Is Quantum Theory Exact?

From quantum foundations to quantum applications

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Wave Equations Derived From First Order Invariance Conditions*

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QUANTUM THEORY OF A FREE PARTICLE:

STARTING FROM SIMMETRY PRINCIPLE:

The Theory is invariant under Galilei transformations

$\Downarrow$ By **mathematical deduction**

$\Downarrow$ methods by Bargmann, Mackey, Wigner

Quantum Theory of a FREE particle:

- Specific Hilbert space $\mathcal{H} = L_2(\mathbf{R}^3, \mathbf{C}^{2s+1})$
- Position operators: $Q_\alpha \psi(\mathbf{x}) = x_\alpha \psi(\mathbf{x})$
- Wave Equation $i\frac{d\psi_t}{dt}(\mathbf{x}) = -\frac{1}{2\mu}\sum_\alpha \frac{\partial^2 \psi_t}{\partial x_\alpha^2}(\mathbf{x})$

THIS THEORY IS EXACT!

Real question:

Can the Quantum Theory of an *interacting* particle be considered exact?

Yes, if a deductive development is discovered that yields such a theory.

For instance, if the currently practized wave equation

$$i\frac{d\psi_t}{dt}(\mathbf{x}) = -\frac{1}{2\mu}\sum_\alpha \left(\frac{\partial}{\partial x_\alpha}(\mathbf{x}) - a_\alpha(\mathbf{x})\right)^2 \psi_t(\mathbf{x}) + \Phi(\mathbf{x})\psi_t(\mathbf{x})$$

can be derived from physical principles.

PROBLEM:

Extension of the deductive method for free particle to an interacting particle does not work:

Galilei group $\mathcal{G}$ is NOT a SYMMETRY group

$\Downarrow$ ~~Theorem of Wigner, Theorem of Mackey~~

- Specific Hilbert space $\mathcal{H} =$?
- No Position operators: Q_α
- No Wave equation: $H =$?

THE APPROACH STOPS!

THE PRESENT WORK IDENTIFIES AN OBJECTIVE CRITERIUM FOR EXACTENESS OF INTERACTING PARTICLE WAVE EQUATIONS, RELATED TO THE "DEGREE" OF INVARIANCE LEFT BY THE SPECIFIC INTERACTION

I. Mathematical tools

II. Quantum Transformations for interacting systems

III. Development of the theory

IV. Exact Wave equations for interacting particle

V. Specific Wave equations

I. MATHEMATICAL TOOLS

NOTATION:

$\mathcal{H}$ Hilbert space of the Quantum Theory

- $\mathcal{U}(\mathcal{H})$ unitary operators
- $\mathcal{S}(\mathcal{H})$ density operators (states)
- $\Omega(\mathcal{H})$ self-adjoint operators (observables)
- $\Pi(\mathcal{H})$ projection operators
- $\mathcal{G}$ Galilei group
- $\mathcal{E} = \mathbf{R}^3 \circledS SO(3)$, Euclide subgroup:

  $g \in \mathcal{E}$, $\mathsf{g}(\mathbf{x}) = R^{-1}\mathbf{x} - R^{-1}\mathbf{a}$

I. MATHEMATICAL TOOLS

Definition: projective representation of a group:

A mapping $U : G \to \mathcal{U}(\mathcal{H})$ with $U(e) = \mathbb{I}$

such that $U_{g_1 g_2} = \sigma(g_1, g_2) U_{g_1} U_{g_2}$, $\sigma(g_1, g_2) \in \mathbb{C}$

$\mathcal{E} = \mathbf{R}^3 \circledS SO(3)$, $\mathsf{g}(\mathbf{x}) = R^{-1}\mathbf{x} - R^{-1}\mathbf{a}$

Let $U : \mathcal{E} \to \mathcal{U}(\mathcal{H})$, $g \to U_g$ be a proj. rep. of $\mathcal{E}$

Definition. Given a projective representation U of $\mathcal{E}$, an *Imprimitivity System* for U is a PV measure $E : \mathcal{B}(\mathbf{R}^3) \to \Pi(\mathcal{H})$ such that

$$U_g E(\Delta) U_g^{-1} = E(\mathsf{g}^{-1}(\Delta))\,, \quad \forall g \in \mathcal{E}$$

I. MATHEMATICAL TOOLS

Mackey's imprimitivity theorem

If $E : \mathcal{B}(\mathbf{R}^3) \to \Pi(\mathcal{H})$ is an imprimitivity system for a *continuous* proj. rep. $U : \mathcal{E} \to \mathcal{U}(\mathcal{H})$

Then a proj.rep. $L : SO(3) \to \mathcal{U}(\mathcal{H}_0)$ exists:

