BEAM INSTABILITIES
IN CIRCULAR PARTICLE ACCELERATORS

E. Métral (CERN, BE/ABP)

3 seminars of 2 hours (3-5/06/19)

- Introduction
- Longitudinal low-intensity
- Transverse low-intensity
- Transverse high-intensity
- Longitudinal high-intensity
- Conclusion

ABSTRACT

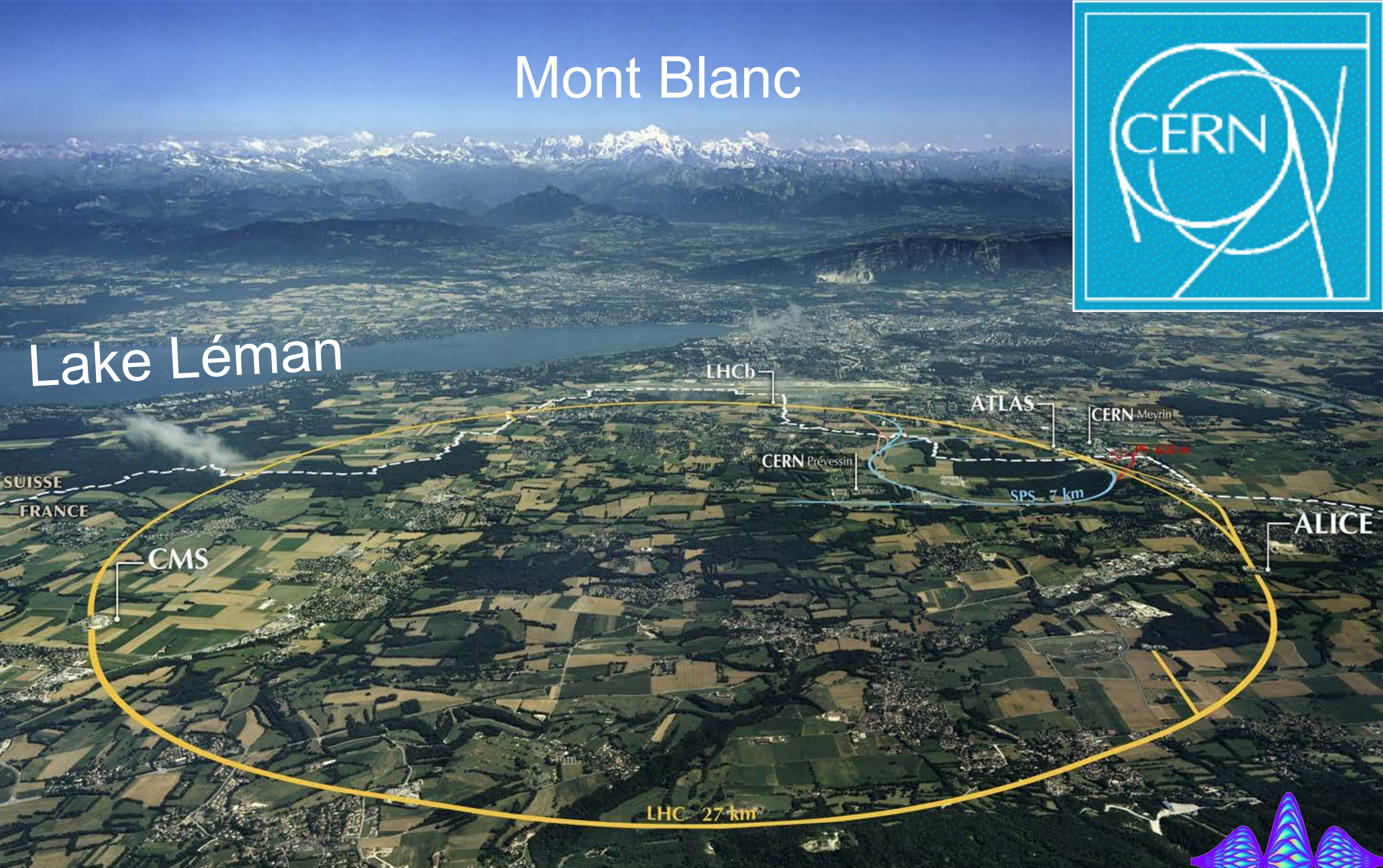
- The theory of impedance-induced bunched-beam coherent instabilities is reviewed following Garnier-Laclare's formalism, adding the effect of an electronic damper in the transverse plane

- Both single-bunch and coupled-bunch instabilities are discussed, both low-intensity and high-intensity regimes are analyzed, both longitudinal and transverse planes are studied, and both short-bunch and long-bunch regimes are considered

- 2 similar approaches for coherent instabilities using the linearised Vlasov equation (and Garnier-Laclare's formalism) are presented, leading to 2 "new" Vlasov solvers
  - For Transverse Instabilities: GALAC-TIC (GArnier-LAclare Coherent Transverse Instabilities Code)
  - For Longitudinal Instabilities: GALAC-LIC (GArnier-LAclare Coherent Longitudinal Instabilities Code)

- Observables and mitigation measures are also briefly examined

INTRODUCTION



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(Hadron Synchrotron Collective/Coherent effects)

https://espace.cern.ch/be-dep-workspace/abp/HSC/SitePages/Home.aspx


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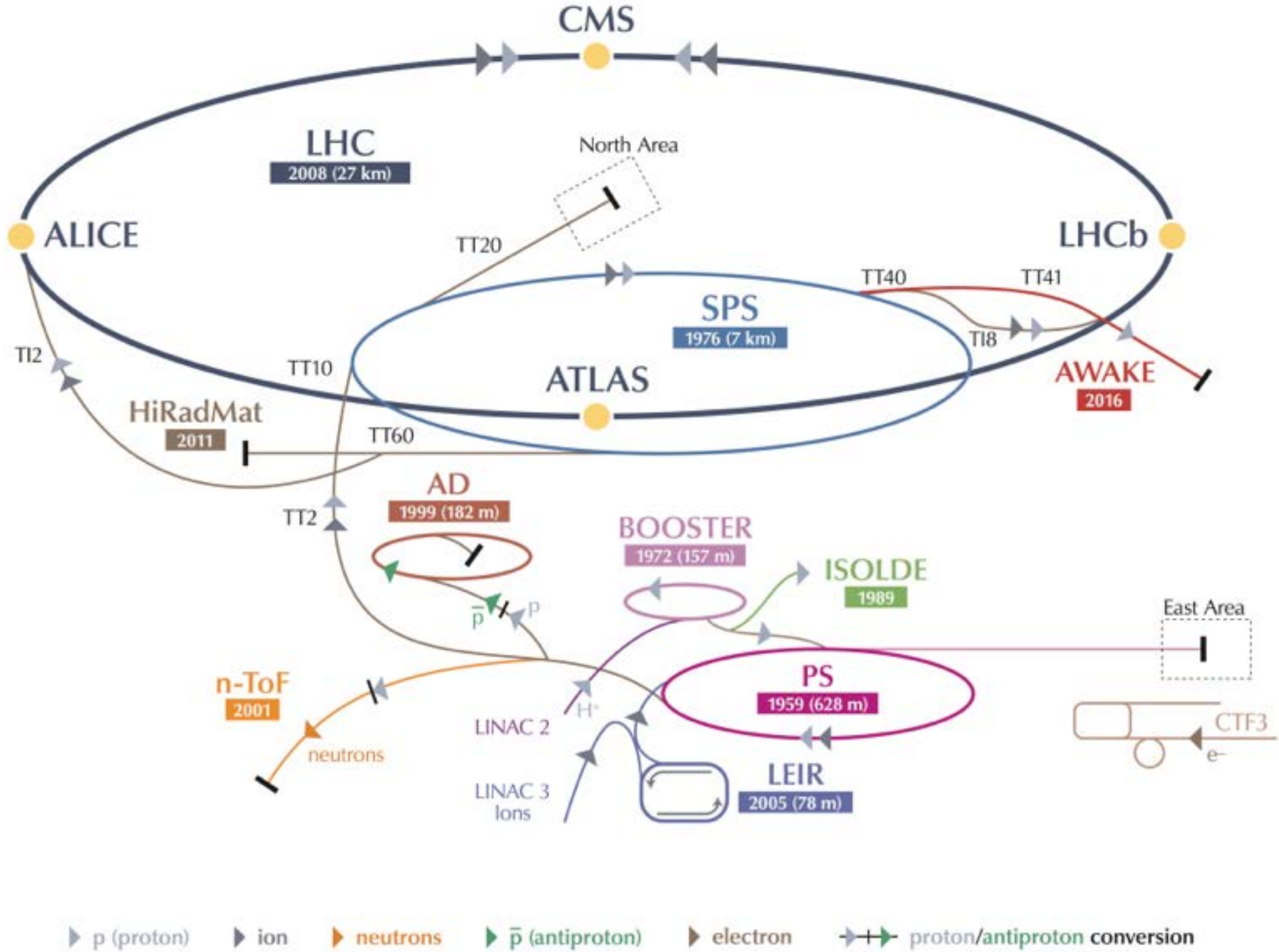
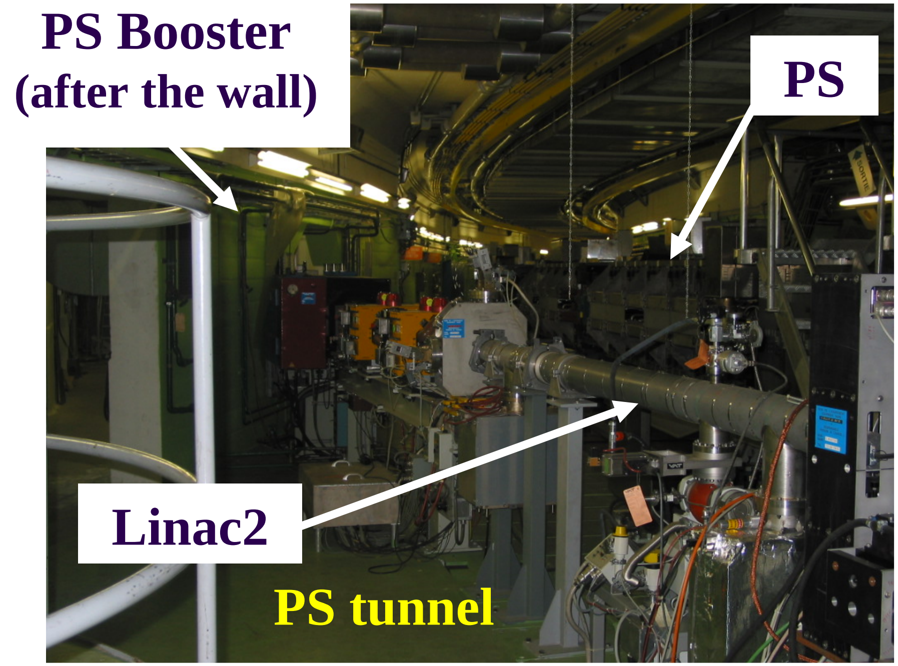
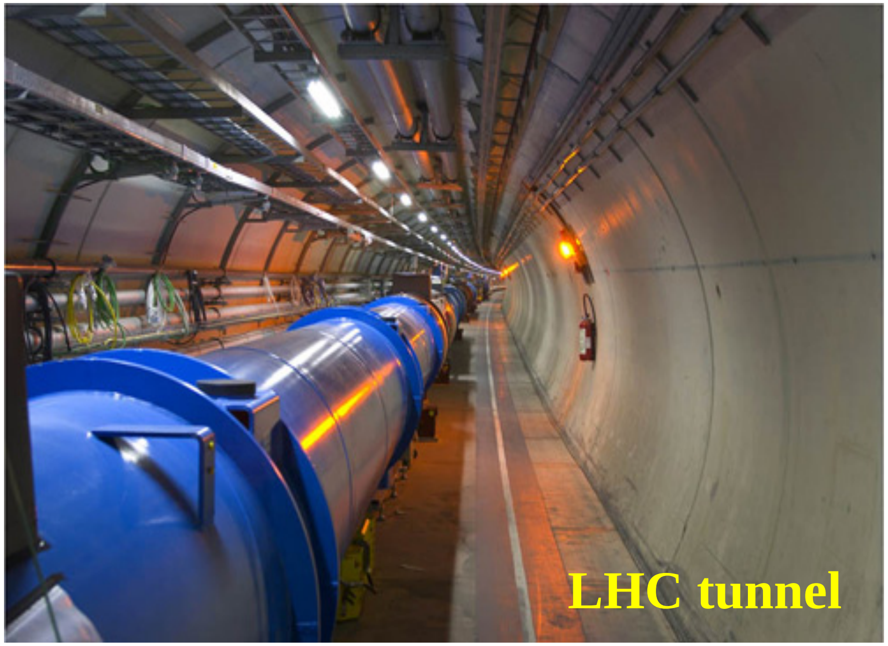


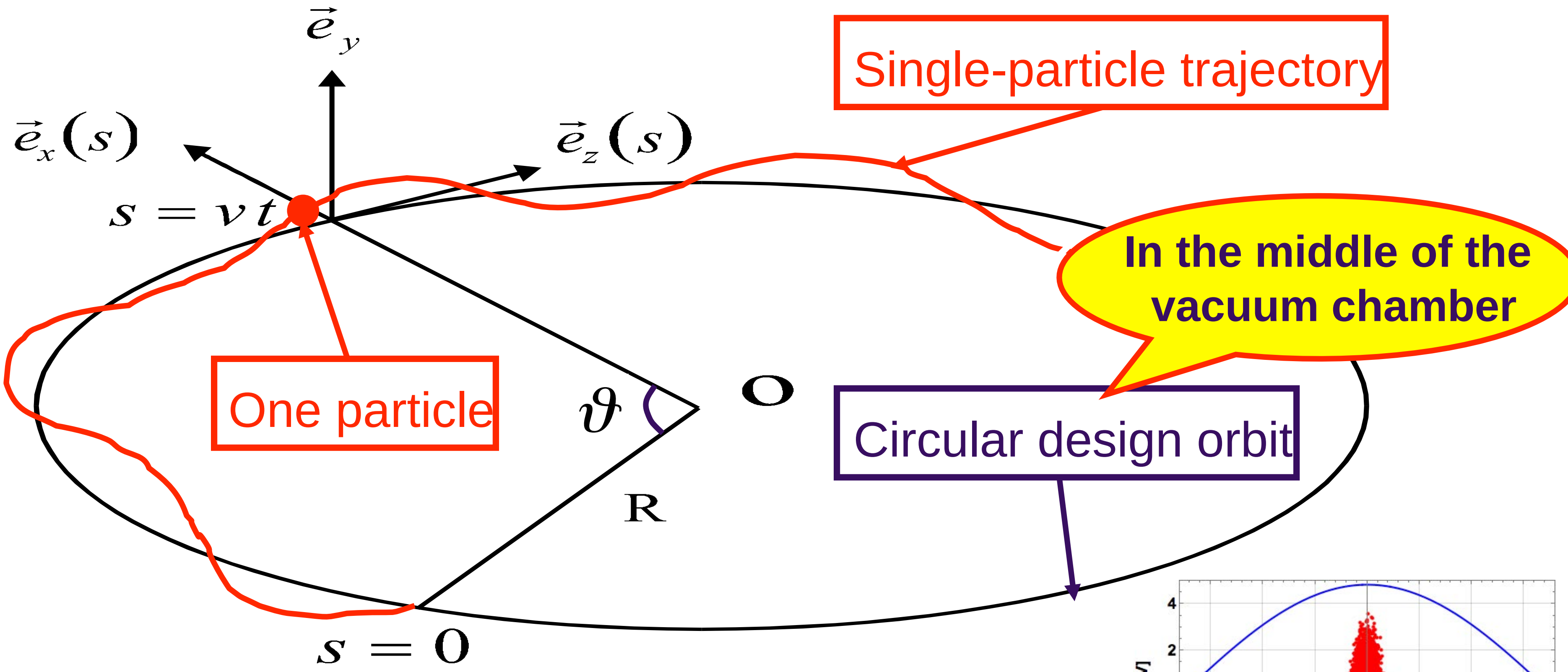
CMS
LHC
2008 (27 km)
North Area
ALICE
LHCb
TT20
TT40
TT41
SPS
1976 (7 km)
TI8
TI2
TT10
ATLAS
AWAKE
2016
HiRadMat
2011
TT60
AD
1999 (182 m)
TT2
BOOSTER
1972 (157 m)
ISOLDE
1989
East Area
p̄
p
n-ToF
2001
PS
1959 (628 m)
H⁻
LINAC 2
CTF3
e⁻
neutrons
LEIR
2005 (78 m)
LINAC 3
Ions
p (proton)
ion
neutrons
p̄ (antiproton)
electron
proton/antiproton conversion


