# Mathematical and Physical Foundations of Extended Gravity (IV)

-Hamiltonian Dynamics and Quantum Cosmology issues-

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# Summary

**I PART (general)**

- The Hamiltonian formulation of General Relativity and the problem of canonical quantization
- The Minisuperspace Approach to Quantum Cosmology
- The Noether Symmetry Approach

**II PART (applications)**

- Extending General Relativity
- Extended Minisuperspace Models
- Discussion and conclusions

Some Big Issues of modern Theoretical Physics are related to the problem of “origins”

- The origin of the Universe
- The origin of Time
- The origin of Large Scale Structure

People search for quantum origins trying to connect General Relativity to Quantum Mechanics!

# The Hamiltonian formulation of General Relativity and the problem of quantization

Following the ADM formalism, the embedding is described by the so-called ( 3 + 1) form of $g_{\mu\nu}$ , that is,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$
$$= -\left(N^2 - N_i N^i\right) dt^2 + 2N_i\, dx^i\, dt + h_{ij}\, dx^i\, dx^j,$$

- $N$ and $N_i$ are the lapse and shift arbitrary functions

The action is the standard one of GR minimally coupled to matter

$$\mathcal{S} = \frac{m_P^2}{16\pi}\left[\int_{\mathcal{M}} d^4x\, \sqrt{-g}\,(R - 2\Lambda) + 2\int_{\partial\mathcal{M}} d^3x\, \sqrt{h} K\right] + \mathcal{S}_{(m)},$$

- $K$ is the trace of the extrinsic curvature $K_{ij}$ at the boundary $\partial M$ of the 4-manifold $M$ and is given by

$$K_{ij} = \frac{1}{2N}\left[-\frac{\partial h_{ij}}{\partial t} + 2D_{(i} N_{j)}\right].$$

# The Hamiltonian formulation of General Relativity and the problem of quantization

The action for the matter scalar field ϕ is

$$\mathcal{S}_{(m)} = -\frac{1}{2}\int d^4x\,\sqrt{-g}\,\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right],$$

which in terms of ( 3 + 1) - variables is

$$\mathcal{S} = \frac{m_P^2}{16\pi}\int d^3x\,dt\,N\sqrt{h}\big[K_{ij}K^{ij} - K^2 + {}^{(3)}R - 2\Lambda\big] + \mathcal{S}_{(m)}.$$

The Hamiltonian 3 -form of the action is then

$$\mathcal{S} = \int d^3x\,dt\big[\dot{h}_{ij}\pi^{ij} + \dot{\Phi}\pi_\Phi - N\mathcal{H} - N^i\mathcal{H}_i\big],$$

- where $\pi_{ij}$ and $\pi_\Phi$ are the conjugate momenta to $h_{ij}$ and $\Phi$ , respectively

The Hamiltonian formulation of General Relativity and the problem of quantization

The momentum constraint is $\mathcal{H}_i = -2D_j\pi_i^j + \mathcal{H}_i^{(m)} = 0,$

while the proper Hamiltonian constraint is

$$\mathcal{H} = \frac{16\pi}{m_P^2} G_{ijkl}\pi^{ij}\pi^{kl} - \frac{m_P^2}{16\pi}\sqrt{h}\left({}^{(3)}R - 2\Lambda\right) + \mathcal{H}^{(m)} = 0$$

- where $G_{ijkl}$ is the so-called *De Witt metric*, explicitly given by

$$G_{ijkl} = \frac{1}{2}\sqrt{h}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$$

These constraints correspond to the time-space and time-time components of the Einstein field equations, respectively.

The canonical quantization procedure is essentially based on them.

# The Hamiltonian formulation of General Relativity and the problem of quantization

- The so-called *Superspace* is the framework where classical dynamics takes place: it is the space of all 3-metric and matter field configurations $h_{ij[}[(\mathbf{x}),\phi(\mathbf{x})]$ defined on a 3-manifold.

- It is infinite dimensional, with a finite number of coordinates $\mathbf{v}[h_{ij}(\mathbf{x}),\phi(\mathbf{x})]$ at every point **x** of the 3-manifold.

- The De Witt metric and the metric on matter fields determine the metric on the Superspace.

- Its signature is hyperbolic at every point in the 3-surface.

- The signature of the De Witt metric does not depend on the signature of the standard space-time.

# The Hamiltonian formulation of General Relativity and the problem of quantization

- The *quantum state* of the system can be represented by a wave functional $\Psi[h_{ij},\phi]$ in the canonical quantization approach.

- The wave function does not depend explicitly on the time coordinate *t.*

- This is because the 3-surfaces are compact, and thus their intrinsic geometry fixes almost uniquely their relative position in the 4-manifold.

- This is the so called *Geometrodynamics.*

Following the *Dirac quantization procedure*, the wave function is assumed to be annihilated by the classical constraints after they have been "transformed" into operators, that is

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}}, \qquad \pi_{\Phi} \to -i\frac{\delta}{\delta \phi}.$$

# The Hamiltonian formulation of General Relativity and the problem of quantization

The equations for $\Psi$ are the momentum constraint

$$\mathcal{H}_i \Psi = 2i\, D_j \frac{\delta \Psi}{\delta h_{ij}} + \mathcal{H}_i^{(m)} \Psi = 0,$$

and the *Wheeler-De Witt equation* (WDW)

$$\mathcal{H}\Psi = \left[ -G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} - \sqrt{h}\left({}^{(3)}R - 2\Lambda\right) + \mathcal{H}^{(m)} \right] \Psi = 0.$$

- The momentum constraint implies that the wave function is the same for configurations {hij (x),Φ(x)} that are related by coordinate transformations in the 3-surface.
- The momentum constraint is the quantum mechanical expression of the invariance of the theory under 3-dimensional diffeomorphisms.
- Similarly, the WDW equation represents the re-parameterization invariance of the theory.
- Such an equation is a second-order hyperbolic functional differential equation describing the dynamical evolution of the wave function in Superspace (the Wave Function of the Universe).

# The Hamiltonian formulation of General Relativity and the problem of quantization

Another approach to the canonical quantization is to derive the wave function by path integrals.

The wave function is an Euclidean functional integral over a class of 4-metrics and matter fields, weighted by $e^{-I}$ , where $I$ is the Euclidean action of gravity plus matter fields, that is,

$$\Psi[\tilde{h}_{ij}, \tilde{\phi}, B] = \sum_{\mathcal{M}} \int \mathcal{D}g_{\mu\nu} \mathcal{D}\phi e^{-I}.$$

The sum is over a given class of manifolds $M$ (where $B$ is their boundary), and over a class of 4-metrics $g_{\mu\nu}$ and matter fields $\phi$ which induce the 3-metric and matter field configuration on the 3-surface $B$

$$\Psi[\tilde{h}_{ij}, \tilde{\Phi}, B] = \int \mathcal{D}N^{\mu} \int \mathcal{D}h_{ij} \mathcal{D}\phi \delta[\dot{N}^{\mu} - \chi^{\mu}] \times \Delta_{\chi} \exp(-I[g_{\mu\nu}, \phi]).$$

Faddeev–Popov determinant

# The Minisuperspace Approach to Quantum Cosmology

Minisuperspaces are restrictions of Superspace where some symmetries are imposed a priori on the metric and the related matter fields.

For example, we can assume a 4-metric, a homogeneous lapse function $N = N(t)$ and the shift functions $N^i = 0$

$$ds^2 = -N^2(t)\,dt^2 + h_{ij}(\mathbf{x},t)\,dx^i\,dx^j$$

Being the 3-metric $h_{ij}$ homogeneous, it is described by a finite number of functions of $t$ , $q^\alpha(t)$ , where $\alpha = 0, 1, 2, \ldots (n-1)$ .

The Hilbert–Einstein action can be recast as

$$\mathcal{S}[h_{ij}, N, N^i] = \frac{m_P^2}{16\pi}\int dt\,d^3x\,N\sqrt{h}\big[K_{ij}K^{ij} - K^2 + {}^{(3)}R - 2\Lambda\big],$$

# The Minisuperspace Approach to Quantum Cosmology

In general, one gets

reduced De Witt metric

$$\mathcal{S}[q^\alpha(t), N(t)] = \int_0^1 dt\, N\left[\frac{1}{2N^2} f_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - U(q)\right] \equiv \int \mathcal{L}\, dt$$

This equation has the form of a relativistic point particle action where the particles moves on a ***n*** -dimensional curved space-time with a self-interaction potential.

