Chiral fermions and trace anomalies

Fiorenzo Bastianelli

Department of Physics and Astronomy, University of Bologna
and INFN, Sezione di Bologna, Italy

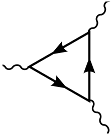
Modena, 9-10 September 2019

Outline

- Trace anomalies and consistency conditions
- Chiral fermions in D=4
- Pauli-Villars regularization
- Trace anomalies of chiral fermions

Anomalies

- Anomalies arise when a classical symmetry is broken by quantization
- The prime example is the chiral anomaly



- Trace anomalies also have extensive applications:

  Classically, conformally invariant theories often exhibit a stress tensor with vanishing trace, $T^{\mu}{}_{\mu}(x) = 0$

  Quantization produces a nonvanishing trace, the trace anomaly $\langle T^{\mu}{}_{\mu}(x) \rangle \neq 0$

Trace anomalies and consistency conditions

The trace anomaly depends on the background geometry.
By dimensional analysis (and covariance)

$$\langle T^{\mu}{}_{\mu}\rangle = \alpha\, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta\, R_{\mu\nu}R^{\mu\nu} + \gamma\, R^2 + \delta\,\Box R + \epsilon\,\sqrt{g}\epsilon_{\mu\nu\lambda\rho}R^{\mu\nu\alpha\beta}R^{\lambda\rho}{}_{\alpha\beta}$$

impose WZ consistency conditions ($[\delta_{\sigma_1(x)},\delta_{\sigma_2(x)}]\Gamma[g]=0$)

$$\langle T^{\mu}{}_{\mu}\rangle = a\,E_4 + c\,C^2 + d\,\Box R + e\,P_4$$

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

$$C^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

$$P_4 = \sqrt{g}\epsilon_{\mu\nu\lambda\rho}R^{\mu\nu\alpha\beta}R^{\lambda\rho}{}_{\alpha\beta} = \sqrt{g}\epsilon_{\mu\nu\lambda\rho}C^{\mu\nu\alpha\beta}C^{\lambda\rho}{}_{\alpha\beta}$$

Trace anomalies

$$\langle T^{\mu}{}_{\mu} \rangle = a\, E_4 + c\, C^2 + d\, \Box R + e\, P_4$$

- $\Box R$ can be eliminated by a counterterm
- trace anomaly parametrized by a and c coefficients, as in the past the Pontryagin term P_4 was not seen to arise ...
- but recently, its relevance has been reconsidered:
  $\star$ Nakayama (2012) stressed that it could be present
  $\star$ Bonora et al., (2014, 2018) found an imaginary contribution proportional to P_4 for massless Weyl fermions
- Claim potentially relevant: it needs an independent check!

Analogous situation in a gauge background

Trace anomaly in a gauge background:
by dimensional analysis (and gauge covariance)

$$\langle T^{\mu}{}_{\mu} \rangle = \alpha F_{\mu\nu} F^{\mu\nu} + \beta \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

solves the WZ consistency conditions.

- Can the Chern-Pontryagin term arise in chiral theories?
- Nakayama (2012) conjectured also its presence
- The situation seems analogous to the one in curved space, and perhaps even simpler to decide

Chiral fermions in $D = 4$

Weyl spinors λ (say $\gamma^5\lambda = \lambda$) coupled to a gauge field A_a

$$\mathcal{L} = -\overline{\lambda}\not{D}(A)\lambda$$

$$\not{D}(A) = \gamma^a D_a(A), \quad D_a(A) = \partial_a + A_a, \quad A_a = -iA_a^\alpha T^\alpha$$

Classically: • covariantly conserved gauge current
• traceless stress tensor

Convenient to embed it into the Bardeen model with Dirac ψ's

$$\mathcal{L} = -\overline{\psi}\not{D}(A, B)\psi$$

$$D_a(A, B) = \partial_a + A_a + B_a\gamma^5$$

A_a couples to vector current $J^{a\alpha} = i\overline{\psi}\gamma^a T^\alpha\psi$
B_a couples to axial current $J_5^{a\alpha} = i\overline{\psi}\gamma^a\gamma^5 T^\alpha\psi$
Projection to the Weyl model: $A_a = B_a \to \frac{A_a}{2}$
Compute anomalies in generalized model
(couple to gravity to study insertion of the stress tensor)

