests
- Gravitational waves and lensing
- Birkhoff 's Theorem
- Torsion

WORK in PROGRESS! (suggestions are welcome!)