The variation with respect to $q^\alpha$ gives the equations of motion

Christoffel symbols derived from the metric $f_{\alpha\beta}$

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{q}^\alpha}{N}\right) + \frac{1}{N^2}\Gamma^\alpha_{\beta\gamma}\dot{q}^\beta\dot{q}^\gamma + f^{\alpha\beta}\frac{\partial U}{\partial q^\beta} = 0$$

Varying with respect to N , one gets

$$\frac{1}{2N^2} f_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + U(q) = 0$$

that is a constraint equation

# The Minisuperspace Approach to Quantum Cosmology

In order to find the Hamiltonian, the canonical momenta have to be defined as

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} = f_{\alpha\beta}\frac{\dot{q}^\beta}{N},$$

and the canonical Hamiltonian is

$$\mathcal{H}_c = p_\alpha \dot{q}^\alpha - \mathcal{L} = N\left[\frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta + U(q)\right] \equiv N\mathcal{H},$$

The Hamiltonian form of the action is

$$\mathcal{S} = \int_0^1 dt\,\left[p_\alpha \dot{q}^\alpha - N\mathcal{H}\right].$$

This equation means that the lapse function *N* is a Lagrange multiplier and then the Hamiltonian constraint has to be

$$\mathcal{H}(q^\alpha, p_\alpha) = \frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta + U(q) = 0.$$

This is the *minisuperspace reduction* equivalent to the Hamiltonian constraint of the full theory, integrated over the spatial hypersurfaces.

# The Minisuperspace Approach to Quantum Cosmology

An important issue has to be addressed

*How to interpret the probability measure in Quantum Cosmology?*

Given a wave function $\Psi(q^\alpha)$, defined in a minisuperspace, one needs to define a suitable probability measure.

The WDW equation is a sort of Klein–Gordon equation and a current can be defined as

$$J = \frac{i}{2}\left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*\right)$$

It is conserved and satisfies the relation $\nabla \cdot J = 0$

thanks to the structure of the WDW equation.

# The Minisuperspace Approach to Quantum Cosmology

- As in the case of Klein–Gordon equation (and, in general, of any hyperbolic equation), the probability derived from such a conserved current can be affected by negative probabilities.

Due to this shortcoming, the correct measure should be $dP = \left|\Psi\left(q^{\alpha}\right)\right|^{2} dV$

- Also this assumption can be problematic since one of the coordinates $q^{\alpha}$ is "time", so that the above equation is the analogue of interpreting $|\Psi(x, t)|^2$ in standard Quantum Mechanics as the probability of finding the particle in the space-time interval *dx dt.*

This means that a careful discussion on the *meaning of time* in Quantum Cosmology has to be pursued.

# The Minisuperspace Approach to Quantum Cosmology

- In summary:
- Minisuperspaces are restrictions of the Superspace of geometrodynamics.
- They are finite-dimensional configuration spaces on which point-like Lagrangians can be defined.
- Cosmological models of physical interest can be defined on such minisuperspaces (e.g. Bianchi models).
- According to the above discussion, a crucial role is played by the conserved currents that allow to interpret the probability measure and then the physical quantities obtained in Quantum Cosmology.
- In this context, the search for general methods to achieve conserved quantities and symmetries become relevant.
- The *Noether Symmetry Approach* is extremely useful to this purpose.

# The Noether Symmetry Approach

By considering a Lagrangian *L* which is a function defined on the tangent space of configurations

$$TQ \equiv \{q_i, \dot{q}_i\}$$

and a vector field X

$$X = \alpha^i(q)\frac{\partial}{\partial q^i} + \dot{\alpha}^i(q)\frac{\partial}{\partial \dot{q}^i},$$

where dot means derivative with respect to *t* , and

$$L_X\mathcal{L} = X\mathcal{L} = \alpha^i(q)\frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}^i(q)\frac{\partial \mathcal{L}}{\partial \dot{q}^i}.$$

The condition $L_X\mathcal{L} = 0,$ implies that the phase flux is conserved along X: this means that a constant of motion exists for *L* and the Noether theorem holds:

*Emmy*

Taking into account the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0,$$

it is easy to show that

$$\frac{d}{dt}\left(\alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) = L_X\mathcal{L}.$$

If $L_X\mathcal{L} = 0,$ holds, $\Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$ is a constant of motion!

# The Noether Symmetry Approach


Alternatively, using the Cartan one-form $\theta_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} dq^i$

and defining the inner derivative $i_X \theta_{\mathcal{L}} = \langle \theta_{\mathcal{L}}, X \rangle$

we get $i_X \theta_{\mathcal{L}} = \Sigma_0$ if condition $L_X \mathcal{L} = 0$, holds.

- This representation is useful to identify cyclic variables!

Using a point transformation on vector field, it is possible to get $\tilde{X} = (i_X dQ^k) \frac{\partial}{\partial Q^k} + \left[ \frac{d}{dt} (i_X dQ^k) \right] \frac{\partial}{\partial \dot{Q}^k}$

If $X$ is a symmetry also $\tilde{X}$ has this property, then it is always possible to choose a coordinate transformation so that $i_X dQ^1 = 1, \quad i_X dQ^i = 0, \quad i \neq 1,$

and then $\tilde{X} = \frac{\partial}{\partial Q^1}, \qquad \frac{\partial \tilde{\mathcal{L}}}{\partial Q^1} = 0$

It is evident that $Q^1$ is the cyclic coordinate and the dynamics can be reduced.

However, the change of coordinates is not unique and a clever choice is always important.

# The Noether Symmetry Approach

- It is possible that more symmetries are found.
- In this case more cyclic variables exist.

For example, if $X_1$,$X_2$ are the Noether vector fields and they commute, $[X_1,X_2] = 0$, we obtain two cyclic coordinates by solving the system

$$i_{X_1} dQ^1 = 1, \qquad i_{X_2} dQ^2 = 1,$$
$$i_{X_1} dQ^i = 0, \quad i \neq 1; \qquad i_{X_2} dQ^i = 0, \quad i \neq 2.$$

If they do not commute, this procedure does not work since commutation relations are preserved by diffeomorphisms

In this case $X_3 = [X_1,X_2]$ is again a symmetry since

$$L_{X_3}\mathcal{L} = L_{X_1} L_{X_2}\mathcal{L} - L_{X_2} L_{X_1}\mathcal{L} = 0.$$

If $X_3$ is independent of $X_1$,$X_2$, we can go on until the vector fields close the Lie algebra.

# The Noether Symmetry Approach

Any symmetry selects a constant conjugate momentum since, by the Euler–Lagrange equations, we have

$$\frac{\partial \tilde{\mathcal{L}}}{\partial Q^i} = 0 \quad \Longleftrightarrow \quad \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}^i} = \Sigma_i \,.$$

Vice versa, the existence of a constant conjugate momentum means that a cyclic variable has to exist.

In other words, a Noether symmetry exists!!!

Further remarks on the form of the Lagrangian $L$ are necessary at this point.

We shall take into account time-independent, non-degenerate Lagrangians $\mathcal{L} = \mathcal{L}(q^i, \dot{q}^j)$ i.e.

$$\frac{\partial \mathcal{L}}{\partial t} = 0, \qquad \det H_{ij} \equiv \det \left\| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0,$$

where $H_{ij}$ is the Hessian determinant.

# The Noether Symmetry Approach

As in analytic mechanics, *L* can be set in the form $\mathcal{L} = T(q^i, \dot{q}^i) - V(q^i)$

The energy function associated with *L* is $E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i - \mathcal{L}(q^j, \dot{q}^j)$

and by the Legendre transformations $\mathcal{H} = \pi_j \dot{q}^j - \mathcal{L}(q^j, \dot{q}^j), \quad \pi_j = \frac{\partial \mathcal{L}}{\partial \dot{q}^j}$

we get the Hamiltonian function and the conjugate momenta. We have all the ingredients to construct Minisuperspaces.

- If *n* is the dimension of the configuration space (i.e. the dimension of the Mini-superspace), we get ***{ 1+n(n+ 1)/ 2}*** partial differential equations whose solutions assign the symmetry.

- Such a symmetry is over-determined and, if a solution exists, it is expressed in terms of integration constants instead of boundary conditions.

# The Noether Symmetry Approach

In the Hamiltonian formalism, we have

$$[\Sigma_j, \mathcal{H}] = 0, \quad 1 \leq j \leq m$$

as it must be for conserved momenta in Quantum Mechanics and the Hamiltonian has to satisfy the relation

$$L_\Gamma \mathcal{H} = 0$$

in order to obtain a Noether symmetry

where $$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i}$$

These considerations can be applied to the minisuperspace models of Quantum Cosmology and to the interpretation of the Wave Function of the Universe.

# The Noether Symmetry Approach

By a straightforward canonical quantization procedure, we have

$$\pi_j \longrightarrow \hat{\pi}_j = -i\partial_j ,$$

$$\mathcal{H} \longrightarrow \hat{\mathcal{H}}(q^j, -i\partial_{q^j})$$

The Hamiltonian constraint gives the WDW equation.

If $|\Psi>$ is a state of the system (i.e. the Wave Function of the Universe), dynamics is given by $H|\Psi>=0$

If Noether symmetries exist, we get

$$\pi_1 \equiv \frac{\partial\mathcal{L}}{\partial\dot{Q}^1} = i_{X_1}\theta_{\mathcal{L}} = \Sigma_1 ,$$

$$\pi_2 \equiv \frac{\partial\mathcal{L}}{\partial\dot{Q}^2} = i_{X_2}\theta_{\mathcal{L}} = \Sigma_2 ,$$

$$\ldots \qquad \ldots \qquad \ldots ,$$

depending on the number of Noether symmetries

After quantization, we get

$$-i\partial_1|\Psi\rangle = \Sigma_1|\Psi\rangle ,$$

$$-i\partial_2|\Psi\rangle = \Sigma_2|\Psi\rangle ,$$

$$\ldots \qquad \ldots ,$$

which are nothing else but translations along the $Q^j$ axis singled out by the corresponding symmetry.