Bardeen model in curved space

$$\mathcal{L} = -e\,\overline{\psi}\,\nabla\!\!\!/(A,B)\psi$$

$$\nabla\!\!\!/(A,B) = \gamma^a e_a{}^\mu\left(\partial_\mu + A_\mu + B_\mu\gamma^5 + \frac{1}{4}\omega_{\mu bc}\gamma^{bc}\right)$$

invariant under:

- diffeomorphisms
- local Lorentz transformations
- vector gauge symmetry
- axial gauge symmetry
- Weyl transformations

We want to calculate corresponding anomalies

Just for the record:

- Weyl symmetry

$$\begin{cases} \delta\psi(x) = -\frac{3}{2}\sigma(x)\psi(x) \,, \qquad \delta\overline{\psi}(x) = -\frac{3}{2}\sigma(x)\overline{\psi}(x) \\ \delta e^a_\mu(x) = \sigma(x) e^a_\mu(x) \end{cases}$$

- Gauge symmetries ($\alpha = -i\alpha^\alpha_a(x)T^\alpha$ and $\beta = -i\beta^\alpha_a(x)T^\alpha$)

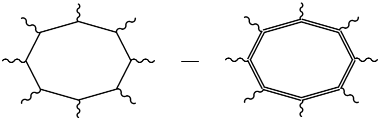
$$\begin{cases} \delta\psi = -(\alpha + \beta\gamma^5)\psi \,, \qquad \delta\overline{\psi} = \overline{\psi}(\alpha - \beta\gamma^5) \\ \delta A_a = \partial_a\alpha + [A_a, \alpha] + [B_a, \beta] \\ \delta B_a = \partial_a\beta + [A_a, \beta] + [B_a, \alpha] \end{cases}$$

- Bardeen curvatures

$$\hat{F}_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b] + [B_a, B_b]$$
$$\hat{G}_{ab} = \partial_a B_b - \partial_b B_a + [A_a, B_b] + [B_a, A_b]$$

Pauli-Villars regularization

We regulate the theory at one-loop by the Pauli-Villars method



• Anomalies come from the non-invariance of the PV mass term, which generates finite contributions at $M \to \infty$
• Can cast the calculation in a simple form, then evaluated using heat kernel formulae
• This method produces consistent anomalies (it computes the variation of the one-loop effective action)

$$\mathcal{L} = \frac{1}{2}\varphi^T T \mathcal{O} \varphi \,, \qquad \delta\varphi = K\varphi$$

$$\mathcal{L}_{PV} = \frac{1}{2}\phi^T T \mathcal{O} \phi + \frac{1}{2} M \phi^T T \phi \,, \quad \delta\phi = K\phi$$

$$\boxed{e^{i\Gamma} = \int D\varphi \; e^{iS} \qquad \to \qquad e^{i\Gamma} = \int D\varphi D\phi \; e^{i(S+S_{PV})}}$$

$$i\delta\Gamma = i\langle \delta S_{PV} \rangle = \lim_{M\to\infty} \; iM\langle \phi^T (TK + \frac{1}{2}\delta T)\phi \rangle$$

$$= -\lim_{M\to\infty} \mathrm{Tr}\left[\left(K + \frac{1}{2} T^{-1}\delta T \right) \left(1 + \frac{\mathcal{O}}{M} \right)^{-1} \right]$$

$$= -\lim_{M\to\infty} \mathrm{Tr}\left[\left(K + \frac{1}{2} T^{-1}\delta T + \frac{1}{2}\frac{\delta\mathcal{O}}{M} \right) \left(1 - \frac{\mathcal{O}^2}{M^2} \right)^{-1} \right]$$

$$= -\lim_{M\to\infty} \mathrm{Tr}\left[\left(K + \frac{1}{2} T^{-1}\delta T + \frac{1}{2}\frac{\delta\mathcal{O}}{M} \right) e^{\frac{\mathcal{O}^2}{M^2}} \right]$$

$$\langle \phi\phi^T \rangle = \frac{i}{T\mathcal{O} + TM}$$

[Diaz et al.(1989)]