PS Booster
(after the wall)
PS
Linac2
PS tunnel

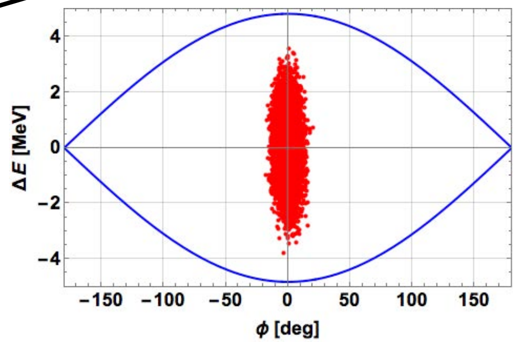

SPS tunnel


TT2 transfer line tunnel

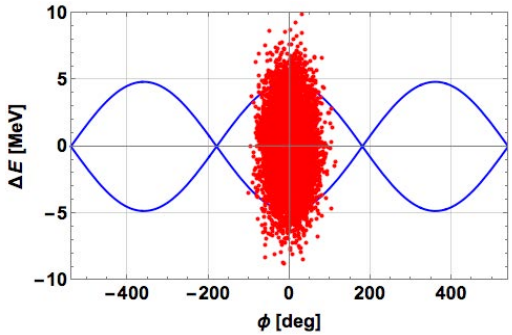

LHC tunnel



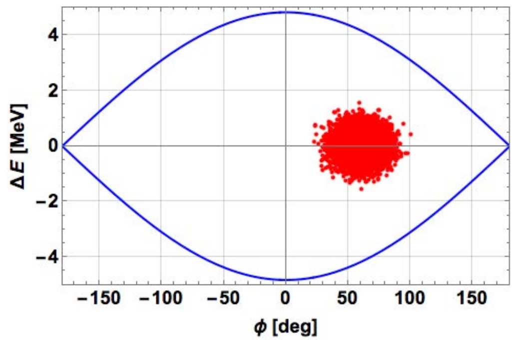
$\vec{e}_y$
$\vec{e}_x(s)$
$\vec{e}_z(s)$
$s = v\,t$
Single-particle trajectory
In the middle of the vacuum chamber
One particle
ϑ
O
R
Circular design orbit
$s = 0$



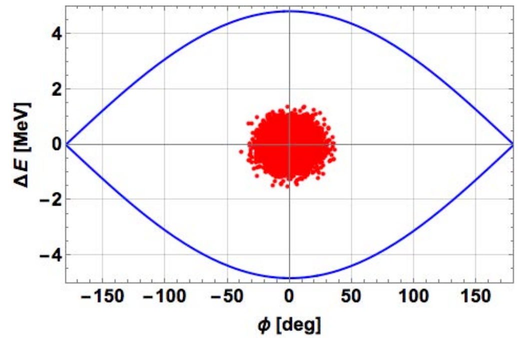
ΔE [MeV]
ϕ [deg]



ΔE [MeV]
ϕ [deg]

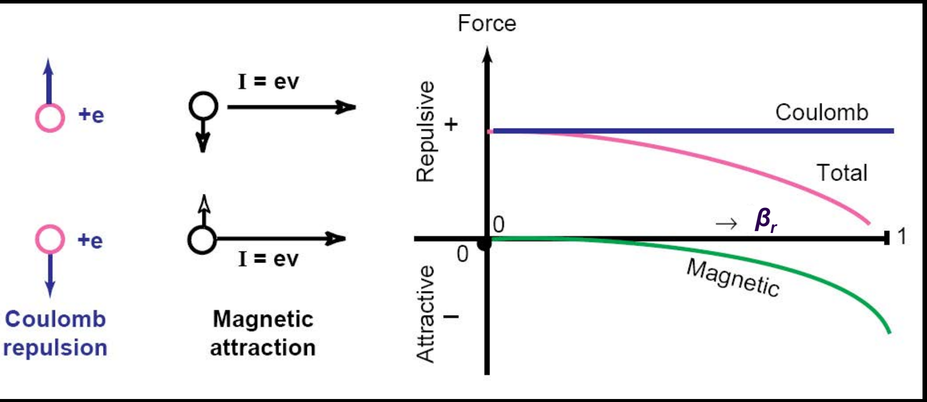


ΔE [MeV]
ϕ [deg]

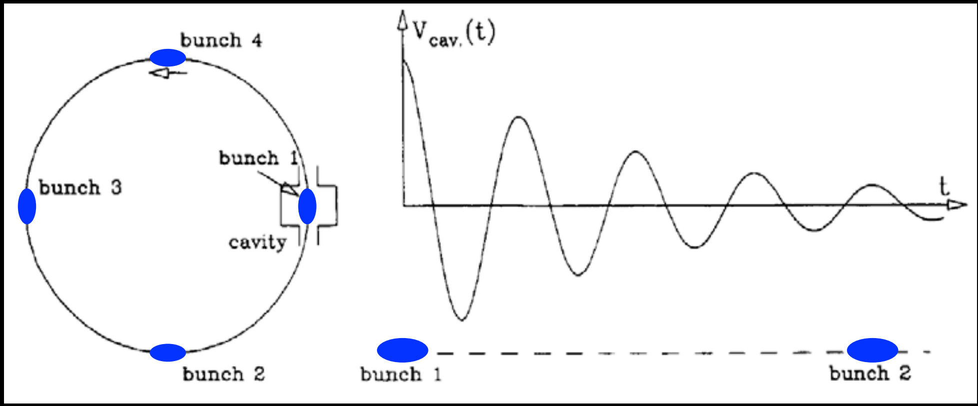


ΔE [MeV]
ϕ [deg]

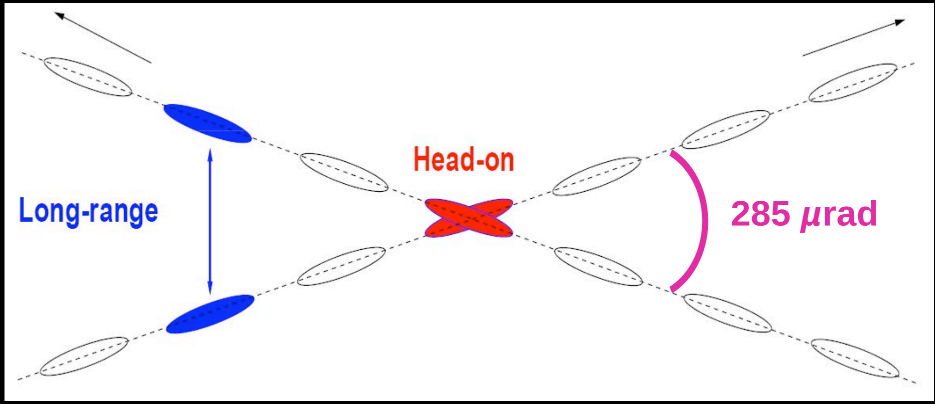
SPACE CHARGE



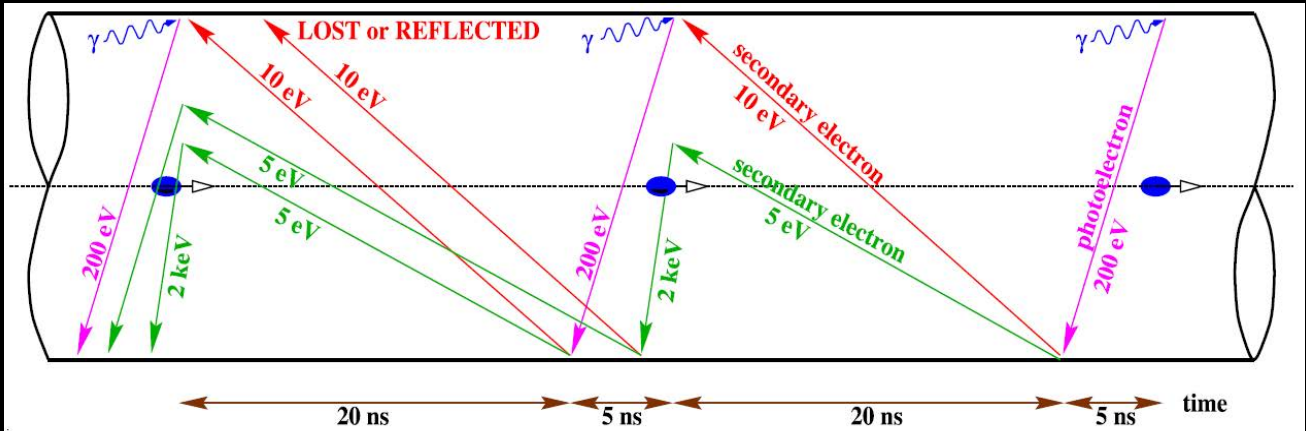
WAKE FIELD / IMPEDANCE



BEAM-BEAM

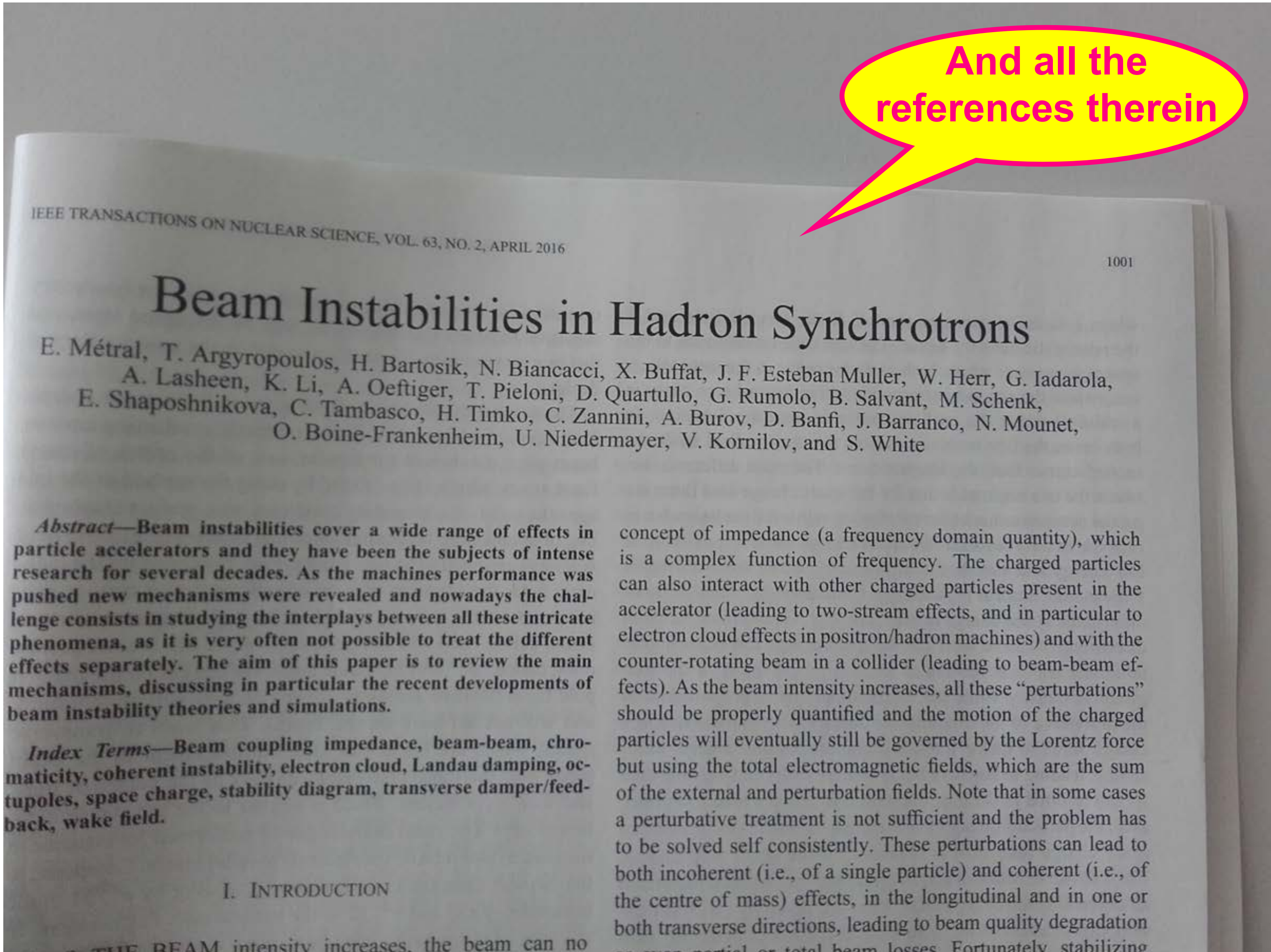


ELECTRON CLOUD



50-page article for a special edition of IEEE Transactions on Nuclear Science for the 50th anniversary of the PAC conference (originally launched by IEEE in 1965)

And all the references therein



IEEE TRANSACTIONS ON NUCLEAR SCIENCE, VOL. 63, NO. 2, APRIL 2016 1001

Beam Instabilities in Hadron Synchrotrons

E. Métral, T. Argyropoulos, H. Bartosik, N. Biancacci, X. Buffat, J. F. Esteban Muller, W. Herr, G. Iadarola, A. Lasheen, K. Li, A. Oeftiger, T. Pieloni, D. Quartullo, G. Rumolo, B. Salvant, M. Schenk, E. Shaposhnikova, C. Tambasco, H. Timko, C. Zannini, A. Burov, D. Banfi, J. Barranco, N. Mounet, O. Boine-Frankenheim, U. Niedermayer, V. Kornilov, and S. White

***Abstract*—Beam instabilities cover a wide range of effects in particle accelerators and they have been the subjects of intense research for several decades. As the machines performance was pushed new mechanisms were revealed and nowadays the challenge consists in studying the interplays between all these intricate phenomena, as it is very often not possible to treat the different effects separately. The aim of this paper is to review the main mechanisms, discussing in particular the recent developments of beam instability theories and simulations.**