Integrating, we obtain oscillatory behavior for $|\Psi>$ in the directions of symmetries

$$|\Psi\rangle = \sum_{j=1}^{m} e^{i\Sigma_j Q^j}|\chi(Q^l)\rangle, \quad m < l \leq n$$

where $m$ is the number of symmetries, $l$ are the directions where symmetries do not exist, $n$ is the total dimension of the minisuperspace

- The $m$ symmetries give first integrals of motion and then the possibility to select classical trajectories according to the Hartle criterion.

# The Noether Symmetry Approach

- In one and two dimensional minisuperspaces, the existence of a Noether symmetry allows the complete solution of the problem and to get the full semi-classical limit of Quantum Cosmology.

- We can state that, in the semi-classical limit of Quantum Cosmology, the reduction procedure of dynamics, connected to the existence of Noether symmetries, allows to select a subset of the solution of WDW equation where oscillatory behaviors are found.

- This fact, in the framework of the Hartle interpretative criterion of the Wave Function of the Universe, gives conserved momenta and trajectories which can be interpreted as classical cosmological solutions.

- Vice versa, if a subset of the solution of WDW equation has an oscillatory behavior, conserved momenta exist and Noether symmetries are present

Noether symmetries select classical universes!

# The Noether Symmetry Approach

- We will show that such a statement holds for general classes of minisuperspaces and allows to select exact classical solutions.

- In this sense, the presence of Noether symmetries is a selection criterion for classical universes.

- Before this, let us discuss the general problem of Extended Theories of Gravity and their conformal properties.

- As we will see, most of theories of gravity can be conformally related to the Einstein one plus a suitable number of scalar fields. In this sense, the above standard minisuperspace approach works for any theory of gravity.

# Extending General Relativity

In Quantum Cosmology, the question of the **effective action of gravity** is crucial since, in general, we do not know the initial conditions from which our classical, observed Universe emerged

This means that general criteria to study minisuperspace models coming from Extended Gravity are extremely relevant towards a full theory of Quantum Gravity

We will consider two main features:

- geometry can couple non-minimally to some scalar field;
- higher-order curvature terms may appear.

In the first case, we have scalar-tensor gravity and in the second case we have higher-order gravity.


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# Extending General Relativity

A general class of higher-order-scalar-tensor theories in four dimensions is given by the action

$$\mathcal{S} = \int d^4x \sqrt{-g}\left[F(R, \Box R, \Box^2 R, \ldots, \Box^k R, \phi) - \frac{\epsilon}{2} g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \mathcal{L}^{(m)}\right]$$

In the metric approach, the field equations are obtained by varying with respect to $g_{\mu\nu}$

$$\begin{aligned} G^{\mu\nu} = \frac{1}{\mathcal{G}}\Bigg[ & \kappa T^{\mu\nu} + \frac{1}{2} g^{\mu\nu}(F - \mathcal{G}R) \\ & + (g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma})\mathcal{G}_{;\lambda\sigma} \\ & + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i}(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma})(\Box^{j-i})_{;\sigma} \\ & \times \left(\Box^{i-j}\frac{\partial F}{\partial \Box^i R}\right)_{;\lambda} - g^{\mu\nu}g^{\lambda\sigma} \\ & \times \left((\Box^{j-1}R)_{;\sigma}\Box^{i-j}\frac{\partial F}{\partial \Box^i R}\right)_{;\lambda}\Bigg], \end{aligned}$$

- $G^{\mu\nu}$ is the Einstein tensor

$$\mathcal{G} \equiv \sum_{j=0}^{n} \Box^j \left(\frac{\partial F}{\partial \Box^j R}\right)$$

# Extending General Relativity

- The simplest extension of GR is achieved assuming *F = f (R), ε = 0,* in the action
- The standard Hilbert–Einstein action is recovered for *f (R) = R*

Varying with respect to $g_{\alpha\beta}$ , we get

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} = \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)$$

and, after some manipulations

$$G_{\mu\nu} = \frac{1}{f'(R)}\left\{\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) + g_{\mu\nu}\frac{[f(R) - f'(R)R]}{2}\right\}$$

where the gravitational contribution due to higher-order terms can be reinterpreted as a stress-energy tensor contribution.

This means that additional and higher-order terms in the gravitational action act, in principle, as a “curvature” stress-energy tensor, related to the form of *f (R)* .



Considering also the standard perfect-fluid matter contribution, we have

$$G_{\alpha\beta} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\Box f'(R)\right\} + \frac{\kappa T^{(m)}_{\alpha\beta}}{f'(R)} = T^{(\mathrm{curv})}_{\alpha\beta} + \frac{T^{(m)}_{\alpha\beta}}{f'(R)}$$

In the case of GR, it identically vanishes while the standard, minimal coupling is recovered for the matter contribution.

It is an effective stress-energy tensor constructed by the extra curvature terms.

The peculiar behavior of $f(R) = R$ is due to the particular form of the Lagrangian itself which, even though it is a second-order Lagrangian, can be non-covariantly rewritten as the sum of a first-order Lagrangian plus a pure divergence term.

# Extending General Relativity

From the general action it is possible to obtain another interesting case by choosing

$$F = F(\phi)R - V(\phi), \qquad \epsilon = -1$$

The variation with respect to $g_{\mu\nu}$ gives the second-order field equations

$$F(\phi)G_{\mu\nu} = F(\phi)\left[R_{\mu\nu} - \frac{1}{2}R_{\mu\nu}\right] = -\frac{1}{2}T^{\phi}_{\mu\nu} - g_{\mu\nu}\Box_g F(\phi) + F(\phi)_{;\mu\nu}$$

The energy-momentum tensor relative to the scalar field is

$$T^{\phi}_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{\alpha}_{;} + g_{\mu\nu}V(\phi)$$

The variation with respect to ϕ provides the Klein–Gordon equation, i.e. the field equation for the scalar field:

$$\Box_g\phi - RF_{\phi}(\phi) + V_{\phi}(\phi) = 0$$

This last equation is equivalent to the Bianchi contracted identity

# Extended Minisuperspace Models

- The existence of a Noether symmetry for a given minisuperspace is *selection rule* to recover classical behavior in cosmic evolution

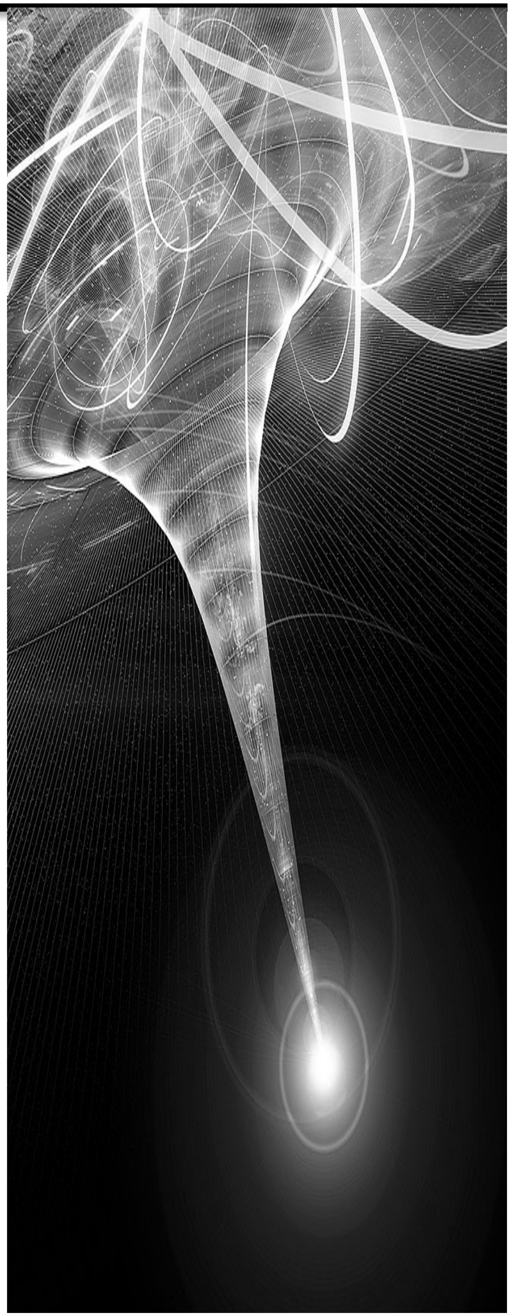
- *Hartle criterion* to select correlated regions in the configuration space of dynamical variables is *directly* connected to the presence of a Noether symmetry. Such a statement works for minisuperspace models coming from Extended Gravity.

- The approach is connected to the search for Lagrange multipliers

- Imposing Lagrange multipliers allows to modify dynamics and select the form of effective potentials.

- By integrating the multipliers, solutions can be achieved.