Thus, we obtain a Fujikawa-like formula

$$\boxed{i\delta\Gamma = i\langle\delta S\rangle = -\lim_{M\to\infty} \mathrm{Tr}\Big[J\, e^{-\frac{R}{M^2}}\Big]}$$

$$J = K + \frac{1}{2}T^{-1}\delta T + \frac{1}{2}\frac{\delta\mathcal{O}}{M}$$

$$R = -\mathcal{O}^2$$

Pauli-Villars for the Bardeen model

How to regulate the Bardeen model?
• Use PV fields: their mass term is totally arbitrary, as long as it regulates (invertible mass matrix)
• The mass term usually cannot maintain all symmetries of the massless lagrangian: it preserves some symmetries while breaking others, the latter can get anomalies!
• We choose a Dirac mass: it preserves the green symmetries and break the red ones.

$$\mathcal{L}_{PV} = -e\overline{\psi}\nabla\!\!\!/(A,B)\psi - eM\overline{\psi}\psi$$

Casting

$$\mathcal{L}_{PV} = -e\,\overline{\psi}\,\nabla\!\!\!\!/\,(A,B)\psi - eM\overline{\psi}\psi$$

in the form

$$\mathcal{L}_{PV} = \frac{1}{2}\phi^T T\mathcal{O}\phi + \frac{1}{2}M\phi^T T\phi\,, \quad \text{with } \phi = \begin{pmatrix} \psi \\ \psi_c \end{pmatrix},\ \psi_c = C^{-1}\overline{\psi}^T$$

one finds (in flat space)

$$T\mathcal{O} = \begin{pmatrix} 0 & C D\!\!\!\!/(-A^T, B^T) \\ C D\!\!\!\!/(A,B) & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$$

$$\mathcal{O} = \begin{pmatrix} D\!\!\!\!/(A,B) & 0 \\ 0 & D\!\!\!\!/(-A^T, B^T) \end{pmatrix}$$

$$\boxed{\mathcal{O}^2 = \begin{pmatrix} D\!\!\!\!/^2(A,B) & 0 \\ 0 & D\!\!\!\!/^2(-A^T, B^T) \end{pmatrix}}$$

The latter is the regulator $R = -\mathcal{O}^2$.

Identify now the breaking term J (the Fujikawa jacobian)

$$J_{axial} = \begin{pmatrix} i\beta^\alpha(x) T^\alpha \gamma^5 & 0 \\ 0 & i\beta^\alpha(x) T^{\alpha T} \gamma^5 \end{pmatrix}$$
$$J_{Weyl} = \begin{pmatrix} \frac{1}{2}\sigma(x) & 0 \\ 0 & \frac{1}{2}\sigma(x) \end{pmatrix}$$

Then, use the heat kernel to find

$$\boxed{\begin{aligned} (D_a \langle J_5^a \rangle)^\alpha &= \frac{i}{(4\pi)^2} \Big[\mathrm{tr}\,[\gamma^5 T^\alpha a_2(R_\psi)] + \mathrm{tr}\,[\gamma^5 T^{\alpha T} a_2(R_{\psi_c})] \Big] \\ \langle T^a{}_a \rangle &= -\frac{1}{2(4\pi)^2} \Big[\mathrm{tr}\, a_2(R_\psi) + \mathrm{tr}\, a_2(R_{\psi_c}) \Big] \end{aligned}}$$

$$R_\psi = -\not{D}^2(A, B) \,, \qquad R_{\psi_c} = -\not{D}^2(-A^T, B^T)$$

The a_2 coefficients are the only ones that survive renormalization and the limit $M \to \infty$.