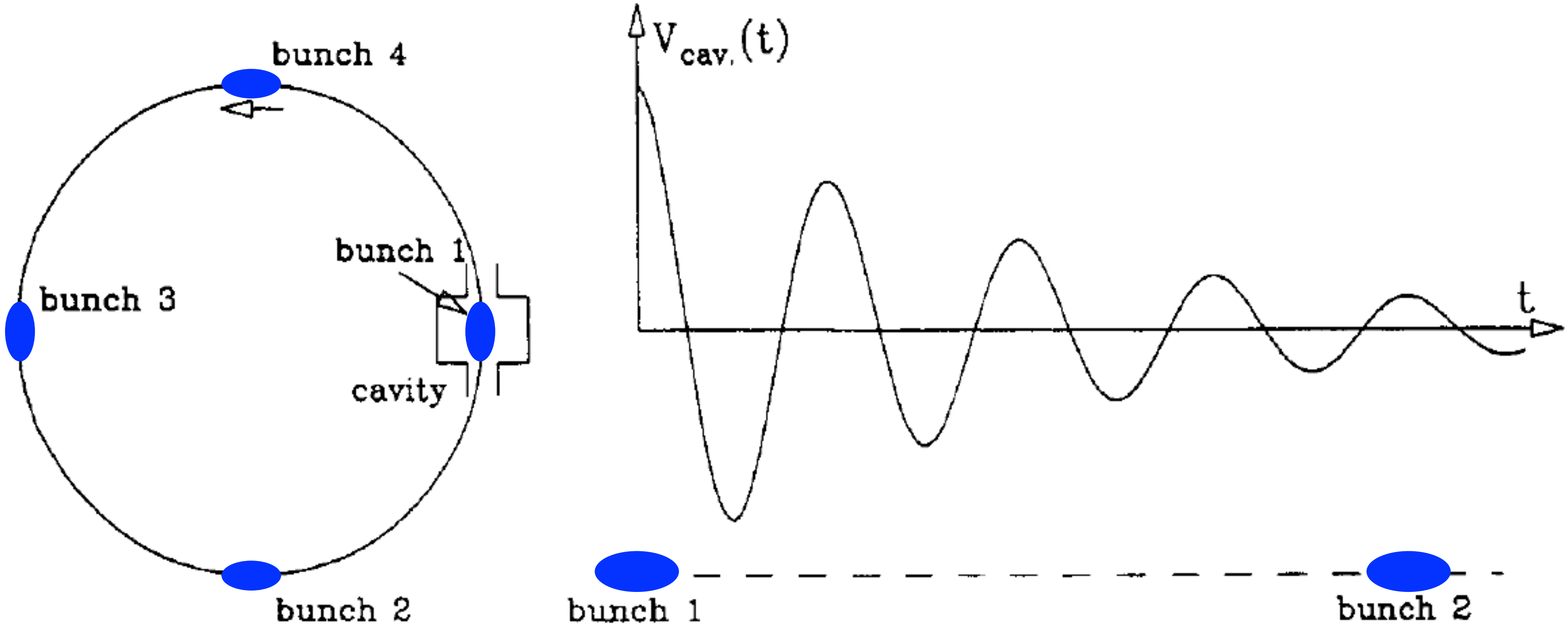
***Index Terms*—Beam coupling impedance, beam-beam, chromaticity, coherent instability, electron cloud, Landau damping, octupoles, space charge, stability diagram, transverse damper/feedback, wake field.**

I. Introduction

AS THE BEAM intensity increases, the beam can no

concept of impedance (a frequency domain quantity), which is a complex function of frequency. The charged particles can also interact with other charged particles present in the accelerator (leading to two-stream effects, and in particular to electron cloud effects in positron/hadron machines) and with the counter-rotating beam in a collider (leading to beam-beam effects). As the beam intensity increases, all these "perturbations" should be properly quantified and the motion of the charged particles will eventually still be governed by the Lorentz force but using the total electromagnetic fields, which are the sum of the external and perturbation fields. Note that in some cases a perturbative treatment is not sufficient and the problem has to be solved self consistently. These perturbations can lead to both incoherent (i.e., of a single particle) and coherent (i.e., of the centre of mass) effects, in the longitudinal and in one or both transverse directions, leading to beam quality degradation or even partial or total beam losses. Fortunately, stabilizing

FOCUS OF THIS COURSE:
IMPEDANCE-INDUCED BEAM INSTABILITIES

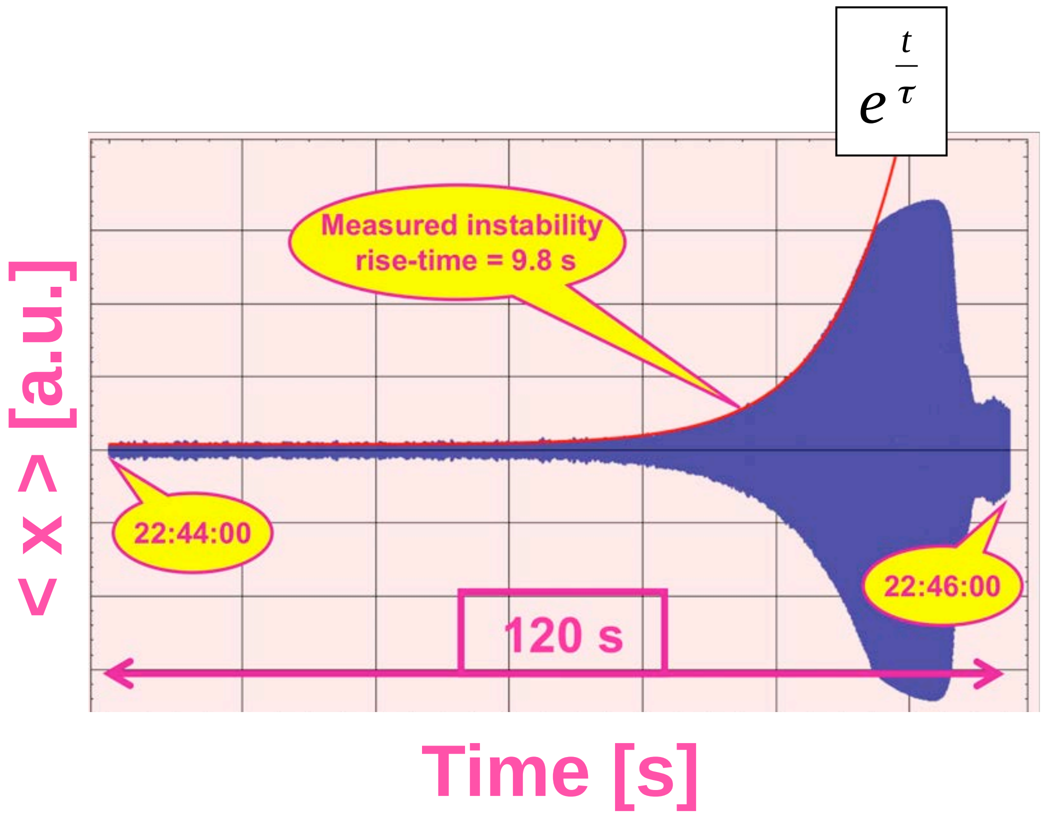


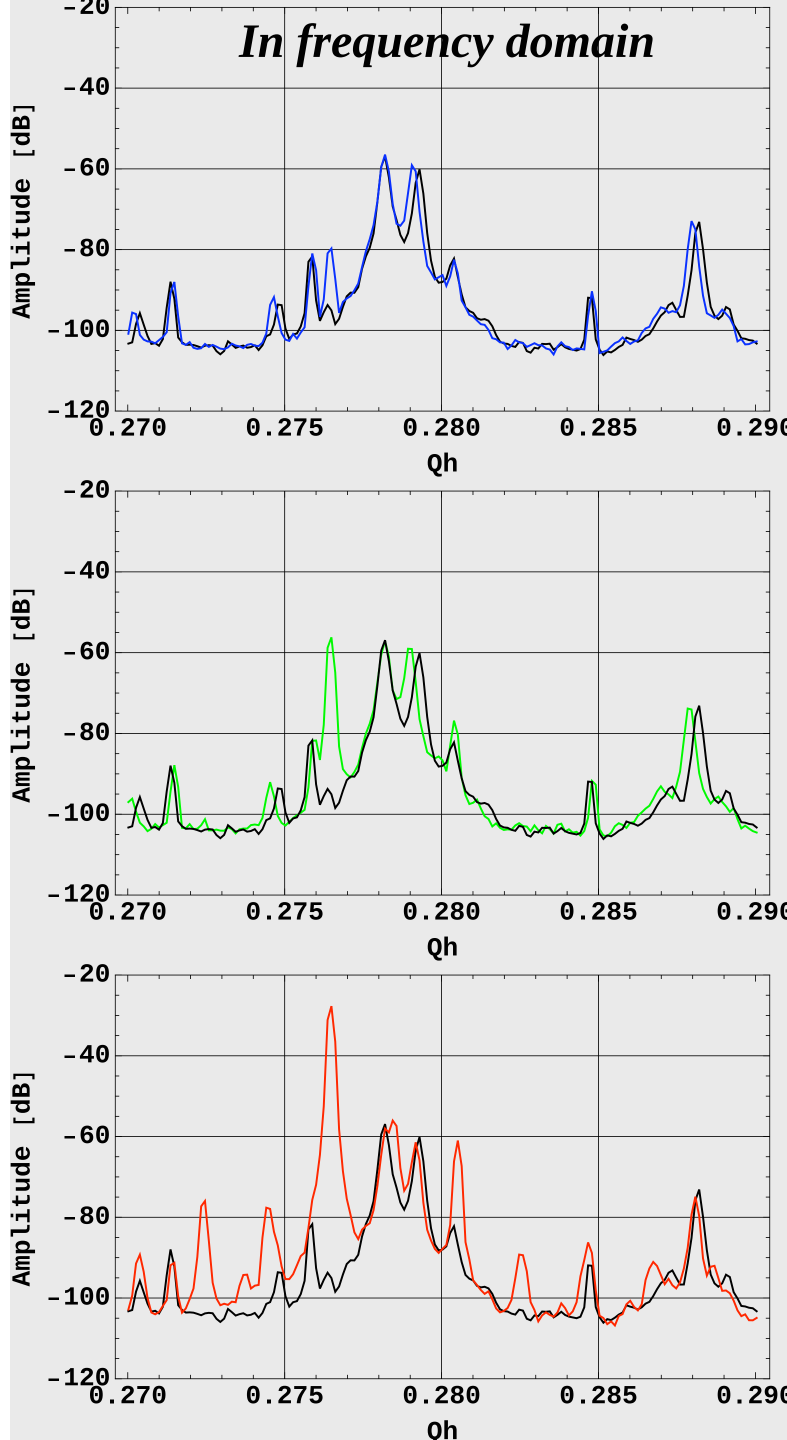
- **Limits performance of ALL machines**
  - **Beam instabilities => Increased beam size, beam losses**
  - **Excessive heating => Deformed / melted components, beam dumps**

- **Each equipment of each accelerator has an impedance => To be characterized and minimized!**