Such solutions are CLASSICAL UNIVERSES!



The case of non-minimally coupled theory of gravity

$$\mathcal{S} = \int d^4x \sqrt{-g}\left[F(\phi)R + \frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu - V(\phi)\right].$$

Let us restrict to a FRW minisuperspace

The Lagrangian becomes point-like, that is

$$X = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\phi}}$$

The configuration space of such a Lagrangian is Q ≡ {a,ϕ} , i.e. a 2-dimensional minisuperspace

A Noether symmetry exists if $L_X\mathcal{L} = 0$

In this case, it has to be

$$X = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\phi}}$$

# Extended minisuperspace models

## Scalar-tensor Gravity Cosmologies

The system of partial differential equations is

$$aF(\phi)\left[\alpha + 2a\frac{\partial\alpha}{\partial a}\right] + aF'(\phi)\left[\beta + a\frac{\partial\beta}{\partial a}\right] = 0,$$

$$3\alpha + 12F'(\phi)\frac{\partial\alpha}{\partial\phi} + 2a\frac{\partial\beta}{\partial\phi} = 0,$$

Prime indicates the derivative with respect to **ϕ**

$$a\beta F''(\phi) + \left[2\alpha + a\frac{\partial\alpha}{\partial a} + \frac{\partial\beta}{\partial\phi}\right]F'(\phi) + 2\frac{\partial\alpha}{\partial\phi}F(\phi) + \frac{a^2}{6}\frac{\partial\beta}{\partial a} = 0$$

$$\left[3\alpha V(\phi) + a\beta V'(\phi)\right]a^2 + 6k\left[\alpha F(\phi) + a\beta F'(\phi)\right] = 0.$$

The number of equations is 4 as it has to be, n = 2 being the **Q** dimension

They select the model since the system gives α,β , F(ϕ) and V (ϕ) .

# Extended minisuperspace models | Scalar-tensor Gravity Cosmologies

For example, if the spatial curvature is k = 0, a solution is

$$\alpha = -\frac{2}{3}p(s)\beta_0 a^{s+1}\phi^{m(s)-1}, \qquad \beta = \beta_0 a^s \phi^{m(s)},$$

$$F(\phi) = D(s)\phi^2, \qquad V(\phi) = \lambda\phi^{2p(s)}, \qquad m(s) = \frac{2s^2+6s+3}{2s+3},$$

$$D(s) = \frac{(2s+3)^2}{48(s+1)(s+2)}, \qquad p(s) = \frac{3(s+1)}{2s+3},$$

s,λ are free parameters

Changing the variables $w = \sigma_0 a^3 \phi^{2p(s)}$ and $z = \dfrac{3}{\beta_0 \chi(s)} a^{-s}\phi^{1-m(s)}$

integration constant

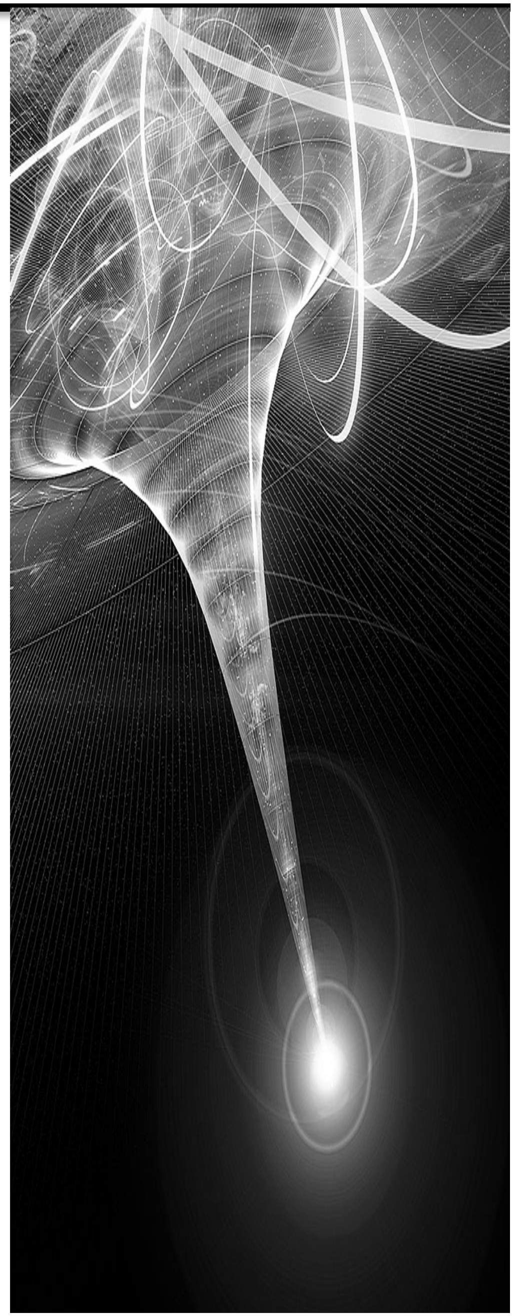
$$\chi(s) = -\frac{6s}{2s+3}$$

The above Lagrangian becomes, for k = 0,

$$\mathcal{L} = \gamma(s) w^{s/3}\dot{z}\dot{w} - \lambda w$$

where z is cyclic and

$$\gamma(s) = \frac{2s+3}{12\sigma_0^2(s+2)(s+1)}$$



## Extended minisuperspace models



The conjugate momenta are

$$\pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \gamma(s) w^{s/3} \dot{w},$$

$$\pi_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = \gamma(s) w^{s/3} \dot{z},$$

and the Hamiltonian is $$\tilde{\mathcal{H}} = \frac{\pi_z \pi_w}{\gamma(s) w^{s/3}} + \lambda w$$

The Noether symmetry is given by $$\pi_z = \Sigma_0$$

Quantizing the conjugate momenta, we get

$$\pi \longrightarrow -i\partial_z, \qquad \pi_w \longrightarrow -i\partial_w$$

and then the WDW equation $$[(i\partial_z)(i\partial_w) + \tilde{\lambda} w^{1+s/3}]|\Psi\rangle = 0$$

$$\tilde{\lambda} = \gamma(s)\lambda$$

The quantum version of Noether symmetry is

$$-i\partial_z|\Psi\rangle = \Sigma_0|\Psi\rangle$$

dynamics results reduced

A straightforward integration gives

$$|\Psi\rangle = \big|\Omega(w)\big\rangle \big|\chi(z)\big\rangle \propto e^{i\Sigma_0 z} e^{-i\tilde{\lambda} w^{2+s/3}}$$

which is an oscillating wave function where the Hartle criterion is recovered

- In the semi-classical limit, we have two first integrals of motion: $\Sigma_0$ (i.e. the equation for $\pi_z$ ) and $E_L = 0$, i.e. the Hamiltonian which becomes the equation for $\pi_w$
- Classical trajectories in the configuration space $\tilde{Q} \equiv \{w,z\}$ are immediately recovered

$$w(t) = [k_1 t + k_2]^{3/(s+3)},$$
$$z(t) = [k_1 t + k_2]^{(s+6)/(s+3)} + z_0,$$

then, going back to $Q \equiv \{a,\phi\}$ , we get the classical cosmological solutions

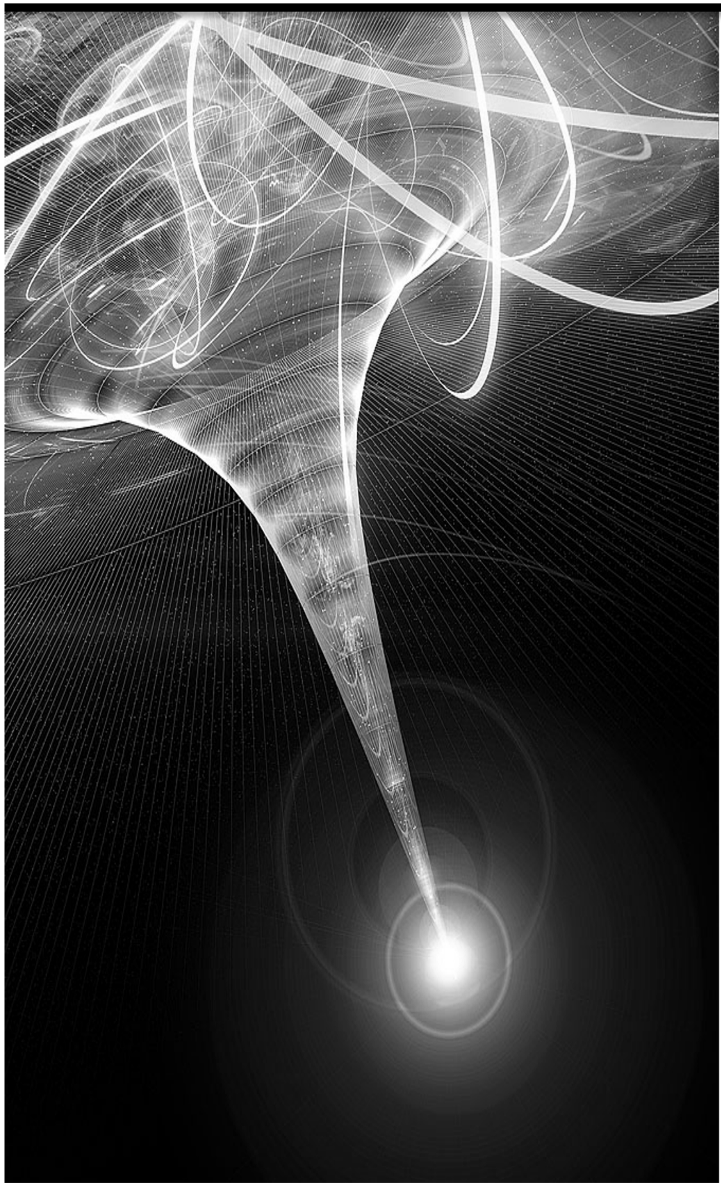
$$a(t) = a_0 (t - t_0)^{l(s)}, \qquad l(s) = \frac{2s^2 + 9s + 6}{s(s+3)}$$
$$\phi(t) = \phi_0 (t - t_0)^{q(s)} \qquad q(s) = -\frac{2s + 3}{s}$$

which means that Hartle criterion selects classical universes

Depending on the value of **s** , we get Friedman, power-law, or pole-like behaviors.