Heat kernel: a quick summary

Given an operator H on flat D-dimensional spacetime

$$H = -\nabla^2 + V$$

V matrix potential, $\nabla^2 = \nabla^a \nabla_a$, $\nabla_a = \partial_a + W_a$

$$[\nabla_a, \nabla_b] = \partial_a W_b - \partial_b W_a + [W_a, W_b] = \mathcal{F}_{ab}$$

The trace of its heat kernel has a small time expansion

$$\mathrm{Tr}\left[J\, e^{-isH}\right] = \int d^D x\, \mathrm{tr}\, \left[J(x) \langle x | e^{-isH} | x \rangle\right]$$

$$= \int \frac{d^D x\, i}{(4\pi i s)^{\frac{D}{2}}} \sum_{n=0}^{\infty} \mathrm{tr}\, [J(x) a_n(x, H)] (is)^n$$

$$= \int \frac{d^D x\, i}{(4\pi i s)^{\frac{D}{2}}} \mathrm{tr}\, [J(x)(a_0(x, H) + a_1(x, H) is + a_2(x, H)(is)^2 + ...)]$$

“tr” is a trace on remaining discrete matrix indices
$J(x)$ is an arbitrary matrix function
$a_n(x, H)$ are the heat kernel (Seeley-DeWitt) coefficients.

Heat kernel: a quick summary

Matrix valued heat kernel coefficients:

$$
\begin{aligned}
a_0(x, H) &= 1 \\
a_1(x, H) &= -V \\
a_2(x, H) &= \frac{1}{2}V^2 - \frac{1}{6}\nabla^2 V + \frac{1}{12}\mathcal{F}_{ab}^2
\end{aligned}
$$

where $\nabla_a V = \partial_a V + [W_a, V]$, etc.

Anomalies in Bardeen model

Chiral anomaly

$$
\begin{aligned}
(D_a\langle J_5^a\rangle)^\alpha = -\frac{1}{(4\pi)^2}\epsilon^{abcd}\,\mathrm{tr}_{YM}\,T^\alpha\Big[&\hat{F}_{ab}\hat{F}_{cd}+\frac{1}{3}\hat{G}_{ab}\hat{G}_{cd}\\
&-\frac{8}{3}(\hat{F}_{ab}B_cB_d+B_a\hat{F}_{bc}B_d+B_aB_b\hat{F}_{cd})+\frac{32}{3}B_aB_bB_cB_d\Big]\\
&+PETs
\end{aligned}
$$

PETs are cohomologically trivial parity-even terms

$$
\begin{aligned}
PETs = \frac{i}{(4\pi)^2}\mathrm{tr}_{YM}\,T^\alpha\Big[&\frac{4}{3}D^2DB+\frac{2}{3}[\hat{F}^{ab},\hat{G}_{ab}]+\frac{8}{3}[D^a\hat{F}_{ab},B^b]\\
&-\frac{4}{3}\{B^2,DB\}+8B_aDBB^a+\frac{8}{3}\{\{B^a,B^b\},D_aB_b\}\Big]\,.
\end{aligned}
$$

Indeed canceled by the chiral gauge variation of a local counterterm

$$
\Gamma_{ct} = \int\frac{d^4x}{(4\pi)^2}\,\mathrm{tr}_{YM}\Big[\frac{2}{3}(D^aB^b)(D_aB_b)+4F^{ab}(A)B_aB_b-\frac{8}{3}B^4+\frac{4}{3}B^aB^bB_aB_b\Big]
$$

Anomalies in Bardeen model

Trace anomaly

$$\langle T^a{}_a \rangle = \frac{1}{(4\pi)^2} \mathrm{tr}_{YM} \left[\frac{2}{3} \hat{F}^{ab} \hat{F}_{ab} + \frac{2}{3} \hat{G}^{ab} \hat{G}_{ab} \right] + CTTs$$

where $CTTs$

$$CTTs = \frac{1}{(4\pi)^2} \left(-\frac{4}{3} \right) \mathrm{tr}_{YM} \left[D^2 B^2 + DBDB - (D^a B^b)(D_b B_a) - 2F^{ab}(A) B_a B_b \right]$$

are canceled by the Weyl variation of

$$\bar{\Gamma}_{ct} = \int \frac{d^4 x \sqrt{g}}{(4\pi)^2} \mathrm{tr}_{YM} \left[\frac{2}{3} (D^\mu B^\nu)(D_\mu B_\nu) + 4F^{\mu\nu}(A) B_\mu B_\nu + \frac{1}{3} R B^2 \right]$$