EXAMPLES OF MEASURED BEAM INSTABILITIES

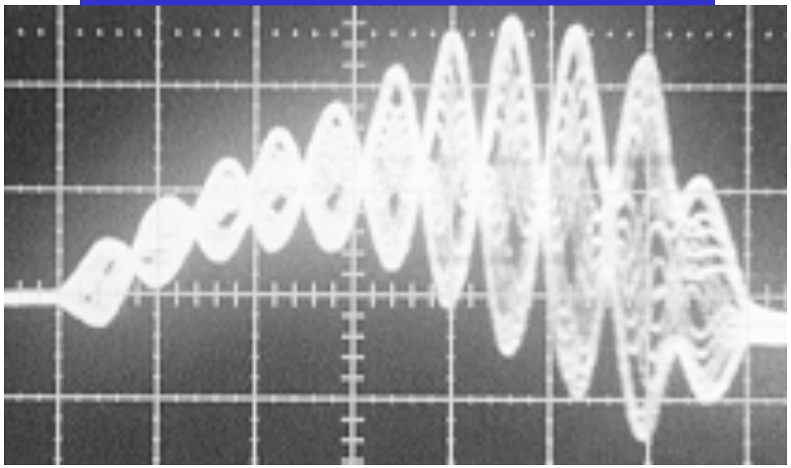
LHC, single bunch, horizontal



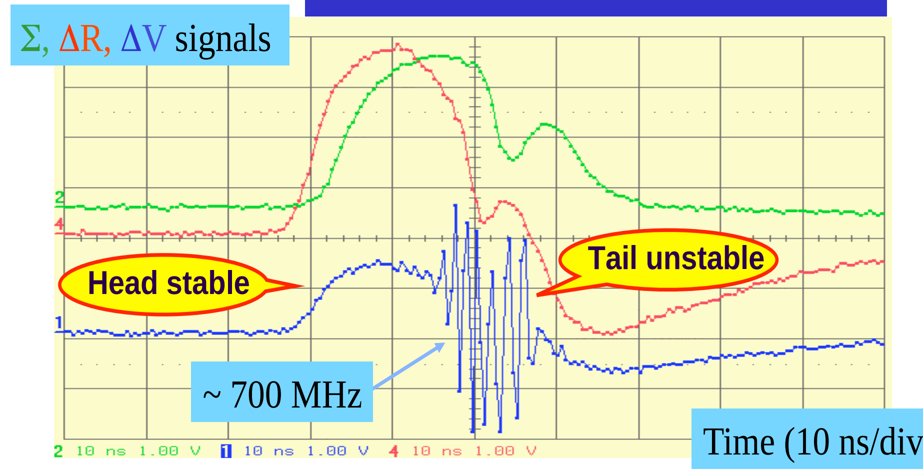


EXAMPLES OF MEASURED BEAM INSTABILITIES

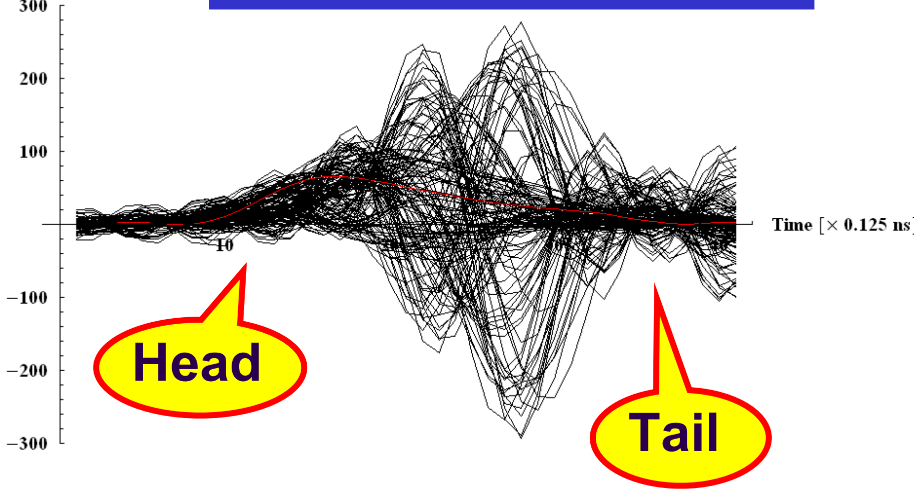
PS, single bunch, horizontal

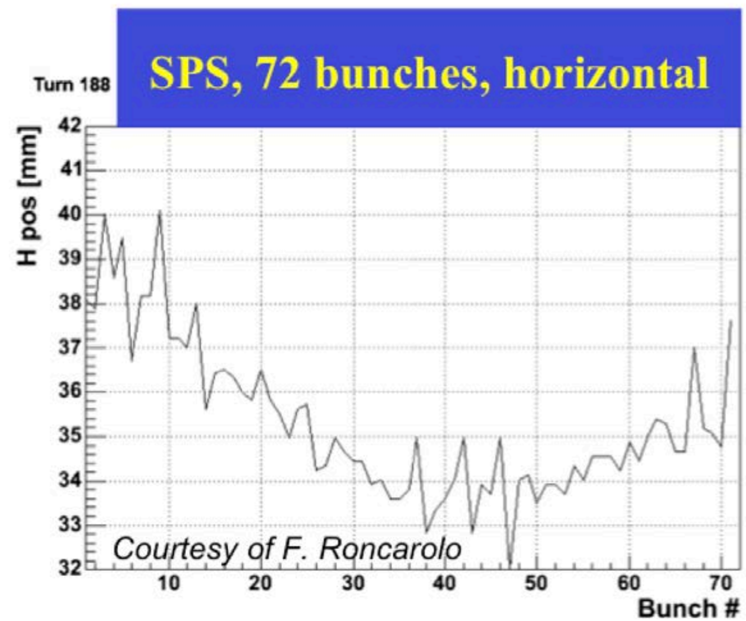


PS, single bunch, vertical



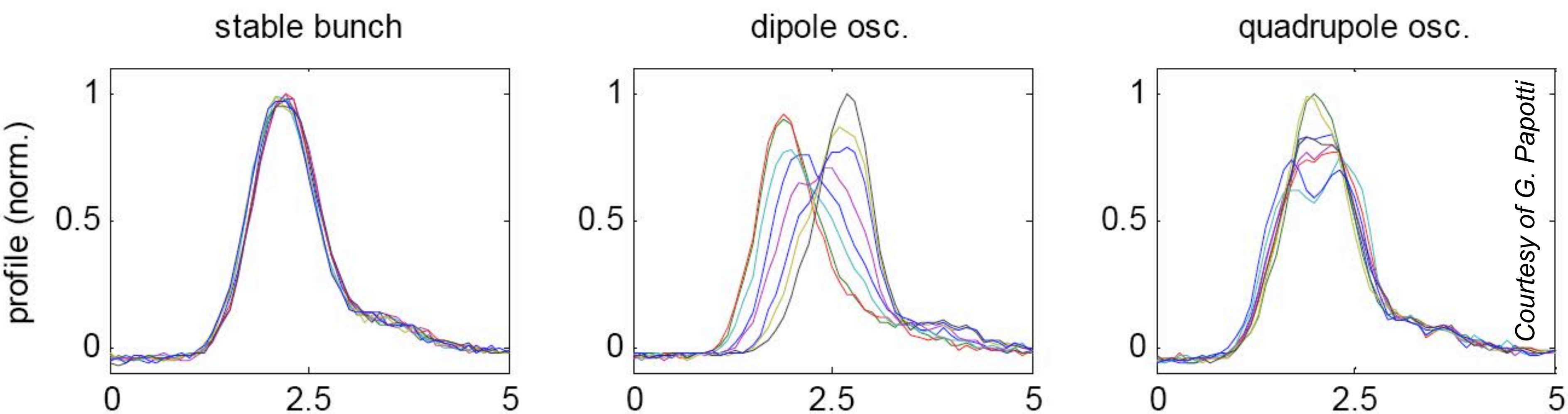
SPS, single bunch, vertical



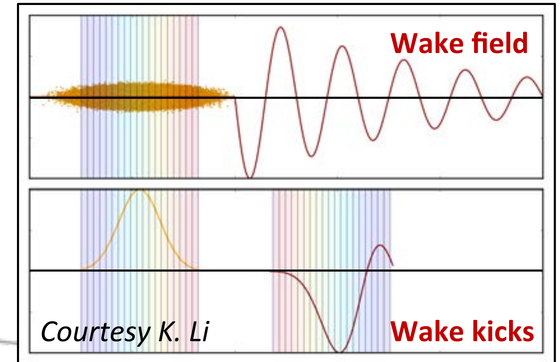


EXAMPLES OF MEASURED BEAM INSTABILITIES

SPS, single bunch, longitudinal

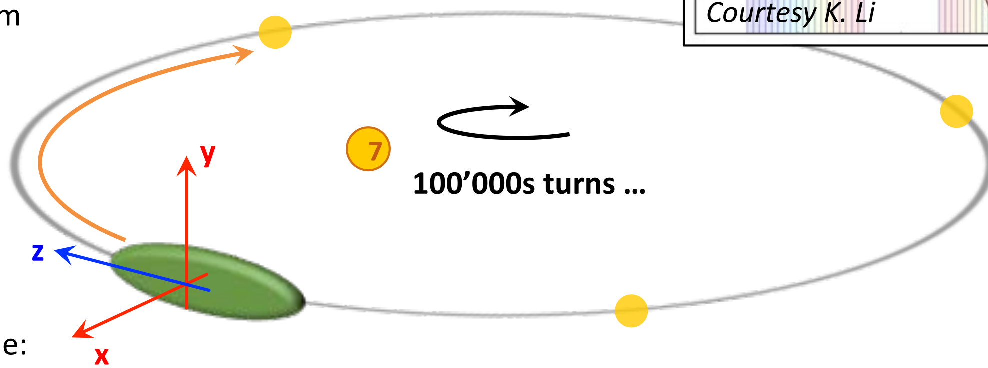


SIMULATION (TRACKING) CODES ARE OFTEN USED: => HEADTAIL / PyHEADTAIL code at CERN



4 **Interaction point**
- Wake field kicks
- Chromaticity
- Octupoles, RF quadrupole
- Electron cloud (PyECLOUD)
- …

3 **Linear periodic maps for transverse tracking** from one IP to the next



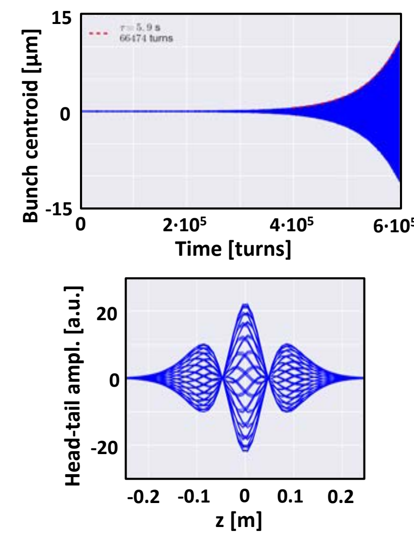
7 **100'000s turns …**

2 **Initialise bunch**
- Typically 10^6 macroparticles
- Various distributions possible: Gaussian, waterbag, matched to rf bucket (longitudinal), …

1 **Divide ring into segments separated by interaction points (IP)**

6 **Once per turn:** Apply (non-)linear synchrotron motion

5 **Data acquisition**



Courtesy of K. Li et al.

PURPOSE OF THIS COURSE

=> **Discuss the theory of bunched-beam coherent instabilities and explain theoretically the measured pictures of instabilities**

- **Longitudinal and transverse**
- **Single-bunch and coupled-bunch**
- **Low-intensity and high-intensity**
- **Short-bunch and long-bunch**

See also last years' seminars:
http://cds.cern.ch/record/2288203/files/CERN-ACC-SLIDES-2017-0010.pdf
http://cds.cern.ch/record/2652200/files/CERN-ACC-SLIDES-2018-0003.pdf

LONGITUDINAL: LOW-INTENSITY

PROCEDURE: BOTH LONGITUDINAL (L) & TRANSVERSE (T)

- Start with the single particle motion => Harmonic oscillator + beam-induced electromagnetic force (L or T)

PROCEDURE: BOTH LONGITUDINAL (L) & TRANSVERSE (T)

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- Study the effect of the impedance on the stationary distribution (for L) => A new fixed point is obtained, with a dependency on the bunch intensity of the synchronous phase, incoherent frequency, effective (total) voltage and bunch length

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- Around the new fixed point (for L), write the perturbation => Coherent with respect to the satellite number *m*

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- Apply the Vlasov equation to first order **=> One ends up with an eigen-value system to solve**
- The result is an infinite number of modes of oscillation ***mq***

azimuthal

radial

SINGLE PARTICLE LONGITUDINAL MOTION (1/2)

$$\boxed{\ddot{\tau} + \omega_{s0}^2 \, \tau = 0}$$

$$\tau = \hat{\tau} \cos\left(\omega_{s0} \, t + \psi_0 \right)$$

$$\omega_{s0} = \Omega_0 \left(- \frac{e \hat{V}_{\mathrm{RF}} \, h \, \eta \cos \phi_{s0}}{2 \pi \beta^2 \, E_{total}} \right)^{1/2}$$

Time interval between the passage of the synchronous particle and the test particle, for a fixed observer at azimuthal position ϑ

$e =$ elementary charge

$R =$ average machine radius $\qquad R \, \Omega_0 = v = \beta c \qquad c =$ speed of light

$p_0 =$ momentum of the synch. particle $\qquad p_0 \, c = \beta \, E_{total}$

$\hat{V}_{\mathrm{RF}} =$ peak RF voltage

$$\eta = \alpha_p - \frac{1}{\gamma^2} = - \frac{\Delta f \, / \, f_0}{\Delta p \, / \, p_0} = \text{slip factor}$$

$h =$ RF harmonic number

$\phi_{s0} =$ RF phase of the synch. particle $\qquad \alpha_p = \dfrac{1}{\gamma_t^2} =$ mom. comp. factor

SINGLE PARTICLE LONGITUDINAL MOTION (2/2)

- Canonical conjugate variables $\left(\tau,\ \dot{\tau} = \dfrac{d\tau}{dt}\right)$ $\qquad \dot{\tau} = \dfrac{d\tau}{dt} = -\dfrac{df}{f_0} = \eta\,\dfrac{dp}{p_0}$

$$\tau^2 + \frac{\dot{\tau}^2}{\omega_{s0}^2} = \hat{\tau}^2$$

- Linear matching condition $\omega_{s0} = \dfrac{2\,|\eta|\,\dfrac{\Delta p}{p_0}}{\tau_b}$ $\qquad \tau_b = 2\,\hat{\tau}_{\max}$

- Effect of the (beam-induced) electromagnetic fields $\dot{\tau} = \eta\,\dfrac{p - p_0}{p_0} \quad \Rightarrow$

$$\boxed{\ddot{\tau} + \omega_{s0}^2\,\tau = \frac{\eta}{p_0}\,\frac{dp}{dt} = \frac{\eta\,e}{p_0}\left[\vec{E} + \vec{v} \times \vec{B}\right]_z \left(t, \vartheta = \Omega_0\left(t - \tau\right)\right)}$$

When following the particle along its trajectory

SINGLE PARTICLE LONGITUDINAL SIGNAL (1/3)

- At time $t = 0$, the synchronous particle starts from $\vartheta = 0$ and reaches the Pick-Up (PU) electrode (assuming infinite bandwidth) at times t_k^0

$$\Omega_0 \, t_k^0 = \vartheta + 2\,k\,\pi, \qquad -\infty \leq k \leq +\infty$$

- The test particle is delayed by τ. It goes through the electrode at times t_k

$$t_k = t_k^0 + \tau$$

- The current signal induced by the test particle is a series of impulses delivered on each passage

$$\boxed{s_z(t, \vartheta) = e \sum_{k=-\infty}^{k=+\infty} \delta\left(t - \tau - \frac{\vartheta}{\Omega_0} - \frac{2\,k\,\pi}{\Omega_0} \right)}$$

Dirac function

SINGLE PARTICLE LONGITUDINAL SIGNAL (2/3)

- **Using the relations**

$$\sum_{k=-\infty}^{k=+\infty} \delta\left(u - \frac{2k\pi}{\Omega_0}\right) = \frac{\Omega_0}{2\pi} \sum_{p=-\infty}^{p=+\infty} e^{j p \Omega_0 u}$$

$$e^{-j u \hat{\tau} \cos(\omega_{s0} t + \psi_0)} = \sum_{m=-\infty}^{m=+\infty} j^{-m} J_m(u \hat{\tau})\, e^{j m (\omega_{s0} t + \psi_0)}$$

Bessel function of mth order

$$\Rightarrow \quad s_z(t, \vartheta) = \frac{e\,\Omega_0}{2\pi} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m} J_m(p\,\Omega_0\,\hat{\tau})\, e^{j(\omega_{pm} t - p\vartheta + m\psi_0)}$$

Fourier transform

$$\boxed{\omega_{pm} = p\Omega_0 + m\omega_{s0}}$$

$$\boxed{s_z(\omega, \vartheta) = \frac{e\,\Omega_0}{2\pi} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m} J_m(p\,\Omega_0\,\hat{\tau})\, e^{-j(p\vartheta - m\psi_0)}\, \delta(\omega - \omega_{pm})}$$

SINGLE PARTICLE LONGITUDINAL SIGNAL (3/3)

- The single particle spectrum is a line spectrum at frequencies

$$\omega_{pm} = p\Omega_0 + m\omega_{s0}$$

- Around every harmonic of the revolution frequency $p\Omega_0$, there is an infinite number of **synchrotron satellites *m***

- The spectral amplitude of the mth satellite is given by $J_m\left(p\,\Omega_0\,\hat{\tau}\right)$

- The spectrum is centered at the origin

- Because the argument of the Bessel functions is proportional to $\hat{\tau}$, the width of the spectrum behaves like $\hat{\tau}^{-1}$

DISTRIBUTION OF PARTICLES (1/2)

$$\Psi\left(\hat{\tau},\psi_0,t\right) = \text{particle density in longitudinal phase space}$$

- Signal induced (at the PU electrode) by the whole beam

$$\boxed{S_z\left(t,\vartheta\right) = \underbrace{N_b}_{\text{Number of particles per bunch}} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} \int_{\psi_0=0}^{\psi_0=2\pi} \Psi\left(\hat{\tau},\psi_0,t\right) s_z\left(t,\vartheta\right) \hat{\tau}\, d\hat{\tau}\, d\psi_0}$$

- Canonically conjugated variables derive from a Hamiltonian $\mathrm{H}\left(q,p,t\right)$ by the canonical equations

$$\dot{q} = \frac{\partial \mathrm{H}\left(q,p,t\right)}{\partial p} \qquad \dot{p} = -\frac{\partial \mathrm{H}\left(q,p,t\right)}{\partial q}$$

DISTRIBUTION OF PARTICLES (2/2)

- According to the **Liouville's theorem**, the particles, in a non-dissipative system of forces, move like an incompressible fluid in phase space. The constancy of the phase space density $\Psi\left(q,\, p,\, t\right)$ is expressed by the equation

$$\frac{d\,\Psi\left(q,\, p,\, t\right)}{d\,t} = 0$$

where the total differentiation indicates that one follows the particle while measuring the density of its immediate neighborhood. This equation, sometimes referred to as the Liouville's theorem, states that the local particle density does not vary with time when following the motion in canonical variables

- As seen by a stationary observer (like a PU electrode) which does not follow the particle => **Vlasov equation**

$$\boxed{\frac{\partial\,\Psi\left(q,\, p,\, t\right)}{\partial\, t} + \dot{q}\,\frac{\partial\,\Psi\left(q,\, p,\, t\right)}{\partial\, q} + \dot{p}\,\frac{\partial\,\Psi\left(q,\, p,\, t\right)}{\partial\, p} = 0}$$

STATIONARY DISTRIBUTION (1/6)

- In the case of a harmonic oscillator $\mathrm{H} = \omega \, \dfrac{q^2 + p^2}{2}$

$$\dot{q} = \frac{\partial \mathrm{H}}{\partial p} = p \, \omega$$

$$\dot{p} = - \frac{\partial \mathrm{H}}{\partial q} = - q \, \omega$$

$$\Rightarrow \qquad \ddot{q} + \omega^2 q = 0$$

- Going to polar coordinates

$$q = r \cos\phi$$

$$p = - r \sin\phi$$

$$\Rightarrow \qquad \frac{\partial \Psi}{\partial t} + \dot{r} \, \frac{\partial \Psi}{\partial r} + \dot{\phi} \, \frac{\partial \Psi}{\partial \phi} = 0$$