# Extended minisuperspace models

## Fourth-order Gravity Cosmologies



The case of fourth-order gravity

$$\mathcal{S} = \int d^4x \sqrt{-g} f(R)$$

Reducing the action to a point-like, FRW one, we have

$$\mathcal{S} = \int dt\, \mathcal{L}(a, \dot{a}; R, \dot{R})$$

where dot means derivative with respect to the cosmic time

The scale factor *a* and the Ricci scalar *R* are the canonical Variables.

The definition of *R* in terms of *a* and its derivatives introduces a constraint which eliminates second and higher order derivatives in the action, and yields a system of second-order differential equations in {a,R} .

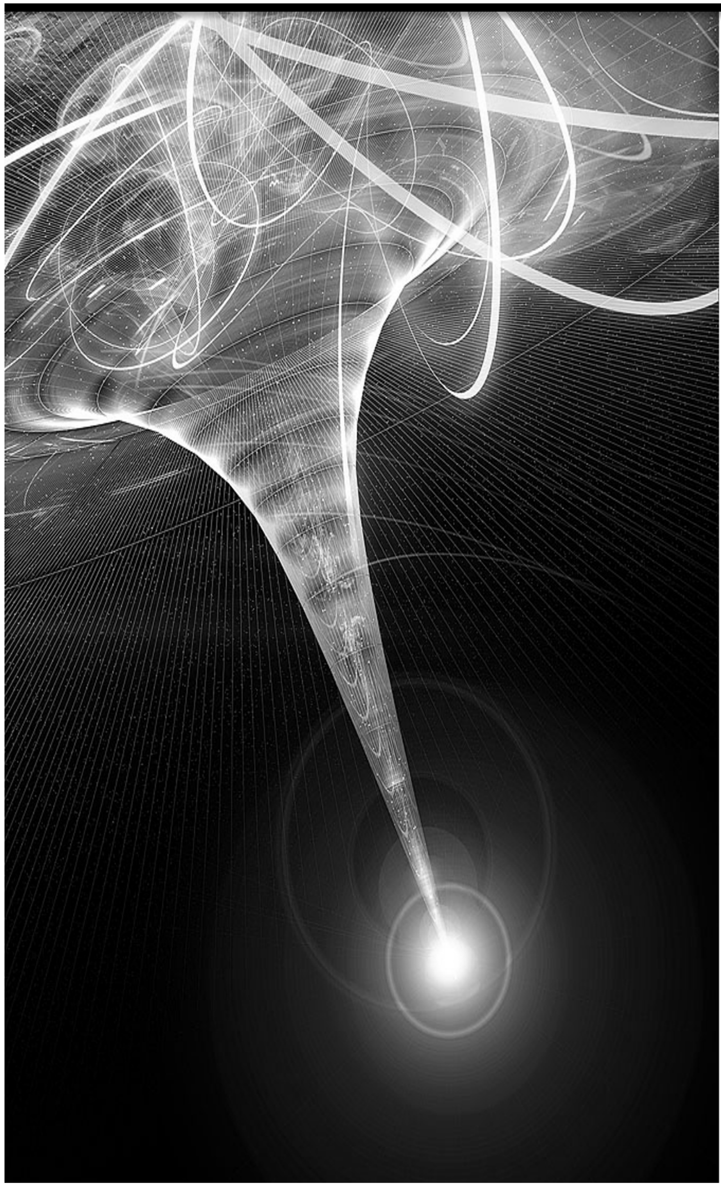
Action can be written as

$$\mathcal{S} = 2\pi^2 \int dt \left\{ a^3 f(R) - \lambda \left[ R + 6\left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\}$$

where the Lagrange multiplier is $\lambda = a^3 f'(R)$

# Extended minisuperspace models

## Fourth-order Gravity Cosmologies



Let us introduce the auxiliary field

$$p \equiv f'(R),$$

so that the Lagrangian becomes

$$\mathcal{L} = 6a\dot{a}^2 p + 6a^2\dot{a}\dot{p} - 6kap - a^3 W(p)$$

This is an Helmholtz-like Lagrangian and *a,p* are independent fields

The potential *W(p)* is defined as $W(p) = h(p)p - r(p)$

is the inverse function of f' $\longleftarrow h(p) = R$

$$r(p) = \int f'(R)\,dR = \int p\,dR = f(R).$$

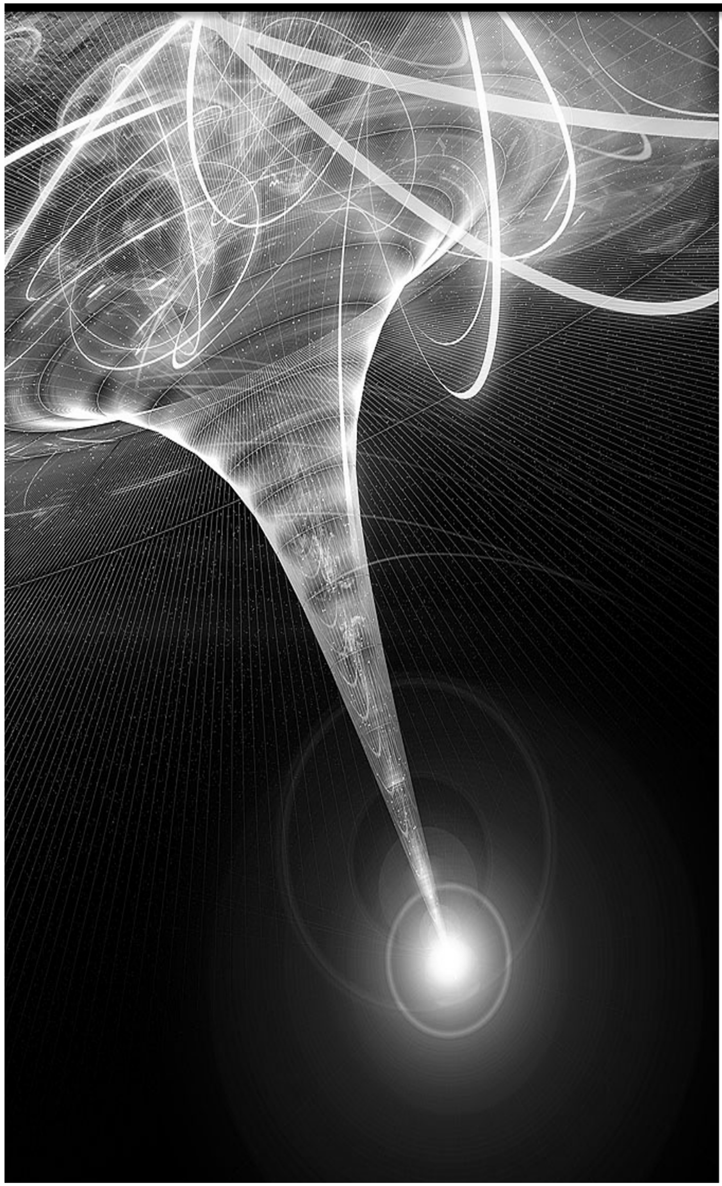
The configuration space is now *Q ≡ {a,p}* and *p* has the same role of the above ϕ

The vector field is

$$X = \alpha(a,p)\frac{\partial}{\partial a} + \beta(a,p)\frac{\partial}{\partial p} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{p}}.$$

# Extended minisuperspace models

## Fourth-order Gravity Cosmologies



The system is

$$a\,p\left[\alpha+2a\frac{\partial\alpha}{\partial a}\right]p+a\left[\beta+a\frac{\partial\beta}{\partial a}\right]=0,$$

$$a^2\frac{\partial\alpha}{\partial p}=0,$$

$$2\alpha+a\frac{\partial\alpha}{\partial a}+2p\frac{\partial\alpha}{\partial p}+a\frac{\partial\beta}{\partial p}=0,$$

$$6k[\alpha p+\beta a]+a^2\left[3\alpha W+a\beta\frac{\partial W}{\partial p}\right]=0.$$