$\mathcal{H} = L_2(\mathbf{R}^3, \mathcal{H}_0)$,

$(U_g\psi)(\mathbf{x}) = L_R\psi(\mathsf{g}(\mathbf{x}))$

$E(\Delta)\psi(\mathbf{x}) = \chi_\Delta(\mathbf{x})\psi(\mathbf{x})$

modulo unitary isomorphisms

II. Quantum Transformations of observables

Wigner Theorem and **Imprimitivity Theorem**

Main tools to derive Quantum Theory

They require that:

i) every $g \in \mathcal{G}$ is a symmetry, to assign U_g unitary (Wigner theorem),

ii) $g \to U_g$ should be a projective representation.

Active interpretation $\Rightarrow$ g is not a symmetry: **(i) fails**

II. Quantum Transformations of observables

Def. $g \in \mathcal{G}$, $\Sigma \xrightarrow{g} \Sigma_g$, $\mathcal{M}_1, \mathcal{M}_2$ measuring devices.

$\mathcal{M}_1, \mathcal{M}_2$ *indistinguishable relative to* (Σ, Σ_g) if $\mathcal{M}_1$ is relatively to Σ identical to what is $\mathcal{M}_2$ relatively to Σ_g.

Quantum Transformation corresponding to $g \in \mathcal{G}$.

$$S_g^{\Sigma} : \Omega(\mathcal{H}) \to \Omega(\mathcal{H}),\ A \to S_g^{\Sigma}[A] \equiv B$$

$B = S_g^{\Sigma}[A]$ *is an observable measurable by a device* $\mathcal{M}_2$
indistinguishable relative to (Σ, Σ_g),
from a device $\mathcal{M}_1$ *that measures* A,

II. Quantum Transformations of observables

General Properties of Quantum Transformations

(S.1) $S_g^\Sigma : \Omega(\mathcal{H}) \to \Omega(\mathcal{H})$ is bijective.

(S.2) If $B = f(A)$ then $f(S_g^\Sigma[A]) = S_g^\Sigma[f(A)]$.

If the device of A is relatively to Σ identical to the device of $S_g^\Sigma[A]$ relatively to Σ_g,
then transforming both outcomes by the same f does not affect relative indistinguishability.

(S.3) $S_{gh}^\Sigma[A] = S_g^{\Sigma_h}[S_h^\Sigma[A]]$

III. DEVELOPMENT OF THE THEORY

For each $g \in \mathcal{G}$, consider S_g^Σ

Theorem. Conditions (S.1), (S.2) imply that Wigner theorem apply, so that an essentially unique operator U_g , unitary or anti-unitary, exists for $g \in \mathcal{G}$ such that

$$S_g^\Sigma[A] = U_g A U_g^*, \qquad \forall A \in \Omega(\mathcal{H}).$$

Furthermore,

if $g \to S_g^\Sigma \mid_{\Pi(\mathcal{H})}$ is Bargmann-continuous

then $g \to U_g$ is continuous and each U_g is unitary.

III. DEVELOPMENT OF THE THEORY

$U : \mathcal{G} \to \mathcal{U}(\mathcal{H})$ exists such that $U_g A U_g^{-1} = S_g^{\Sigma}[A]$

But $g \to U_g$ is NOT a projective representation:

Imprimitivity theorem does not apply!

Idea:

σ-**conversion** $\{g \to U_g\} \to \{g \to \widehat{U}_g\}$

where V_g is a unitary and continuous in g such that

$g \to \widehat{U}_g = V_g U_g$ is a projective represenation

Remark: A σ- conversion always exists.

III. DEVELOPMENT OF THE THEORY

$g \to \widehat{U}_g = V_g U_g$ continuous proj. representation

$\Rightarrow$ 9 generators $\widehat{P}_\alpha$, $\widehat{J}_\alpha$, $\widehat{G}_\alpha$ exist such that

$[\widehat{P}_\alpha, \widehat{P}_\beta] = \mathbf{0}$, $[\widehat{J}_\alpha, \widehat{J}_\beta] = i\epsilon_{\alpha\beta\gamma}\widehat{J}_\gamma$, $[\widehat{J}_\alpha, \widehat{P}_\beta] = i\epsilon_{\alpha\beta\gamma}\widehat{P}_\gamma$,

$[\widehat{G}_\alpha, \widehat{G}_\beta] = \mathbf{0}, [\widehat{J}_\alpha, \widehat{G}_\beta] = i\epsilon_{\alpha\beta\gamma}\widehat{G}_\gamma$,

$[\widehat{G}_\alpha, \widehat{P}_\beta] = i\delta_{\alpha,\beta}\mu$.