STATIONARY DISTRIBUTION (2/6)

- As *r* is a constant of motion $\Rightarrow \quad \dot{r} = 0$

$$\Rightarrow \quad \frac{\partial \Psi}{\partial t} + \omega \, \frac{\partial \Psi}{\partial \phi} = 0 \qquad \text{with} \qquad \phi = \omega \, t$$

$$\Rightarrow \quad \frac{\partial \Psi}{\partial t} = - \omega \, \frac{\partial \Psi}{\partial \phi} = - \frac{\partial \Psi}{\partial t} \qquad \Rightarrow \qquad \frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial \phi} = 0$$

$$\Rightarrow \quad \Psi\left(r\right)$$

A stationary distribution is any function of *r*, or equivalently any function of the Hamiltonian H

STATIONARY DISTRIBUTION (3/6)

- In our case $q = \tau$ $\quad r = \hat{\tau}$ $\qquad \boxed{\Psi_0\left(\hat{\tau}, \psi_0, t\right) = g_0\left(\hat{\tau}\right)}$

  $p = \dot{\tau}$ $\quad \phi = \psi_0$

$$\Rightarrow \quad \boxed{S_{z0}\left(\omega, \vartheta\right) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_0\left(p\right) \delta\left(\omega - p\,\Omega_0\right) e^{-j\,p\,\vartheta}}$$

$$I_b = N_b\, e\, \Omega_0 / 2\pi$$

Amplitude of the spectrum

with $$\boxed{\sigma_0\left(p\right) = \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_0\left(p\,\Omega_0\,\hat{\tau}\right) g_0\left(\hat{\tau}\right) \hat{\tau}\, d\hat{\tau}}$$

STATIONARY DISTRIBUTION (4/6)

- Let ' s assume a parabolic amplitude density $$\boxed{g_0\left(\hat{z}\right)=\frac{2}{\pi\left(\frac{\tau_b}{2}\right)^2}\left(1-\hat{z}^2\right)}$$

$$\hat{z}\equiv\hat{\tau}/\left(\tau_b/2\right)$$

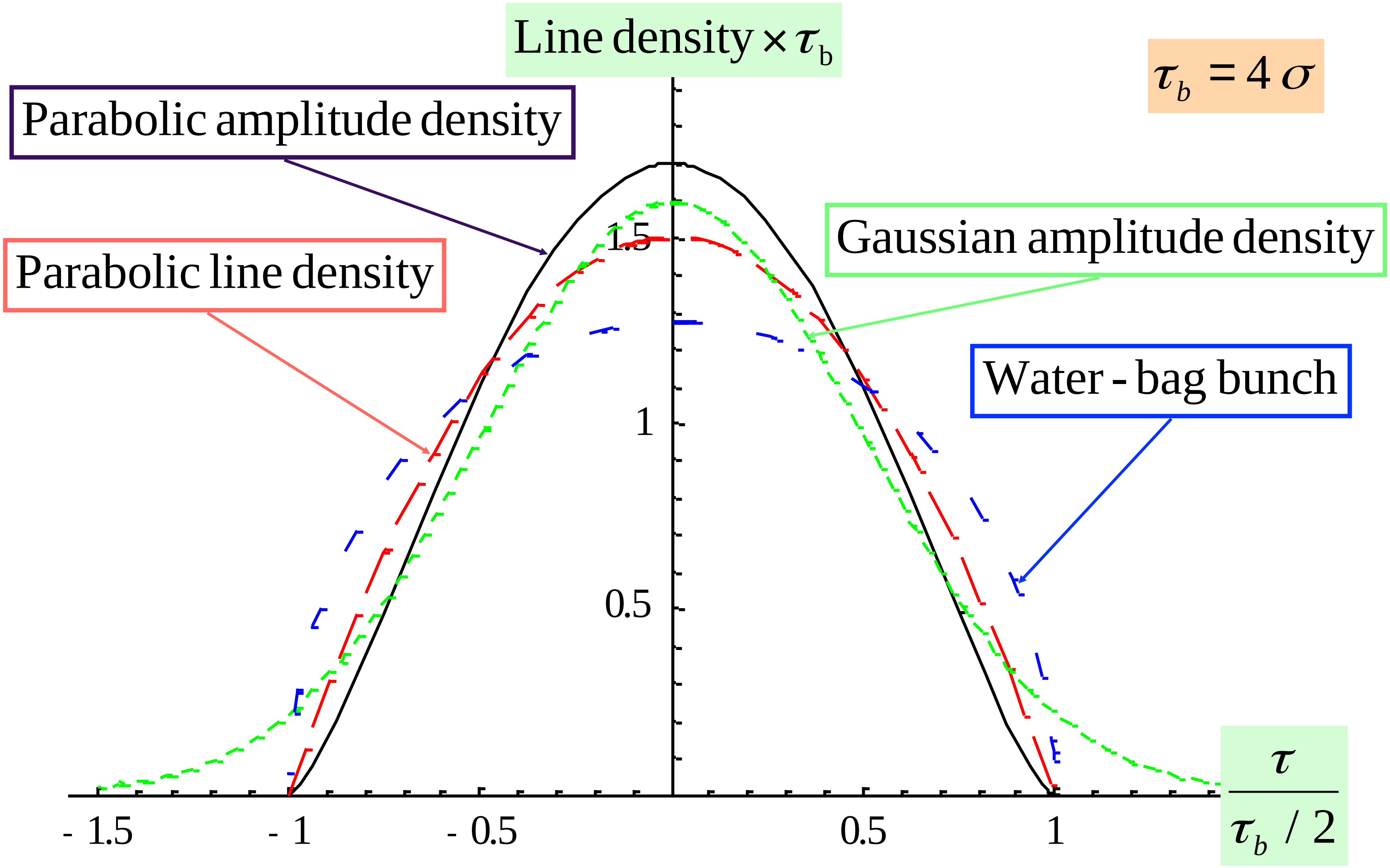
- The line density $\lambda\left(\tau\right)$ is the projection of the distribution $g_0\left(\hat{\tau}\right)$ on the τ axis

$$\lambda\left(\tau\right)=\int g_0\left(\hat{\tau}\right)\frac{d\dot{\tau}}{\omega_{s0}}$$

$$\int\lambda\left(\tau\right)d\tau=1 \qquad \Rightarrow \qquad \boxed{\lambda\left(z\right)=\frac{8}{3\pi\left(\frac{\tau_b}{2}\right)}\left(1-z^2\right)^{3/2}}$$

$$z\equiv\tau/\left(\tau_b/2\right)$$

STATIONARY DISTRIBUTION (5/6)



STATIONARY DISTRIBUTION (6/6)

- **Using the relations**

$$\int_{u'=0}^{u'=u} J_0(u')\, u'\, du' = u\, J_1(u)$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2\,n}{x} J_n(x)$$

$$\int x^3 J_0(x)\, dx = x^2 \left[2\, J_2(x) - x\, J_3(x) \right]$$

$$\Rightarrow \quad \boxed{\sigma_0(p) = \frac{4}{\pi\,(p\,\pi\,B)^2} J_2(p\,\pi\,B)}$$

$$B = \tau_b\, \Omega_0 \,/\, 2\pi$$

Bunching factor

and

$$\boxed{S_{z0}(\omega, \vartheta) = 8\, I_b \sum_{p=-\infty}^{p=+\infty} \delta(\omega - p\,\Omega_0)\, e^{-j\,p\,\vartheta}\, \frac{J_2(p\,\pi\,B)}{(p\,\pi\,B)^2}}$$

LONGITUDINAL IMPEDANCE

$$\boxed{2\pi R \left[\vec{E} + \vec{v} \times \vec{B} \right]_z (t, \vartheta) = - \int_{\omega=-\infty}^{\omega=+\infty} Z_l(\omega)\, S_z(\omega, \vartheta)\, e^{j\omega t}\, d\omega}$$

All the properties of the electromagnetic response of a given machine to a passing particle is gathered into the impedance (complex function => in Ω)

EFFECT OF THE STATIONARY DISTRIBUTION (1/9)

$$\ddot{\tau} + \omega_{s0}^2 \, \tau = F_0 = \frac{\eta \, e}{p_0} \left[\vec{E} + \vec{v} \times \vec{B} \right]_{z0} \left(t, \vartheta = \Omega_0 \left(t - \tau \right) \right)$$

$$\left[\vec{E} + \vec{v} \times \vec{B} \right]_{z0} \left(t, \vartheta = \Omega_0 \left(t - \tau \right) \right) = - \frac{1}{2 \, \pi \, R} \int_{\omega = -\infty}^{\omega = +\infty} Z_l \left(\omega \right) S_{z0} \left(\omega, \vartheta = \Omega_0 \left(t - \tau \right) \right) e^{\, j \omega t} \, d\omega$$

$$\Rightarrow \quad \boxed{\ddot{\tau} + \omega_{s0}^2 \, \tau = F_0 = \frac{2 \, \pi \, I_b \, \omega_{s0}^2}{\Omega_0 \, \hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \sum_{p = -\infty}^{p = +\infty} Z_l \left(p \right) \sigma_0 \left(p \right) e^{\, j \, p \, \Omega_0 \, \tau}}$$

$p \, \Omega_0$

EFFECT OF THE STATIONARY DISTRIBUTION (2/9)

- Expanding the exponential in series (for small amplitudes)

$$\ddot{\tau} + \omega_{s0}^2 \, \tau = \frac{2 \, \pi \, I_b \, \omega_{s0}^2}{\Omega_0 \, \hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} Z_l(p) \, \sigma_0(p) \left[1 + j \, p \, \Omega_0 \, \tau - \frac{(p \, \Omega_0 \, \tau)^2}{2} + \dots \right]$$

Synchronous phase shift

Incoherent frequency shift (potential-well distortion)

Nonlinear terms introducing some synchrotron frequency spread

EFFECT OF THE STATIONARY DISTRIBUTION (3/9)

- Synchronous phase shift

$$\ddot{\tau} + \omega_{s0}^2 \, \tau = \frac{2\,\pi\, I_b\, \omega_{s0}^2}{\Omega_0\, \hat{V}_{\mathrm{RF}}\, h \cos\phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \mathrm{Re}\left[Z_l(p) \right] \sigma_0(p)$$

$$\tau = t_p - t_{s0}$$

Test particle (t_p)   **Synchronous particle** (t_{s0})

$$\ddot{t}_{s0} = 0$$

$$\Rightarrow \quad \ddot{t}_p + \omega_{s0}^2 \, t_p = \omega_{s0}^2 \, t_{s0} + \frac{2\,\pi\, I_b\, \omega_{s0}^2}{\Omega_0\, \hat{V}_{\mathrm{RF}}\, h \cos\phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \mathrm{Re}\left[Z_l(p) \right] \sigma_0(p)$$

$$\Rightarrow \quad \ddot{t}_p + \omega_{s0}^2 \, t_p = \omega_{s0}^2 \, t_s$$

with $\quad \Delta t_s = t_s - t_{s0} = \dfrac{2\,\pi\, I_b}{\Omega_0\, \hat{V}_{\mathrm{RF}}\, h \cos\phi_{s0}} \displaystyle\sum_{p=-\infty}^{p=+\infty} \mathrm{Re}\left[Z_l(p) \right] \sigma_0(p)$

EFFECT OF THE STATIONARY DISTRIBUTION (4/9)

$$\phi = \omega_{\mathrm{RF}}\, t \qquad \omega_{\mathrm{RF}} = h\, \Omega_0$$

$$\phi_s = \omega_{\mathrm{RF}}\, t_s$$

$$\Delta\phi_s = \phi_s - \phi_{s0} = \omega_{\mathrm{RF}}\, \Delta t_s$$

Only for the small amplitudes. For the power loss of the whole bunch an averaging is needed!

$$\Rightarrow \quad \boxed{\Delta\phi_s = \phi_s - \phi_{s0} = \frac{2\,\pi\, I_b}{\hat{V}_{\mathrm{RF}}\,\cos\phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \mathrm{Re}\left[Z_l(p) \right] \sigma_0(p)}$$

Can be used to probe the resistive part of the longitudinal impedance

EFFECT OF THE STATIONARY DISTRIBUTION (5/9)

- Incoherent synchrotron frequency shift (potential-well distortion)

$$\ddot{\tau} + \omega_{s0}^2 \, \tau = \frac{2 \, \pi \, I_b \, \omega_{s0}^2}{\Omega_0 \, \hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} Z_l \left(p \right) \sigma_0 \left(p \right) j \, p \, \Omega_0 \, \tau$$

$$\Longrightarrow \quad \ddot{\tau} + \omega_s^2 \, \tau = 0$$

with $\omega_s^2 = \omega_{s0}^2 \left[1 - \dfrac{2 \, \pi \, I_b}{\hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \displaystyle\sum_{p=-\infty}^{p=+\infty} j \, Z_l \left(p \right) p \, \sigma_0 \left(p \right) \right]$

- If the impedance is constant (in the frequency range of interest)

$$\omega_s^2 = \omega_{s0}^2 \left\{ 1 - \frac{2 \, \pi \, I_b}{\hat{V}_{\mathrm{RF}} \, h \cos \phi_{s0}} \left[j \, \frac{Z_l \left(p \right)}{p} \right]_{const} \sum_{p=-\infty}^{p=+\infty} p^2 \, \sigma_0 \left(p \right) \right\}$$

EFFECT OF THE STATIONARY DISTRIBUTION (6/9)

- **Using the relation** $\boxed{\sum_{p=-\infty}^{p=+\infty} J_2(p\,x) = \frac{2}{x}}$ $\Rightarrow \sum_{p=-\infty}^{p=+\infty} p^2\,\sigma_0(p) = \frac{8}{\pi^4\,B^3}$

For the parabolic amplitude density

$$\Rightarrow \quad \boxed{\Delta = \frac{\omega_s^2 - \omega_{s0}^2}{\omega_{s0}^2} = -\frac{16\,I_b}{\pi^3\,B^3\,\hat{V}_{\mathrm{RF}}\,h\cos\phi_{s0}}\left[j\,\frac{Z_l(p)}{p}\right]_{const}}$$

The change in the RF slope corresponds to the **effective (total) voltage**

$$\hat{V}_{\mathrm{T}} = \hat{V}_{\mathrm{RF}}\left(\frac{\omega_s}{\omega_{s0}}\right)^2$$

EFFECT OF THE STATIONARY DISTRIBUTION (7/9)

- **Bunch lengthening / shortening** (as a consequence of the shifts of the synchronous phase and incoherent frequency)

  - **Electrons** — **The equilibrium momentum spread is imposed by synchrotron radiation**

$$\boxed{\frac{\Delta p}{p_0} = \left(\frac{\Delta p}{p_0} \right)_0} \quad \Rightarrow \quad \frac{B}{B_0} = \frac{\omega_{s0}}{\omega_s} \sqrt{\left| \frac{\cos \phi_{s0}}{\cos \phi_s} \right|}$$

Neglecting the (usually small) synchronous phase shift

$$\Rightarrow \quad \frac{B}{B_0} = \left(\frac{B}{B_0} \right)^3 + \Delta_0$$

with

$$\boxed{\Delta_0 = \Delta_{B = B_0}}$$

EFFECT OF THE STATIONARY DISTRIBUTION (8/9)