The solution of this system, i.e. the existence of a Noether symmetry, gives α , β and *W(p)*, it is satisfied for

$$\alpha=\alpha(a),\qquad \beta(a,p)=\beta_0 a^s p,$$

In particular, we obtain the solutions

$$s=0\longrightarrow\alpha(a)=-\frac{\beta_0}{3}a,\quad \beta(p)=\beta_0 p,$$

$$W(p)=W_0 p,\quad k=0,$$

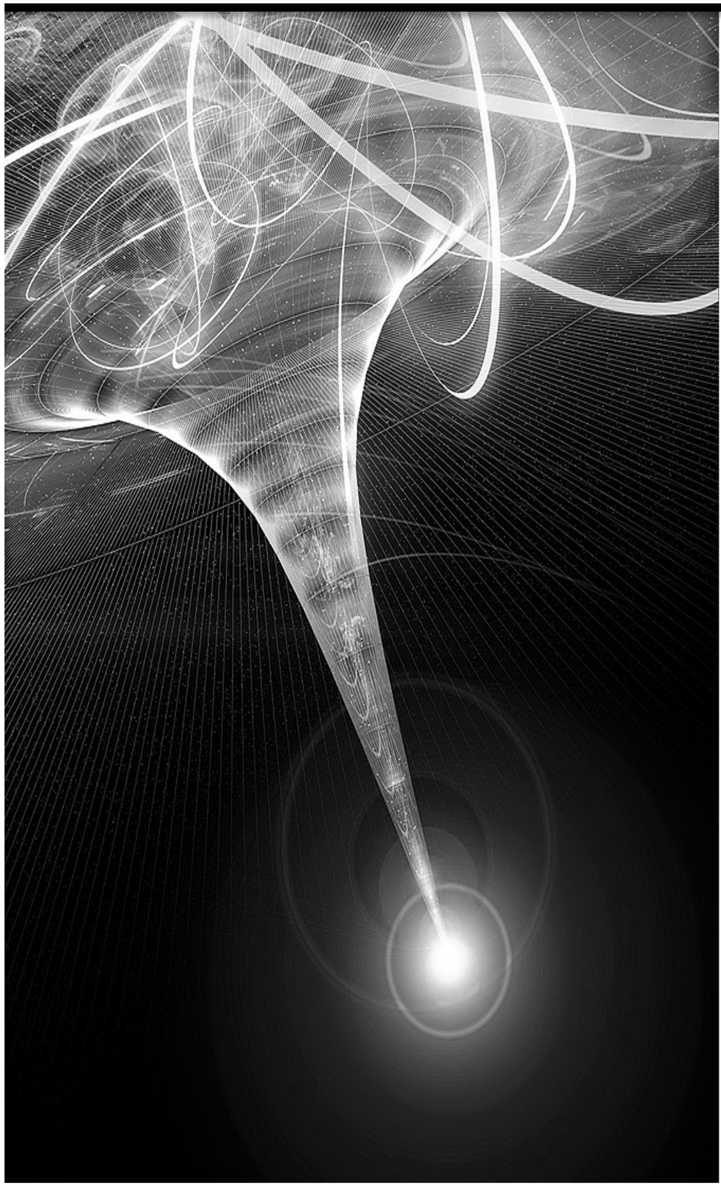
$$s=-2\longrightarrow\alpha(a)=-\frac{\beta_0}{a},\quad \beta(a,p)=\beta_0\frac{p}{a^2},$$

$$W(p)=W_1 p^3,\quad \forall k,$$

Let us discuss separately the solutions

# Extended minisuperspace models

## Fourth-order Gravity Cosmologies



- The case *s* = ***0***

The induced change of variables

$$\mathcal{Q} \equiv \{a, p\} \longrightarrow \tilde{\mathcal{Q}} \equiv \{w, z\}$$

can be written $\quad w(a, p) = a^3 p, \qquad z(p) = \ln p.$

Lagrangian (whit z cyclic cariable) becomes

$$\tilde{\mathcal{L}}(w, \dot{w}, \dot{z}) = \dot{z}\dot{w} - 2w\dot{z}^2 + \frac{\dot{w}^2}{w} - 3W_0 w,$$

The conjugate momenta are

$$\pi_z \equiv \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}} = \dot{w} - 4\dot{z} = \Sigma_0,$$

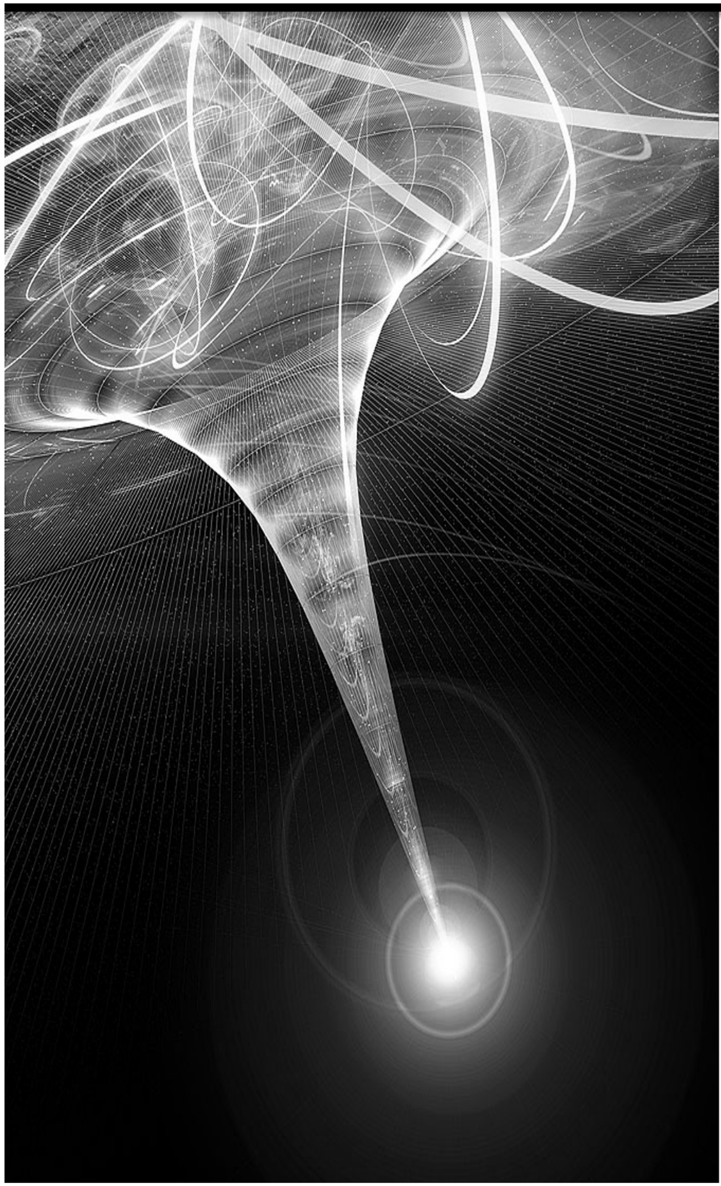
$$\pi_w \equiv \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{w}} = \dot{z} + 2\frac{\dot{w}}{w},$$

and the Hamiltonian is

$$\mathcal{H}(w, \pi_w, \pi_z) = \pi_w \pi_z - \frac{\pi_z^2}{w} + 2w\pi_w^2 + 6W_0 w$$

# Extended minisuperspace models

## Fourth-order Gravity Cosmologies



By canonical quantization, reduced dynamics is given by

$$\left[\partial_z^2 - 2w^2\partial_w^2 - w\partial_w\partial_z + 6W_0w^2\right]|\Psi\rangle = 0,$$

$$-i\partial_z|\Psi\rangle = \Sigma_0|\Psi\rangle.$$

the wave function has an oscillatory factor, being

$$|\Psi\rangle \sim e^{i\Sigma_0 z}\left|\chi(w)\right\rangle$$

The function $|\chi>$ satisfies the Bessel differential equation

$$\left[w^2\partial_w^2 + i\frac{\Sigma_0}{2}w\partial_w + \left(\frac{\Sigma_0^2}{2} - 3W_0w^2\right)\right]\chi(w) = 0,$$