$$\Rightarrow \quad \widehat{U}_g \mathbf{F} \widehat{U}_g^{-1} = \mathsf{g}(\mathbf{F}), \quad \mathbf{F} = \frac{\widehat{\mathbf{G}}}{\mu}, \quad g \in \mathcal{E} \qquad (Cov)$$

Given $\Delta \to E(\Delta)$ common PV measure of $\mathbf{F}$

$\widehat{U}_g \mathbf{F} \widehat{U}_g^{-1} = \mathsf{g}(\mathbf{F})$ (Cov) $\Rightarrow \widehat{U}_g E(\Delta) \widehat{U}_g^{-1} = \mathsf{g}^{-1}(\Delta)$

$\Delta \to E(\Delta)$ imprimitivity system for $\widehat{U}\mid_{\mathcal{E}}$:

III. DEVELOPMENT OF THE THEORY

Now Imprimitivity Theorem applies:

A proj. rep. $L : SO(3) \to \mathcal{U}(\mathcal{H}_0)$ exists so that

$\mathcal{H} = L_2(\mathbf{R}^3, \mathcal{H}_0)$, $\quad (F_\alpha \psi)(\mathbf{x}) = x_\alpha \psi(\mathbf{x})$,

If $\mathsf{g}(\mathbf{x}) = R^{-1}\mathbf{x} - R^{-1}\mathbf{a}$, $(\hat{U}_g \psi)(\mathbf{x}) = L_R \psi(\mathsf{g}(\mathbf{x}))$

Irred. representations $\leftrightarrow$ elementary particle:

$\mathcal{H}_0 = \mathbb{C}^{2s+1}$, $\hat{J}_\alpha = F_\beta \hat{P}_\gamma - F_\gamma \hat{P}_\beta + S_\alpha$

S_α spin operators in $\mathbb{C}^{2s+1}$

Quantum Theory of a Localizable Particle

Formalism is obtained, but the position operators $\mathbf{Q}$ not identified: then it is meaningless!

Def. (Position operator).

For any $g \in \mathcal{G}$, let $\mathsf{g}_t : \mathbf{R}^3 \to \mathbf{R}^3$ be its function.

Position a time t observable is a tern

$$\mathbf{Q}^{(t)} = (Q_1^{(t)}, Q_2^{(t)}, Q_3^{(t)}); \qquad \mathbf{Q}^{(0)} \equiv \mathbf{Q}.$$

such that $\quad S_g^{\Sigma}[\mathbf{Q}^{(t)}] = \mathsf{g}_t(\mathbf{Q}^{(t)}),$

i.e. $$U_g \mathbf{Q}^{(t)} U_g^{-1} = \mathsf{g}_t(\mathbf{Q}^{(t)})$$

III. DEVELOPMENT OF THE THEORY

To attain an effective theory of a particle

- To concretely identify $\mathbf{Q}$
- to determine the wave equation

$\mathbf{Q} = \mathbf{F}$?

Theorem. Let $\mathbf{Q}$ be position at time 0 operators.

$\mathbf{Q} = \mathbf{F}$ if and only if $\hat{U}_g \mathbf{Q} \hat{U}_g^{-1} = S_g^{\Sigma}[\mathbf{Q}] = U_g \mathbf{Q} U_g^{-1}$

$U_g \to \hat{U}_g$ preserves covariance properties of $\mathbf{Q}$:

"Q-covariant σ-conversion"

IV. EXACT WAVE EQUATION

Theorem. If the interaction admits Q-covariant σ-conversion then $f_\alpha(\mathbf{x}) \in \Omega(\mathcal{H}_0)$ and $\eta_\alpha(\mathbf{x}) \in \Omega(\mathcal{H}_0)$ exist such that

$$i[H, \mu Q_\alpha - \eta_\alpha(\mathbf{Q})] = \widehat{P}_\alpha - f_\alpha(\mathbf{Q})$$

Different specific forms of H satisfy (DynEq)

$H = -\frac{1}{2\mu}\sum_\alpha \frac{\partial^2}{\partial x_\alpha^2}$, $H = \frac{1}{2\mu}\sum_\alpha \left(-i\frac{\partial}{\partial x_\alpha}\right)^2 + \Phi(\mathbf{x})$ ...