- **Protons**

The longitudinal emittance is invariant

$$\boxed{\tau_b \, \frac{\Delta p}{p_0} = \tau_{b0} \left(\frac{\Delta p}{p_0} \right)_0} \quad \Rightarrow \quad \left(\frac{B}{B_0} \right)^2 = \frac{\omega_{s0}}{\omega_s} \sqrt{\left| \frac{\cos \phi_{s0}}{\cos \phi_s} \right|}$$

Again, neglecting the (usually small) synchronous phase shift

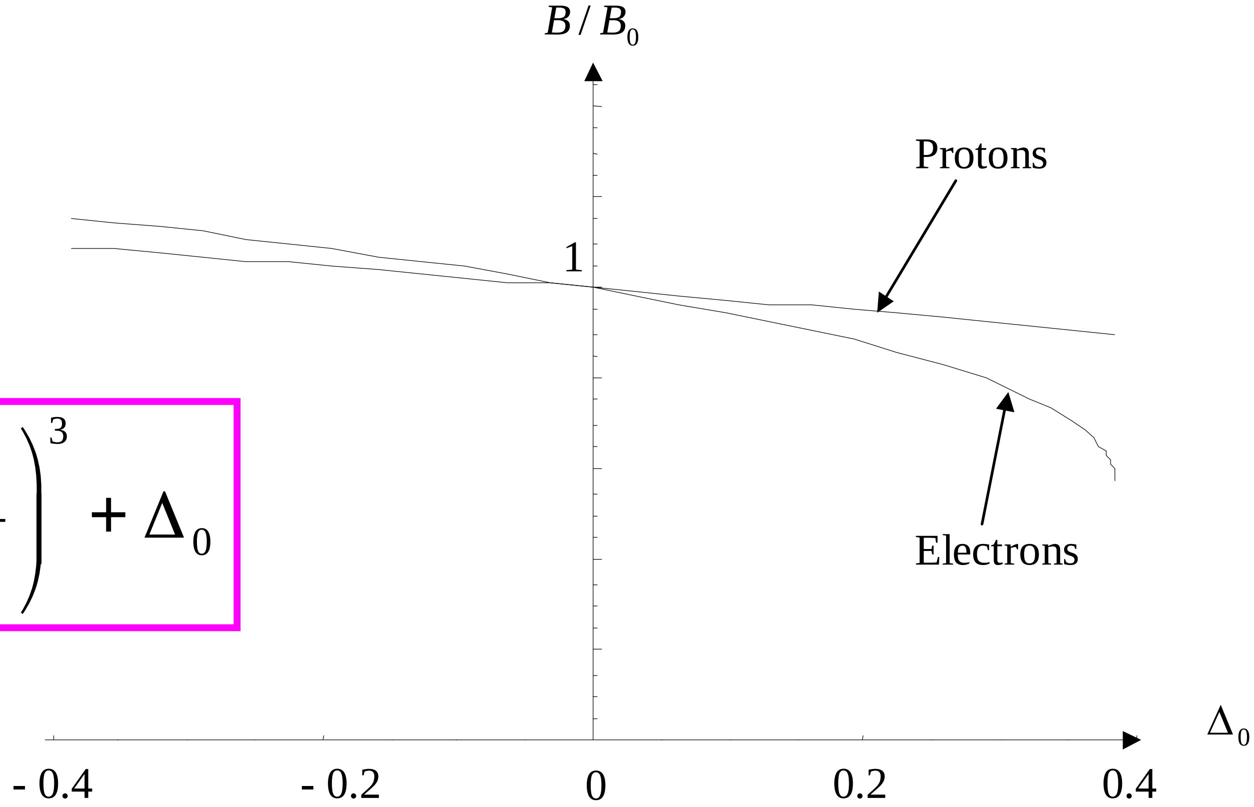
$$\Rightarrow \quad \left(\frac{B}{B_0} \right)^{-1} = \left(\frac{B}{B_0} \right)^3 + \Delta_0$$

EFFECT OF THE STATIONARY DISTRIBUTION (9/9)

- **General formula**

+ for electrons and – for protons

$$\boxed{\left(\frac{B}{B_0}\right)^{\pm 1} = \left(\frac{B}{B_0}\right)^3 + \Delta_0}$$



◆ Conclusion of the effect of the stationary distribution: **New fixed point**

- Synchronous phase shift $\phi_{s0} \Rightarrow \phi_s \left(I_b \right)$
- Potential-well distortion

$$\hat{V}_{\mathrm{RF}} \Rightarrow \hat{V}_{\mathrm{T}} \left(I_b \right) \qquad \omega_{s0} \Rightarrow \omega_s \left(I_b \right)$$

$$\left.\right\} \; B_0 \Rightarrow B \left(I_b \right)$$

PERTURBATION DISTRIBUTION (1/2)

Around the new fixed point

- The form is suggested by the single-particle signal

$$s_z(t,\vartheta) = \frac{e\,\Omega_0}{2\pi} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m}\, J_m\left(p\,\Omega_0\,\hat{\tau}\right) e^{j\left(\omega_{pm}\,t - p\,\vartheta + m\,\psi_0\right)}$$

  - **Low-intensity** $\boxed{\Delta\Psi\left(\hat{\tau},\psi_0,t\right) = g_m\left(\hat{\tau}\right) e^{-j\,m\,\psi_0}\, e^{j\,\Delta\omega_{cm}\,t}}$ $\quad m \neq 0$

$$\Delta\omega_{cm} = \omega_c - m\,\omega_s << \omega_{s0}$$

Coherent synchrotron frequency shift to be determined

Therefore, the spectral amplitude is maximum for satellite number *m* and null for the other satellites

PERTURBATION DISTRIBUTION (2/2)

$$\Rightarrow \quad \boxed{\Delta S_{zm}(\omega,\vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_m(p)\,\delta\left[\,\omega - \left(p\,\Omega_0 + m\,\omega_s + \Delta\omega_{cm}\right)\right] e^{-j\,p\,\vartheta}}$$

with
$$\boxed{\sigma_m(p) = j^{-m} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_m(p\,\Omega_0\,\hat{\tau})\, g_m(\hat{\tau})\,\hat{\tau}\,d\hat{\tau}}$$

Amplitude of the perturbation spectrum

$$\omega_s = \Omega_0 \left(- \frac{e\hat{V}_{\mathrm{T}}\,h\,\eta\cos\phi_s}{2\pi\beta^2\,E_{total}} \right)^{1/2}$$

- **High-intensity** $\boxed{\Delta\Psi(\hat{\tau},\psi_0,t) = \sum_m g_m(\hat{\tau})\,e^{-j\,m\,\psi_0}\;e^{\,j\,\Delta\omega_{cm}\,t}}$

EFFECT OF THE PERTURBATION (1/10)

$$\Psi\left(\hat{\tau},\psi_0,t\right)=\Psi_0+\Delta\Psi=g_0\left(\hat{\tau}\right)+\sum_m g_m\left(\hat{\tau}\right)e^{-j\,m\psi_0}\;e^{j\,\Delta\omega_{cm}\,t}$$

- Vlasov equation with variables $\left(\hat{\tau},\psi_0\right)$

$$\frac{\partial\Psi}{\partial t}+\left(\frac{dg_0}{d\hat{\tau}}+\frac{\partial\Delta\Psi}{\partial\hat{\tau}}\right)\frac{d\hat{\tau}}{dt}+\frac{\partial\Delta\Psi}{\partial\psi_0}\frac{d\psi_0}{dt}=0$$

$\Longrightarrow$ **Linearized Vlasov equation** $\boxed{\dfrac{\partial\Psi}{\partial t}=-\dfrac{dg_0}{d\hat{\tau}}\dfrac{d\hat{\tau}}{dt}}$

$$\Longrightarrow\quad j\sum_m g_m\left(\hat{\tau}\right)e^{-j\,m\psi_0}\;\Delta\omega_{cm}\;e^{j\,\Delta\omega_{cm}\,t}=-\frac{dg_0}{d\hat{\tau}}\frac{d\hat{\tau}}{dt}$$

EFFECT OF THE PERTURBATION (2/10)

$$\boxed{\frac{d\hat{\tau}}{dt} = \frac{d}{dt}\left(\sqrt{\tau^2 + \frac{\dot{\tau}^2}{\omega_s^2}}\right) = -\frac{F_c}{\omega_s}\sin\left(\omega_s t + \psi_0\right)}$$

with $\ddot{\tau} + \omega_s^2\,\tau = F_c = \dfrac{\eta\, e}{p_0}\left[\vec{E} + \vec{v} \times \vec{B}\right]_{zc}\left(t, \vartheta = \Omega_0\left(t - \tau\right)\right)$

$$\Rightarrow \qquad \boxed{F_c = \frac{2\pi I_b\, \omega_s^2}{\Omega_0 \hat{V}_{\mathrm{T}}\; h \cos\phi_s}\, e^{\,j\,\omega_c\, t} \sum_{p=-\infty}^{p=+\infty} Z_l\left(p\right) e^{\,j\,p\,\Omega_0\,\tau}\; \sigma\left(p\right)}$$

with $\boxed{\sigma\left(p\right) = \sum_m \sigma_m\left(p\right)}$

Spectrum amplitude at frequency $p\,\Omega_0 + \omega_c$

EFFECT OF THE PERTURBATION (3/10)

◆ Expanding the product $\sin\psi \, e^{\,j\,p\,\Omega_0\,\tau}$ (**using previously given relations**)

$$\psi = \omega_s \, t + \psi_0$$

$$\boxed{\sin\psi \, e^{\,j\,p\,\Omega_0\,\tau} = \sum_{m=-\infty}^{m=+\infty} j^{\,m} \, e^{-j\,m\,\psi} \, \frac{m}{p\,\Omega_0\,\hat{\tau}} \, J_m\left(p\,\Omega_0\,\hat{\tau}\right)}$$

$\Longrightarrow$ **Final form of the equation of coherent motion of a single bunch:**

$$\Delta\omega_{cm} = \omega_c - m\,\omega_s$$

Contribution from all the modes *m*

$$j\,\Delta\omega_{cm}\, j^{-m}\, g_m\left(\hat{\tau}\right)\hat{\tau} = \frac{2\pi I_b\, m\,\omega_s}{\Omega_0^2 \hat{V}_{\mathrm{T}}\; h\cos\phi_s}\,\frac{dg_0}{d\hat{\tau}} \sum_{p=-\infty}^{p=+\infty} \frac{Z_l\left(p\right)}{p}\, J_m\left(p\,\Omega_0\,\hat{\tau}\right)\sigma\left(p\right)$$

EFFECT OF THE PERTURBATION (4/10)

- Coherent modes of oscillation **at low intensity** (i.e. considering only a single mode *m*)

$$j\,\Delta\omega_{cm}\;j^{-m}\;g_m\left(\hat{\tau}\right)\hat{\tau}=\frac{2\pi\,I_b\;m\,\omega_s}{\Omega_0^2\,\hat{V}_{\mathrm{T}}\;h\cos\phi_s}\;\frac{dg_0}{d\hat{\tau}}\sum_{p=-\infty}^{p=+\infty}\frac{Z_l\left(p\right)}{p}\;J_m\left(p\,\Omega_0\,\hat{\tau}\right)\sigma_m\left(p\right)$$

Multiplying both sides by $J_m\left(l\,\Omega_0\,\hat{\tau}\right)$ and integrating over $\hat{\tau}$

$$\Rightarrow\qquad \boxed{\Delta\omega_{cm}\;\sigma_m\left(l\right)=\sum_{p=-\infty}^{p=+\infty}K_{lp}^{m}\;\sigma_m\left(p\right)}$$

$$\boxed{K_{lp}^{m}=-\frac{2\pi\,I_b\;m\,\omega_s}{\Omega_0^2\,\hat{V}_{\mathrm{T}}\;h\cos\phi_s}\;j\,\frac{Z_l\left(p\right)}{p}\int_{\hat{\tau}=0}^{\hat{\tau}=+\infty}\frac{dg_0}{d\hat{\tau}}\;J_m\left(p\,\Omega_0\,\hat{\tau}\right)J_m\left(l\,\Omega_0\,\hat{\tau}\right)d\hat{\tau}}$$

EFFECT OF THE PERTURBATION (5/10)

- The procedure to obtain first order exact solutions, with realistic modes and a general interaction, thus consists of finding the eigenvalues and eigenvectors of the infinite complex matrix whose elements are K_{lp}^{m}

- The result is an infinite number of modes mq of oscillation (as there are 2 degrees of freedom $\left(\hat{\tau}, \psi_0\right)$)

- To each mode, one can associate:

The imaginary part tells us if this mode is stable or not

  - a coherent frequency shift $\Delta\omega_{cmq} = \omega_{cmq} - m\,\omega_s$ (**qth eigenvalue**)
  - a coherent spectrum $\sigma_{mq}\left(p\right)$ (**qth eigenvector**)
  - a perturbation distribution $g_{mq}\left(\hat{\tau}\right)$

- For numerical reasons, the matrix needs to be truncated, and thus only a finite frequency domain is explored

EFFECT OF THE PERTURBATION (6/10)

- The longitudinal signal at the PU electrode is given by

$$\boxed{S_{mq}(t,\vartheta) = S_{z0}(t,\vartheta) + \Delta S_{zmq}(t,\vartheta)}$$

$$S_{z0}(t,\vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_0(p)\, e^{j p \Omega_0 t}\, e^{-j p \vartheta}$$

$$\Delta S_{zmq}(t,\vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_{mq}(p)\, e^{j\left(p\Omega_0 + m\,\omega_s + \Delta\omega_{cmq}\right)t}\, e^{-j p \vartheta}$$

- For the case of the parabolic amplitude distribution

$$g_0(\hat{z}) = \frac{2}{\pi\left(\frac{\tau_b}{2}\right)^2}\left(1-\hat{z}^2\right) \qquad S_{z0}(t,\vartheta) = 8\, I_b \sum_{p=-\infty}^{p=+\infty} e^{j p \Omega_0 t}\, e^{-j p \vartheta}\, \frac{J_2(p\pi B)}{(p\pi B)^2}$$

EFFECT OF THE PERTURBATION (7/10)

$$K_{lp}^{m} = \frac{128\, I_b\; m\, \omega_s}{\Omega_0^2\, \hat{V}_{\mathrm{T}}\;\, h \cos\phi_s\;\, \tau_b^4}\; j\, \frac{Z_l(p)}{p} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_m\left(p\, \Omega_0\, \hat{\tau}\right) J_m\left(l\, \Omega_0\, \hat{\tau}\right) \hat{\tau}\, d\hat{\tau}$$

- Low order eigenvalues and eigenvectors of the matrix can be found quickly by computation, **using the relations**