whose solutions are linear combinations of Bessel functions $Z_\nu(w)$

$$\chi(w) = w^{1/2 - i\Sigma_0/4}Z_\nu(\lambda w)$$

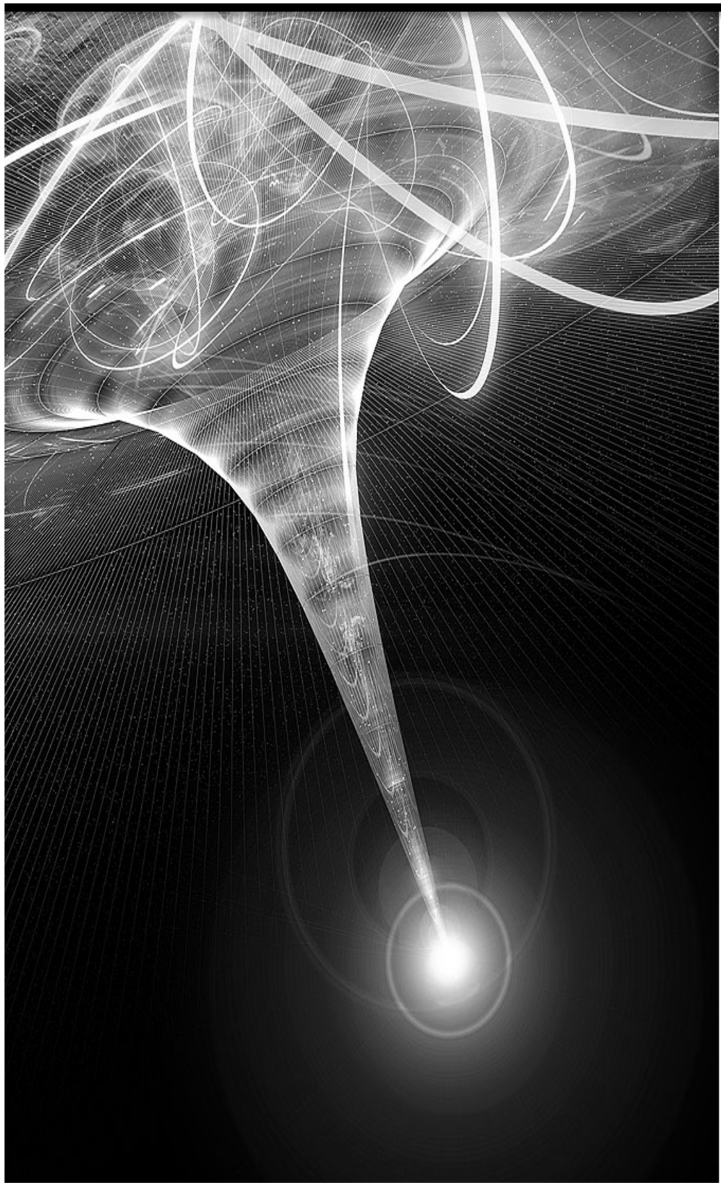
where

$$\nu = \pm\frac{1}{4}\sqrt{4 - 9\Sigma_0^2 - i4\Sigma_0}, \qquad \lambda = \pm 9\sqrt{\frac{W_0}{2}}.$$

The oscillatory regime for this component depends on the reality of $\nu$ and $\lambda$

# Extended minisuperspace models

## Fourth-order Gravity Cosmologies



The wave function of the universe, from Noether symmetry, is then

$$\Psi(z,w) \sim e^{i\,\Sigma_0[z-(1/4)\ln w]} w^{1/2} Z_\nu(\lambda w)$$

For large *w* , the Bessel functions have an exponential behavior, so that the wave function can be written as

$$\Psi \sim e^{i[\Sigma_0 z-(\Sigma_0/4)\ln w \pm \lambda w]}$$

Due to the oscillatory behavior of Ψ , Hartle's criterion is immediately recovered.

By identifying the exponential factor of with $S_0$ , we can recover the conserved momenta $\pi_z$,$\pi_w$ and select classical trajectories.

Going back to the old variables, we get the cosmological solutions

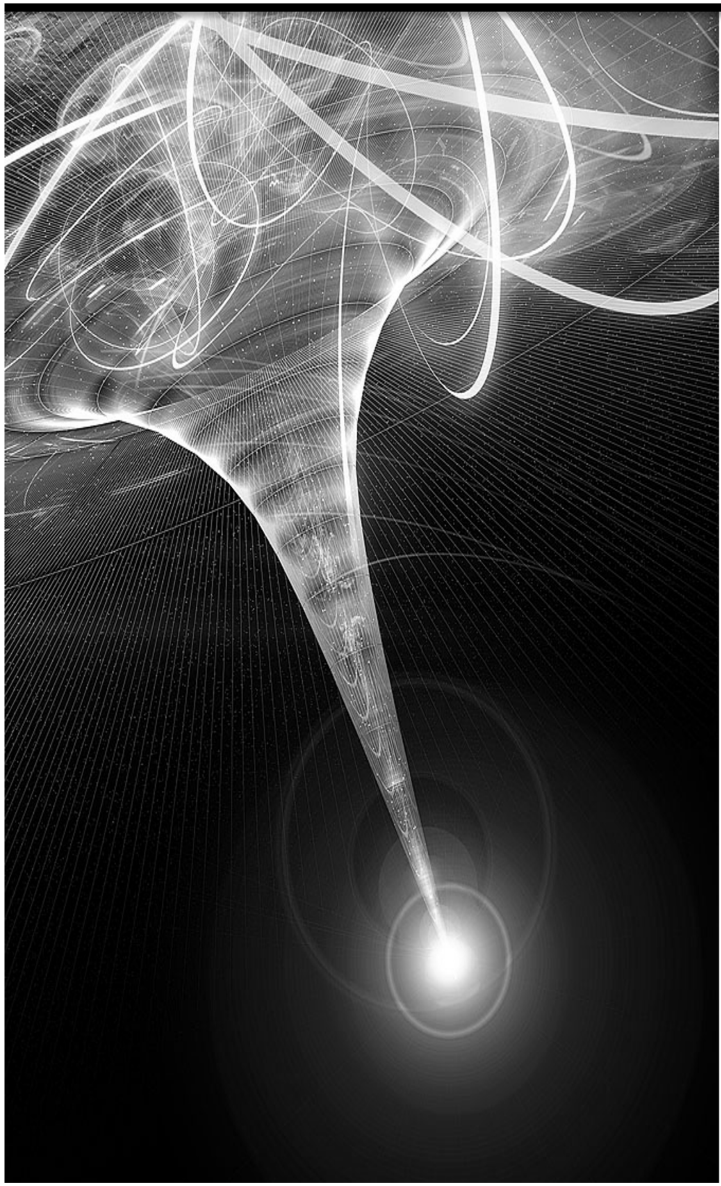
$$a(t) = a_0 e^{(\lambda/6)t} \exp\left\{-\frac{z_1}{3} e^{-(2\lambda/3)t}\right\}$$

$$p(t) = p_0 e^{(\lambda/6)t} \exp\left\{z_1 e^{-(2\lambda/3)t}\right\},$$

It is clear that λ plays the role of a cosmological constant and inflationary behavior is asymptotically recovered.

# Extended minisuperspace models

## Fourth-order Gravity Cosmologies



The case s = **−2**

The new variables adapted to the foliation for the solution are now

$$w(a,p) = ap, \qquad z(a) = a^2$$

and Lagrangian assumes the form

$$\tilde{\mathcal{L}}(w, \dot{w}, \dot{z}) = 3\dot{z}\dot{w} - 6kw - W_1 w^3$$

The conjugate momenta are

$$\pi_z = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}} = 3\dot{w} = \Sigma_1,$$

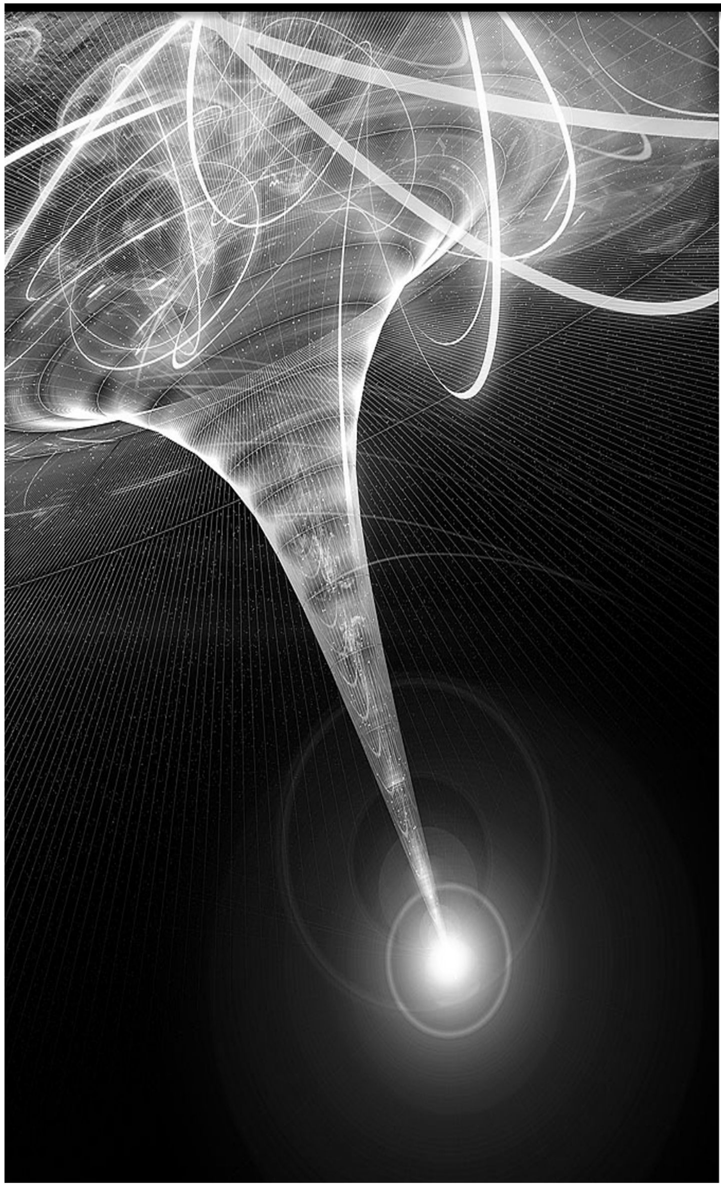
$$\pi_w = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{w}} = 3\dot{z}.$$