Problem: characterize them *Physically*

V. DERIVING SPECIFIC WAVE EQUATIONS

The general law

$$i[H, \mu Q_\alpha - \eta_\alpha(\mathbf{Q})] = \widehat{P}_\alpha - f_\alpha(\mathbf{Q})$$

was implied by the invariance of the covariance properties of $\mathbf{Q}$ after σ-conversion:

$$U_g \mathbf{Q} U_g^{-1} = S_g^\Sigma[\mathbf{Q}] \quad \rightarrow(\sigma\text{-conv}) \rightarrow \quad \hat{U}_g \mathbf{Q} \hat{U}_g^{-1} = S_g^\Sigma[\mathbf{Q}]$$

V. SPECIFIC WAVE EQUATIONS

RESULT OF THE RESENT WORK:

The different SPECIFIC forms of Wave Equations (H) are determined by *approximate* invariance of the covariance properties of $\mathbf{Q}^{(t)}$ (position at time t) with respect to SPECIFIC subgroups of $\mathcal{G}$.

Different specific wave equations correspond to different subgroups of (Ist order) invariance

V. SPECIFIC WAVE EQUATIONS

Let the σ-conv. does not affect the covariance properties of $\mathbf{Q}^{(t)}$ with respect to boosts at first order, i.e.

$$e^{i\widehat{G}_\alpha u} Q^{(t)}_\beta e^{-i\widehat{G}_\alpha u} = S^\Sigma_g[\mathbf{Q}^{(t)}] + o^{(t)}(u)$$

$$= Q^{(t)}_\beta - \delta_{\alpha\beta} u t \mathbb{I} + o^{(t)}(u) \qquad (\mathcal{B})$$

Theorem. (Electromagnetic interaction)

If $(\mathcal{B})$ holds then

$$i\frac{d\psi_t}{dt}(\mathbf{x}) = -\frac{1}{2\mu}\sum_\alpha \left(\frac{\partial}{\partial x_\alpha}(\mathbf{x}) - a_\alpha(\mathbf{x})\right)^2 \psi_t(\mathbf{x}) + \Phi(\mathbf{x})\psi_t(\mathbf{x})$$

where $a_\gamma(\mathbf{x}), \Phi(\mathbf{x}) \in \Omega(\mathcal{H}_0 = \mathbb{C}^{2s+1})$

V. SPECIFIC WAVE EQUATIONS

Invariance of covariance properties of $\mathbf{Q}^{(t)}$

with respect to spatial translations, $\widehat{U}_g = e^{-i\widehat{P}_\alpha u}$

i.e. $\widehat{U}_g \mathbf{Q}^{(t)} \widehat{U}_g^{-1} = S_g^\Sigma[\mathbf{Q}^{(t)}]$, at first order:

$$e^{-i\widehat{P}_\alpha a} Q_\beta^{(t)} e^{i\widehat{P}_\alpha a} = Q_\beta^{(t)} - \delta_{\alpha\beta} a \mathbb{I} + o^{(t)}(a) \qquad (\mathcal{T})$$

Theorem. If $(\mathcal{T})$ holds then

$$i\frac{d\psi_t}{dt}(\mathbf{x}) = F(-i\nabla)\psi_t(\mathbf{x}) + \Psi(\mathbf{x})\psi_t(\mathbf{x})$$

where $F(\mathbf{p}), \Psi(\mathbf{x}) \in \Omega(\mathcal{H}_0 = \mathbb{C}^{2s+1})$

V. SPECIFIC WAVE EQUATIONS

Invariance of covariance properties of $\mathbf{Q}^{(t)}$

with respect to both:

$$e^{-i\widehat{P}_\alpha a} Q_\beta^{(t)} e^{i\widehat{P}_\alpha a} = Q_\beta^{(t)} - \delta_{\alpha\beta} a \mathbb{I} + o^{(t)}(a) \qquad (\mathcal{T})$$

$$e^{i\widehat{G}_\alpha u} Q_\beta^{(t)} e^{-i\widehat{G}_\alpha u} = Q_\beta^{(t)} - \delta_{\alpha\beta} ut \mathbb{I} + o^{(t)}(u) \qquad (\mathcal{B})$$

Theorem. If $(\mathcal{T})$ and $(\mathcal{B})$ hold then

$$i\frac{d\psi_t}{dt}(\mathbf{x}) = -\frac{1}{2\mu}\sum_\alpha \left(\frac{\partial}{\partial x_\alpha} - \hat{a}_\alpha\right)^2 \psi_t(\mathbf{x}) + \Phi(\mathbf{x})\psi_t(\mathbf{x})$$

where $\hat{a}_\gamma \in \Omega(\mathcal{H}_0 = \mathbb{C}^{2s+1})$

standard but non magnetic interaction

CONCLUSIONS

Different equations for different first order invariance subgroups.

Wave equations without first order invariance?

Extension to the relativistic case.

Problem: Covariance properties of $\mathbf{Q}^{(t)}$ with respect to Lorentz boosts not available.