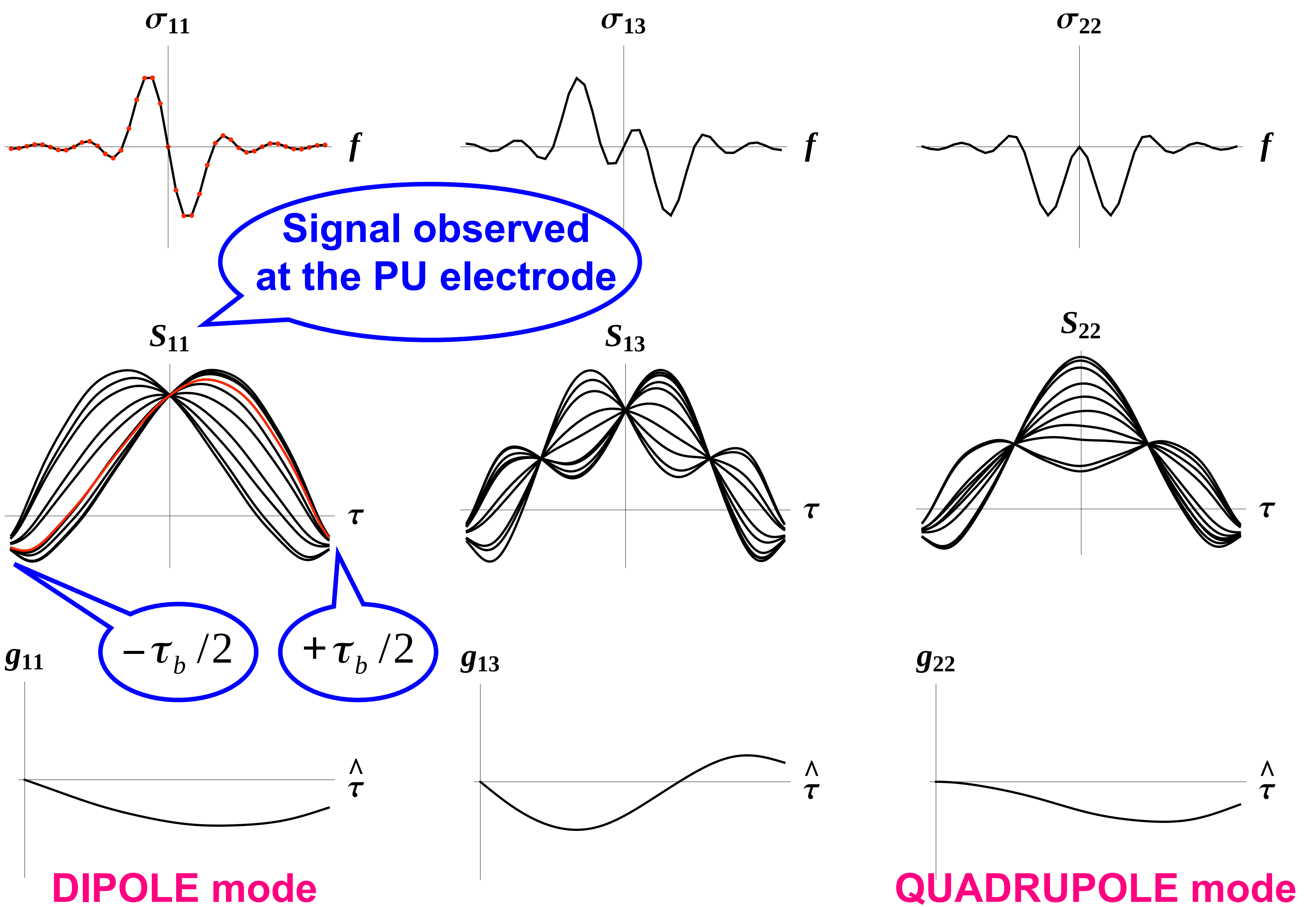
$$\boxed{\int_0^X J_m^2(a\,x)\, x\, dx = \frac{X^2}{2}\left[J_m'(a\,X)\right]^2 + \frac{1}{2}\left[X^2 - \frac{m^2}{a^2}\right] J_m^2(a\,X)}$$

$$\boxed{\int_0^X x\, J_m(a\,x)\, J_m(b\,x)\, dx = \frac{X}{a^2 - b^2}\left[a\, J_m(b\,X)\, J_{m+1}(a\,X) - b\, J_m(a\,X)\, J_{m+1}(b\,X)\right]}$$

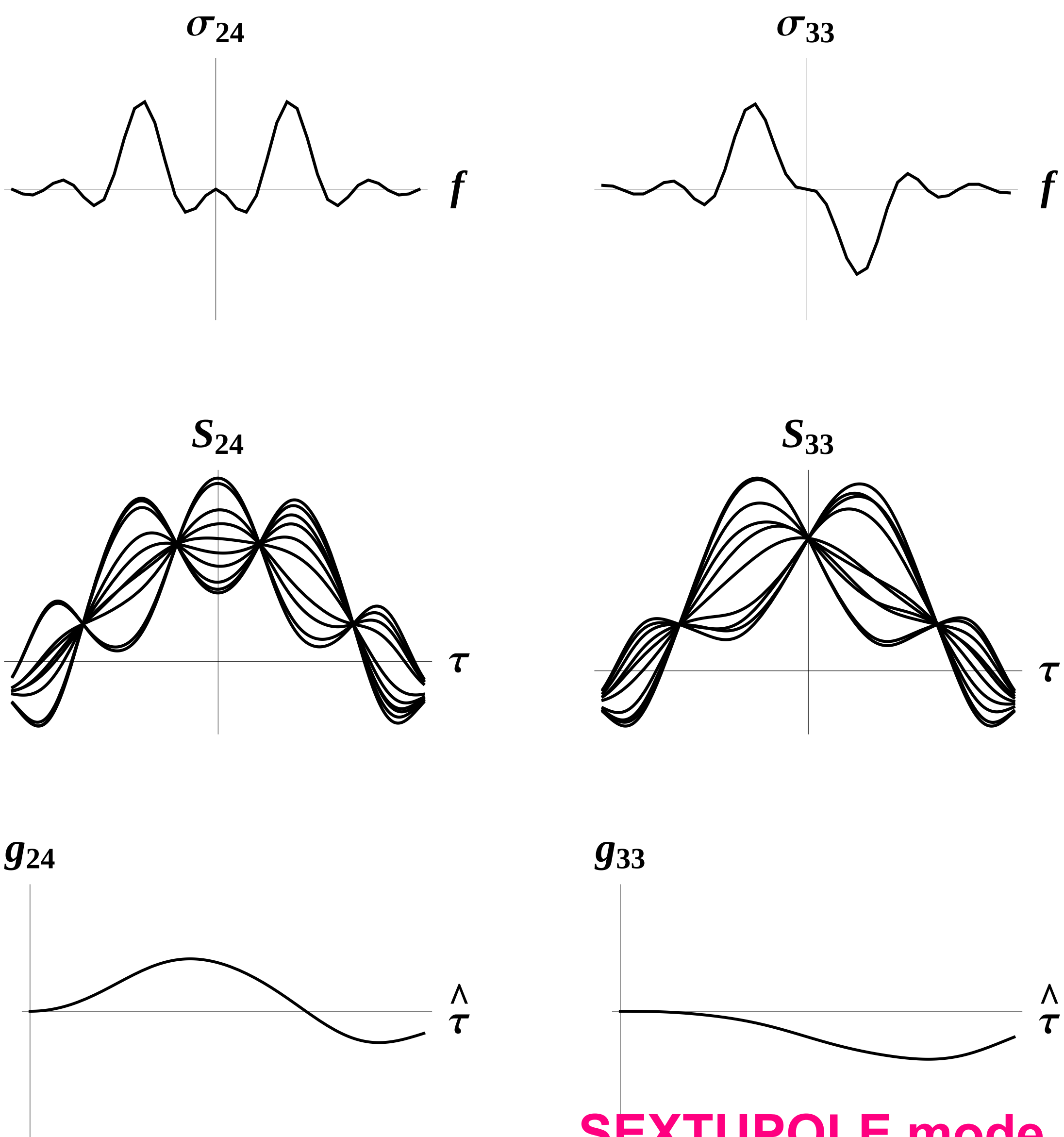
$$\boxed{a^2 \neq b^2}$$

- The case of a **constant inductive impedance** is solved in the next slides, and the signal at the PU shown for several superimposed turns

EFFECT OF THE PERTURBATION (8/10)



EFFECT OF THE PERTURBATION (9/10)



- **The spectrum of mode *mq***

  is peaked at $\boxed{f_q \approx \frac{q+1}{2\,\tau_b}}$

  and extends $\boxed{\sim \pm\, \tau_b^{-1}}$

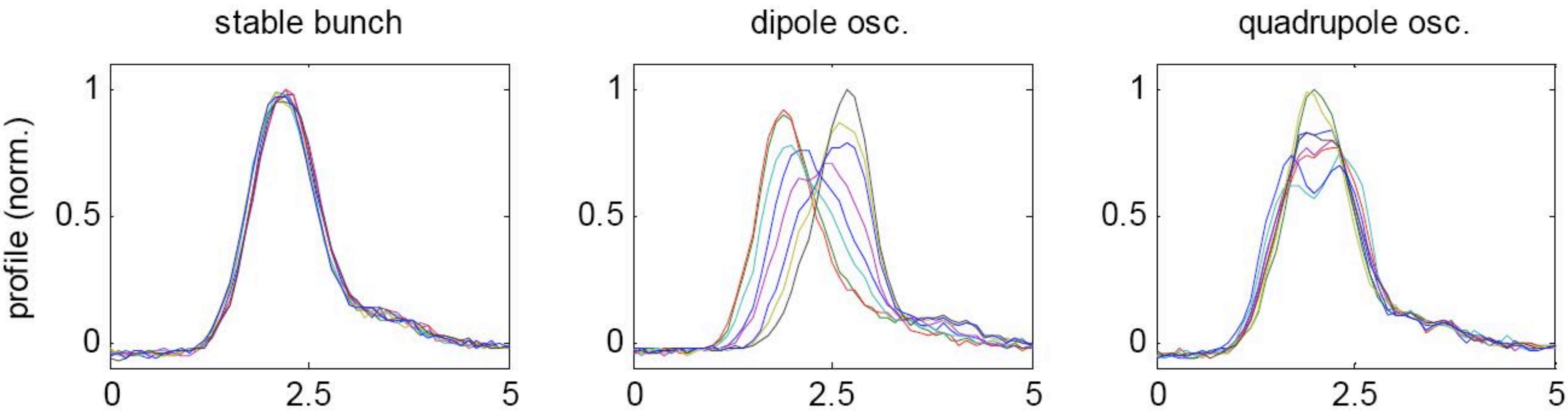
$\boxed{q \equiv |m| + 2k}$ $\quad 0 \leq k < +\infty$

- There are ***q* nodes** on these "**standing-wave**" patterns

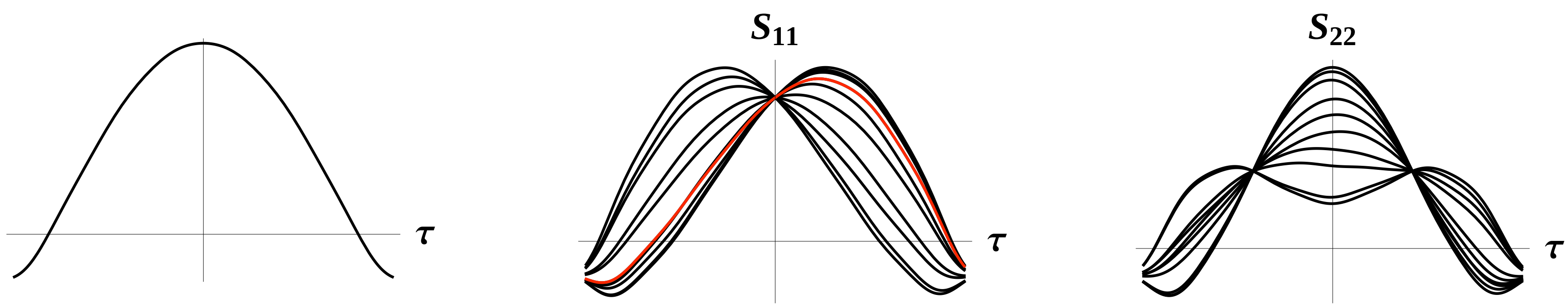
SEXTUPOLE mode

EFFECT OF THE PERTURBATION (10/10)

Observations in the CERN SPS in 2007



(Garnier-Laclare's) theory



TRANSVERSE: LOW-INTENSITY

SINGLE PARTICLE TRANSVERSE MOTION (1/3)

- A purely linear synchrotron oscillation around the synchronous particle is assumed (with no coherent oscillations)

$$\ddot{\tau} + \omega_s^2 \, \tau = 0 \qquad \tau = \hat{\tau} \cos\left(\omega_s \, t + \psi_0 \right)$$

- For the transverse betatron oscillation, the equation of unperturbed motion, e.g. in the horizontal plane, is written as

$$\boxed{x = \hat{x} \cos\left[\varphi_x \left(t \right) \right]} \qquad x^2 + \frac{\dot{x}^2}{\dot{\varphi}_x^2} = \hat{x}^2$$

- The horizontal betatron frequency is given by $\dot{\varphi}_x = Q_x \, \Omega$

Chromaticity

with $\xi_x = \dfrac{\Delta Q_x / Q_{x0}}{\Delta p / p_0} = \dfrac{Q'_x}{Q_{x0}}$ $\qquad \eta = - \dfrac{\Delta \Omega / \Omega_0}{\Delta p / p_0} = \dfrac{\dot{\tau}}{\Delta p / p_0}$

SINGLE PARTICLE TRANSVERSE MOTION (2/3)

$$\Longrightarrow \qquad Q_x\left(p\right) = Q_{x0}\left(1 + \xi_x \, \frac{\Delta p}{p_0}\right) \qquad \Omega\left(p\right) = \Omega_0\left(1 - \eta \, \frac{\Delta p}{p_0}\right)$$

$$\Longrightarrow \qquad \dot{\varphi}_x = Q_x \, \Omega \approx Q_{x0} \, \Omega_0 \left[1 - \dot{\tau}\left(1 - \frac{\xi_x}{\eta}\right)\right]$$

and $\boxed{\varphi_x = Q_{x0} \, \Omega_0 \left(t - \tau\right) + \omega_{\xi_x} \, \tau + \varphi_{x0}}$ $\boxed{\omega_{\xi_x} = Q_{x0} \, \Omega_0 \, \frac{\xi_x}{\eta}}$

Horiz. chromatic frequency

SINGLE PARTICLE TRANSVERSE MOTION (3/3)

- In the absence of perturbation, the horizontal coordinate satisfies

$$\ddot{x} - \frac{\ddot{\varphi}_x}{\dot{\varphi}_x}\,\dot{x} + \dot{\varphi}_x^2\; x = 0$$

- In the presence of electromagnetic fields induced by the beam, the equation of motion writes

$$\boxed{\ddot{x} - \frac{\ddot{\varphi}_x}{\dot{\varphi}_x}\,\dot{x} + \dot{\varphi}_x^2\; x = F_x = \frac{e}{\gamma\, m_0}\left[\,\vec{E} + \vec{v} \times \vec{B}\,\right]_x \Big(t, \vartheta = \Omega_0\left(t - \tau\right)\Big)}$$

When following the particle along its trajectory

SINGLE PARTICLE TRANSVERSE SIGNAL (1/2)

- The horizontal signal induced at a perfect PU electrode (infinite bandwidth) at angular position ϑ in the ring by the off-centered test particle is given by

$$\boxed{s_x(t,\vartheta) = s_z(t,\vartheta)\, x(t) = s_z(t,\vartheta)\, \hat{x}\cos(\varphi_x)}$$

$$\Rightarrow \quad s_x(t,\vartheta) = e\, \hat{x}\cos(\varphi_x) \sum_{k=-\infty}^{k=+\infty} \delta\left(t - \tau - \frac{\vartheta}{\Omega_0} - \frac{2k\pi}{\Omega_0}\right)$$