The Hamiltonian is given by

$$\mathcal{H}(w, \pi_w, \pi_z) = \frac{1}{3}\pi_z \pi_w + 6kw + W_1 w^3$$

# Extended minisuperspace models

## Fourth-order Gravity Cosmologies



Going over the same steps as above, the wave function of the universe is given by

$$\Psi(z, w) \sim e^{i[\Sigma_1 z + 9kw^2 + (3W_1/4)w^4]}$$

and the classical cosmological solutions are

$$a(t) = \pm\sqrt{h(t)}, \qquad p(t) = \pm\frac{c_1 + (\Sigma_1/3)t}{\sqrt{h(t)}}$$

where $\rightarrow$

$$h(t) = \left(\frac{W_1\Sigma_1^3}{36}\right)t^4 + \left(\frac{W_1 w_1 \Sigma_1}{6}\right)t^3 + \left(k\Sigma_1 + \frac{W_1 w_1^2 \Sigma_1}{2}\right)t^2 + w_1(6k + W_1 w_1^2)t + z_2.$$

$w_1$, $z_1$ and $z_2$ are integration constants

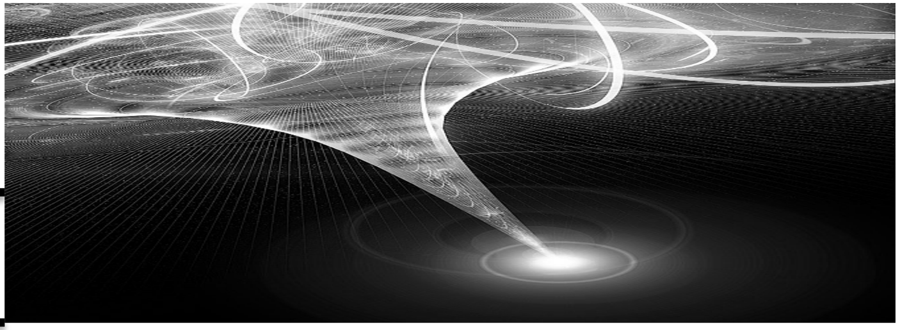
Immediately we see that, for large t

$$a(t) \sim t^2, \qquad p(t) \sim \frac{1}{t},$$

which is a power-law inflationary behavior.

# Extended minisuperspace models

## Higher than fourth-order Gravity Cosmologies



The case of higher than fourth-order theories of gravity

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\,f(R,\Box R)$$

The configuration space is Q ≡ {a,R, □R} considering □R as an independent degree of freedom.

The FRW point-like Lagrangian is formally $\mathcal{L} = \mathcal{L}\big(a, \dot{a}, R, \dot{R}, \Box R, (\dot{\Box R})\big)$

and the constraints $R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$ and $\Box R = \ddot{R} + 3\frac{\dot{a}}{a}\dot{R}$

Using the above Lagrange multipliers, we get the Helmholtz point-like Lagrangian

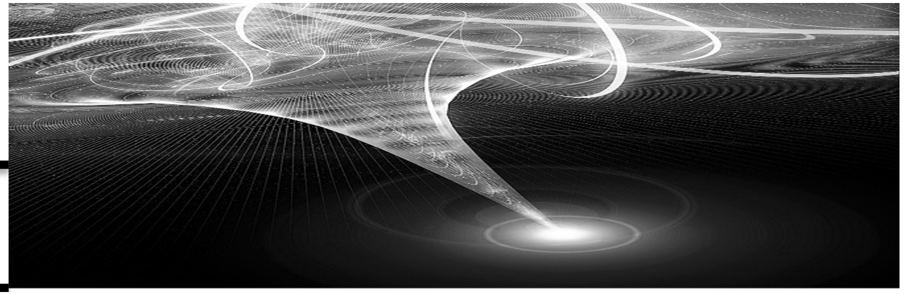
$$\mathcal{L} = 6a\dot{a}^2 p + 6a^2\dot{a}\dot{p} - 6kap - a^3\dot{h}q - a^3 W(p,q)$$

where $p \equiv \frac{\partial f}{\partial R}$, $q \equiv \frac{\partial f}{\partial \Box R}$, and $W(p,q) = h(p)p + g(q)q - f$,

and $h(p) = R$, $g(q) = \Box R$, $f = f(R,\Box R)$

Now the minisuperspace is 3-dimensional but, again, Noether symmetries can be selected.

# Extended minisuperspace models



## Higher than fourth-order Gravity Cosmologies

A case of physical interest is $f(R, \Box R) = F_0 R + F_1\sqrt{R\Box R},$

we get ➡ $\tilde{\mathcal{L}} = 3\left[w\dot{w}^2 - kw\right] - F_1\left[3w\dot{w}^2 u + 3w^2\dot{w}\dot{u} + \frac{w^3\dot{z}\dot{u}}{2u^2} - 3kwu\right]$

where we assume $F_0 = -1/2$, the standard Einstein coupling, z is the cyclic variable and

$z = R, \qquad u = \sqrt{\frac{\Box R}{R}}, \qquad w = a.$ and the conserved quantity is $\Sigma_0 = \frac{w^3\dot{u}}{2u^2}$

By canonical quantization and WDW equation, the Wave Function of the Universe is

$$|\Psi\rangle \sim e^{i\Sigma_0 z}\left|\chi(u)\right\rangle\left|\Theta(w)\right\rangle.$$

where χ and Θ are combinations of Bessel functions. The oscillatory subset of the solution is evident and the Hartle criterion is recovered. Classical cosmological solutions are

$$a(t) = a_0 t, \qquad a(t) = a_0 t^{1/2}, \qquad a(t) = a_0 e^{k_0 t}$$

## Discussion and conclusions

- We have discussed the Minisuperspace Approach to Quantum Cosmology

- This one does not give a satisfactory solution to the full Quantum Gravity problem, however, it is a useful scheme to set the problem of boundary conditions from which should emerge classical universes, that is, cosmological dynamical models that could be reasonably observed with standard astrophysical tools.

- A main role in this approach is played by the identification of conserved quantities that give rise to peaked behaviors in the Wave Function of the Universe.

- Such a function is the solution of the WDW equation, the corresponding of

Schrödinger equation in Quantum Cosmology.

# Discussion and conclusions

- Peaked behaviors mean correlations among variables and then the possibility to obtain classical universes according to the Hartle interpretative criterion.

- These conserved quantities can naturally be related to the Noether symmetries of the theory.

- In this sense, the Noether symmetries allow to reduce the dynamics and recover classical solutions.

# Discussion and conclusions

- The emergence of singularities at finite for such solutions means that symmetries are broken for certain values of the parameters.

- Reversing the argument, if the wave function of the universe is related to the probability to get a classical cosmological solution, the existence of Noether symmetries tell us when the Hartle criterion works.

Some remarks are necessary at this point

- We have to stress that the wave function is only related to the probability to get a certain behavior but it is not the probability amplitude since, till now, Quantum Cosmology is not a unitary theory.

- The Hartle criterion works in the context of an Everett-type interpretation of Quantum Cosmology which assumes the ideas that the Universe branches into a large number of copies of itself whenever a measurement is made.

- This point of view is called Many Worlds interpretation of Quantum Cosmology

# Discussion and conclusions

- Such an interpretation is just one way of thinking and gives a formulation of Quantum Mechanics designed to deal with correlations internal to individual, isolated systems.

- The Hartle criterion gives an operative interpretation of such correlations.

- In particular, if the wave function is strongly peaked in some region of configuration space, we predict that we will observe the correlations which characterize that region.

- if the wave function is smooth in some region, we predict that correlations which characterize that region are precluded to the observations.

- If the wave function is neither peaked nor smooth, no predictions are possible from observations. In other words, we can read the correlations of some region of minisuperspace as causal connection.

- The analogy with standard Quantum Mechanics is straightforward!

# Discussion and conclusions

- By considering the case in which the individual system consists of a large number of identical subsystems, one can derive from the above interpretation, the usual probabilistic interpretation of Quantum Mechanics for the subsystems.

- What we proposed is an approach by which the Hartle criterion can be recovered without arbitrariness.

- If a Noether symmetry (or more than one) is selected for a given minisuperspace model, then strongly peaked (oscillatory) subsets of the wave function of the Universe are found.

- Vice versa, oscillatory parts of the wave function can be always connected to conserved momenta and then to Noether symmetries.