N.B.: counterterms merge consistently into a unique counterterm

$$\Gamma_{ct}^{tot} = \int \frac{d^4 x \sqrt{g}}{(4\pi)^2} \mathrm{tr}_{YM} \left[\frac{2}{3} (D_\mu B_\nu)^2 + 4F^{\mu\nu}(A) B_\mu B_\nu + \frac{1}{3} R B^2 - \frac{8}{3} B^4 + \frac{4}{3} B^\mu B^\nu B_\mu B_\nu \right]$$

Anomalies in Bardeen model

$$
\begin{aligned}
(D_a\langle J_5^a\rangle)^\alpha &= -\frac{1}{(4\pi)^2}\epsilon^{abcd}\,\mathrm{tr}_{YM}\,T^\alpha\Big[\hat{F}_{ab}\hat{F}_{cd}+\frac{1}{3}\hat{G}_{ab}\hat{G}_{cd} \\
&\quad -\frac{8}{3}(\hat{F}_{ab}B_cB_d+B_a\hat{F}_{bc}B_d+B_aB_b\hat{F}_{cd})+\frac{32}{3}B_aB_bB_cB_d\Big] \\
\langle T^a{}_a\rangle &= \frac{1}{(4\pi)^2}\mathrm{tr}_{YM}\left[\frac{2}{3}\hat{F}^{ab}\hat{F}_{ab}+\frac{2}{3}\hat{G}^{ab}\hat{G}_{ab}\right]
\end{aligned}
$$

Trace anomaly is gauge invariant and without parity-odd terms

Anomalies of Weyl fermions

Take chiral limit $A_a = B_a \to \frac{1}{2} A_a$
$\hat{F}_{ab} = \hat{G}_{ab} \to \frac{1}{2} F_{ab}(A)$, $J^a = J^a_5 \to J_a = \frac{1}{2}(J^a + J^a_5)$

$$(D_a \langle J^a \rangle)^\alpha = -\frac{1}{(4\pi)^2} \epsilon^{abcd} \, \mathrm{tr}_{YM} T^\alpha \partial_a \left[\frac{2}{3} A_b \partial_c A_d + \frac{1}{3} A_b A_c A_d \right]$$
$$\langle T^a{}_a \rangle = \frac{1}{(4\pi)^2} \mathrm{tr}_{YM} \left[\frac{1}{3} F^{ab} F_{ab} \right]$$

The chiral anomaly is the standard one
The trace anomaly is our result, which shows the absence of parity-odd terms
It is just half the trace anomaly of non-chiral Dirac fermions
F.B. and M. Broccoli, arXiv:1908.03750, 1808.03489

Weyl fermions in curved space

Same procedure can be applied to Weyl fermions in curved space:

- no contribution from Pontryagin term to the trace anomaly
  F.B. and R. Martelli, arXiv:1610.02304
- tension with the alternative calculation of
  L. Bonora et al., arXiv:1403.2606, 1703.10473, 1807.01249
- last two references introduce a metric-axial-tensor (MAT) background

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu}\gamma^5$$

  as an extension of Bardeen's method to curved space
- fermions in MAT background can also be regulated with PV fields. We anticipate that no parity-odd term arises in the trace anomaly of Weyl fermions F.B. and M. Broccoli, in progress

Conclusions

- Described the trace anomalies of Weyl fermions
- Obtained the trace anomaly of Weyl fields in abelian and non-abelian gauge backgrounds: no parity-odd terms
- Similar findings for Weyl fermions on curved background
- ... but clash with different derivations by Bonora et al.
- Alternative calculations are welcome!