- Developing $\cos(\varphi_x)$ into exponential functions and using relations given in the longitudinal course, yields

$$s_x(t,\vartheta) = \frac{e\,\Omega_0}{4\pi}\, \hat{x}\, e^{j\left(Q_{x0}\,\Omega_0\, t + \varphi_{x0}\right)} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m} J_{m,x}(p,\hat{\tau})\, e^{j\left[\omega_{pm} t + m\psi_0 - p\vartheta\right]}$$

$$+\, c.c.$$

Complex conjugate

$$\boxed{\omega_{pm} = p\Omega_0 + m\,\omega_s}$$

SINGLE PARTICLE TRANSVERSE SIGNAL (2/2)

with $$\boxed{J_{m,x}(p,\hat{\tau}) = J_m\left\{\left[\left(p+Q_{x0}\right)\Omega_0 - \omega_{\xi_x}\right]\hat{\tau}\right\}} \quad \Rightarrow$$

$$\boxed{\begin{aligned} s_x(\omega,\vartheta) &= \frac{e\,\Omega_0}{4\pi}\,\hat{x}\,e^{j\varphi_{x0}} \\ &\sum_{p,m=-\infty}^{p,m=+\infty} j^{-m}\,J_{m,x}(p,\hat{\tau})\,\delta\left\{\omega-\left[\left(p+Q_{x0}\right)\Omega_0+m\,\omega_s\right]\right\}e^{j\left(m\psi_0-p\vartheta\right)} + c.c.\end{aligned}}$$

- The spectrum is a line spectrum at frequencies $\left(p+Q_{x0}\right)\Omega_0 + m\,\omega_s$
- Around every betatron line $\left(p+Q_{x0}\right)\Omega_0$, there is an infinite number of **synchrotron satellites *m***
- The spectral amplitude of the mth satellite is given by $J_{m,x}(p,\hat{\tau})$
- **The spectrum is centered at the chromatic frequency** $\omega_{\xi_x} = Q_{x0}\,\Omega_0\,\dfrac{\xi_x}{\eta}$

STATIONARY DISTRIBUTION (1/2)

- In the absence of perturbation, $\hat{x}$ and $\hat{\tau}$ are constants of the motion

- Therefore, the stationary distribution is a function of the peak amplitudes only

$$\Psi_{x0}\left(\hat{x},\ \hat{\tau}\right)$$

- No correlation between horizontal and longitudinal planes is assumed and the stationary part is thus written as the product of 2 stationary distributions, one for the longitudinal phase space and one for the horizontal one

$$\boxed{\Psi_{x0}\left(\hat{x},\ \hat{\tau}\right) = f_0\left(\hat{x}\right) g_0\left(\hat{\tau}\right)}$$

$$\int_{\hat{x}=0}^{\hat{x}=+\infty} f_0\left(\hat{x}\right) \hat{x}\, d\hat{x} = \frac{1}{2\pi} \qquad \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} g_0\left(\hat{\tau}\right) \hat{\tau}\, d\hat{\tau} = \frac{1}{2\pi}$$

STATIONARY DISTRIBUTION (2/2)

- Since on average, the beam center of mass is on axis, the horizontal signal induced by the stationary distribution is null

$$S_{x0}\left(t,\vartheta\right)=N_b \int_{\hat{x}=0}^{\hat{x}=+\infty}\int_{\varphi_{x0}=0}^{\varphi_{x0}=2\pi}\int_{\hat{\tau}=0}^{\hat{\tau}=+\infty}\int_{\psi_0=0}^{\psi_0=2\pi} f_0\left(\hat{x}\right) g_0\left(\hat{\tau}\right) s_x\left(t,\vartheta\right) \hat{x}\,\hat{\tau}\, d\hat{x}\, d\hat{\tau}\, d\varphi_{x0}\, d\psi_0$$

$$=0$$

PERTURBATION DISTRIBUTION (1/3)

- In order to get some dipolar fields, density perturbations $\Delta\Psi_x$ that describe beam center-of-mass displacements along the bunch are assumed

- The mathematical form of the perturbations is suggested by the single-particle signal

$$s_x(t,\vartheta) = \frac{e\Omega_0}{4\pi}\,\hat{x} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m}\, J_{m,x}(p,\hat{\tau})\, e^{j(\varphi_{x0}+m\psi_0)}\, e^{-jp\vartheta}\, e^{j[(p+Q_{x0})\Omega_0+m\omega_s]t}$$

$$+\ c.c.$$

- **Low-intensity** $$\boxed{\Delta\Psi_x = h_m(\hat{x},\hat{\tau})\, e^{-j(\varphi_{x0}+m\psi_0)}\, e^{j\Delta\omega_{cm}^x t}}$$

$$\Delta\omega_{cm}^x = \omega_c - m\,\omega_s << \omega_s$$

Coherent betatron frequency shift to be determined

PERTURBATION DISTRIBUTION (2/3)

- In the time domain, the horizontal signal takes the form (for a single value *m*)

$$S_x(t,\vartheta) = 2\,\pi^2\, I_b \sum_{p=-\infty}^{p=+\infty} e^{-j\,p\,\vartheta}\,\sigma_{x,m}(p)\, e^{\,j\left[\,(p+Q_{x0})\,\Omega_0 + \omega_c\right]t}$$

Fourier transform

$$\boxed{S_x(\omega,\vartheta) = 2\,\pi^2\, I_b \sum_{p=-\infty}^{p=+\infty} e^{-j\,p\,\vartheta}\,\sigma_{x,m}(p)\,\delta\left\{\omega - \left[\,(p+Q_{x0})\,\Omega_0 + \omega_c\right]\right\}}$$

with $$\boxed{\sigma_{x,m}(p) = j^{-m}\int_{\hat{x}=0}^{\hat{x}=+\infty}\int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} h_m(\hat{x},\hat{\tau})\, J_{m,x}(p,\hat{\tau})\,\hat{x}^2\, d\hat{x}\;\hat{\tau}\;d\hat{\tau}}$$

PERTURBATION DISTRIBUTION (3/3)

- **High-intensity**

$$\boxed{\Delta\Psi_x = \sum_m h_m\left(\hat{x}\,,\hat{\tau}\right) e^{-j\left(\varphi_{x0}+m\psi_0\right)}\; e^{\,j\,\Delta\omega_{cm}^{x}\,t}}$$

TRANSVERSE IMPEDANCE

$$\boxed{\left[\vec{E}+\vec{v}\times\vec{B}\right]_x(t,\vartheta)=\frac{-j\beta}{2\pi R}\int Z_x(\omega)\,S_x(\omega,\vartheta)\,e^{j\omega t}\,d\omega}$$

All the properties of the electromagnetic response of a given machine to a passing particle is gathered into the transverse impedance (complex function => in Ω / m)

EFFECT OF THE PERTURBATION (1/10)

$$\Psi_x\left(\hat{x}\,,\,\varphi_{x0}\,,\,\hat{\tau}\,,\,\psi_0\,,\,t\right)=\Psi_{x0}+\Delta\Psi_x$$

$$\Rightarrow \qquad \Psi_x = f_0\left(\hat{x}\right) g_0\left(\hat{\tau}\right) + \sum_m h_m\left(\hat{x}\,,\hat{\tau}\right) e^{-j\left(\varphi_{x0}+m\psi_0\right)}\, e^{\,j\,\Delta\omega_{cm}^{x}\,t}$$

- Vlasov equation

$$\frac{\partial \Psi_x}{\partial t}+\frac{\partial \Psi_x}{\partial \hat{x}}\,\dot{\hat{x}}+\frac{\partial \Psi_x}{\partial \varphi_{x0}}\,\dot{\varphi}_{x0}+\frac{\partial \Psi_x}{\partial \hat{\tau}}\,\dot{\hat{\tau}}+\frac{\partial \Psi_x}{\partial \psi_0}\,\dot{\psi}_0=0$$

$\Rightarrow$ **Linearized Vlasov equation** $\boxed{\dfrac{\partial \Psi_x}{\partial t}=-\dfrac{df_0\left(\hat{x}\right)}{d\hat{x}}\,g_0\left(\hat{\tau}\right)\dot{\hat{x}}}$

$$\Rightarrow \quad j\sum_m h_m\left(\hat{x}\,,\hat{\tau}\right) e^{-j\left(\varphi_{x0}+m\psi_0\right)}\,\Delta\omega_{cm}^{x}\,e^{\,j\,\Delta\omega_{cm}^{x}\,t}=-\frac{df_0\left(\hat{x}\right)}{d\hat{x}}\,g_0\left(\hat{\tau}\right)\dot{\hat{x}}$$

EFFECT OF THE PERTURBATION (2/10)

- The expression of $\dot{\hat{x}}$ can be drawn from the single-particle horizontal equation of motion

$$\dot{\hat{x}} = \frac{d}{dt}\left(\hat{x}\right) = \frac{d}{dt}\left[x^2 + \left(\frac{\dot{x}}{\dot{\varphi}_x}\right)^2\right]^{1/2} = F_x \; \frac{\dot{x}}{\hat{x}\;\dot{\varphi}_x^2}$$

$$\frac{\dot{x}}{\hat{x}\;\dot{\varphi}_x} = -\sin\left(\varphi_x\right)$$

$$\Rightarrow \quad \boxed{\dot{\hat{x}} = -\frac{\sin\left(\varphi_x\right)}{\dot{\varphi}_x}\,F_x}$$

EFFECT OF THE PERTURBATION (3/10)

- Using the definition of the transverse impedance, the force can be written

$$F_x = -\frac{j\, e\, \beta\, \pi\, I_b}{R\, \gamma\, m_0} \sum_{p=-\infty}^{p=+\infty} Z_x\left(p\right) \sigma_{x,m}\left(p\right) e^{-j\, p\, \Omega_0 \left(t - \tau\right)}\, e^{j\left[\left(p + Q_{x0}\right)\Omega_0 + \omega_c\right] t}$$

$$\left(p + Q_{x0}\right)\Omega_0 + \omega_c$$

- Developing the $\sin\left(\varphi_x\right)$ into exponential functions, keeping then only the slowly varying term, making the approximation $\dot{\varphi}_x \approx Q_{x0}\, \Omega_0$ and **using the relations** $\boxed{J_{-m}\left(-x\right) = J_m\left(x\right)}$ and one from the longitudinal course, yields

$$\dot{\hat{x}} = -\frac{e\, \pi\, I_b}{2\, \gamma\, m_0\, c\, Q_{x0}} \sum_{p,m=-\infty}^{p,m=+\infty} Z_x\left(p\right) \sigma_{x,m}\left(p\right) j^m\, J_{m,x}\left(p,\, \hat{\tau}\right) e^{-j\left(\varphi_{x0} + m\, \psi_0\right)}\, e^{j\, \Delta\omega_{cm}^x\, t}$$

EFFECT OF THE PERTURBATION (4/10)

$\Longrightarrow$ For each mode *m*, one has

$$j \, h_m(\hat{x}, \hat{\tau}) \, \Delta\omega_{cm}^{x} = \frac{e \, \pi \, I_b}{2 \, \gamma \, m_0 \, c \, Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p) \, \sigma_x(p) \, j^m \, J_{m,x}(p, \hat{\tau}) \frac{df_0(\hat{x})}{d\hat{x}} \, g_0(\hat{\tau})$$

with $$\boxed{\sigma_x(p) = \sum_m \sigma_{x,m}(p)}$$

Spectrum amplitude at frequency

$(p + Q_{x0}) \, \Omega_0 + \omega_c$

Multiplying both sides by $\hat{x}^2$ and integrating over $\hat{x}$ $\Longrightarrow$

$$j \, \Delta\omega_{cm}^{x} \int_{\hat{x}=0}^{\hat{x}=+\infty} h_m(\hat{x}, \hat{\tau}) \, \hat{x}^2 \, d\hat{x} = - \frac{e \, I_b}{2 \, \gamma \, m_0 \, c \, Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p) \, \sigma_x(p) \, j^m \, J_{m,x}(p, \hat{\tau}) \, g_0(\hat{\tau})$$

using the relation $$\int_{\hat{x}=0}^{\hat{x}=+\infty} \frac{df_0(\hat{x})}{d\hat{x}} \, \hat{x}^2 \, d\hat{x} = -2 \int_{\hat{x}=0}^{\hat{x}=+\infty} f_0(\hat{x}) \, \hat{x} \, d\hat{x} = -\frac{1}{\pi}$$

EFFECT OF THE PERTURBATION (5/10)

- Note that the horizontal stationary distribution disappeared and only the longitudinal one remains => Only the beam center of mass is important (in our case). This should also be valid for the perturbation, which can be written

$$\boxed{\int_{\hat{x}=0}^{\hat{x}=+\infty} h_m(\hat{x}\,,\,\hat{\tau})\,\hat{x}^2\,d\hat{x} = g_0(\hat{\tau})\,\hat{x}_m(\hat{\tau})}$$

Averaged peak betatron amplitude

$\Longrightarrow$ **Final form of the equation of coherent motion of a single bunch:**

$$\Delta\omega_{cm}^x = \omega_c - m\,\omega_s$$

Contribution from all the modes *m*

$$\boxed{j\,\Delta\omega_{cm}^x\,\hat{x}_m(\hat{\tau}) = -\frac{e\,I_b}{2\,\gamma\,m_0\,c\,Q_{x0}}\sum_{p=-\infty}^{p=+\infty} Z_x(p)\,\sigma_x(p)\,j^m\,J_{m,x}(p,\,\hat{\tau})}$$

EFFECT OF THE PERTURBATION (6/10)

with $$\sigma_{x,m}(p) = j^{-m} \int_{\hat{x}=0}^{\hat{x}=+\infty} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} h_m(\hat{x}, \hat{\tau}) \, J_{m,x}(p, \hat{\tau}) \, \hat{x}^2 \, d\hat{x} \; \hat{\tau} \; d\hat{\tau}$$

$$= j^{-m} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_{m,x}(p, \hat{\tau}) \, g_0(\hat{\tau}) \, \hat{x}_m(\hat{\tau}) \, \hat{\tau} \; d\hat{\tau}$$

- Coherent modes of oscillation **at low intensity** (i.e. considering only a single mode *m*)

$$j \, \Delta\omega_{cm}^{x} \, \hat{x}_m(\hat{\tau}) = - \frac{e \, I_b}{2 \, \gamma \, m_0 \, c \, Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p) \, \sigma_{x,m}(p) \, j^m \, J_{m,x}(p, \hat{\tau})$$

Multiplying both sides by $j^{-m} \, J_{m,x}(l, \hat{\tau}) \, g_0(\hat{\tau}) \, \hat{\tau}$ and integrating over $\hat{\tau}$

EFFECT OF THE PERTURBATION (7/10)

$$\Rightarrow \quad \boxed{\Delta\omega_{cm}^{x}\ \sigma_{x,m}(l) = \sum_{p=-\infty}^{p=+\infty} K_{lp}^{x,m}\ \sigma_{x,m}(p)}$$

$$\boxed{K_{lp}^{x,m} = \frac{j\, e\, I_b}{2\,\gamma\, m_0\, c\, Q_{x0}}\, Z_x(p) \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_{m,x}(l,\ \hat{\tau})\, J_{m,x}(p,\ \hat{\tau})\, g_0(\hat{\tau})\, \hat{\tau}\, d\hat{\tau}}$$

- Following the same procedure as for the longitudinal plane, the horizontal coherent oscillations (over several turns) of a **"water-bag" bunch** interacting with a **constant inductive impedance** are shown in the next slides for the first **head-tail modes**

** Note that the index x has been removed for clarity*

$$g_0(\hat{\tau}) = 4 / \left(\pi\, \tau_b^2\right)$$

EFFECT OF THE PERTURBATION (8/10)

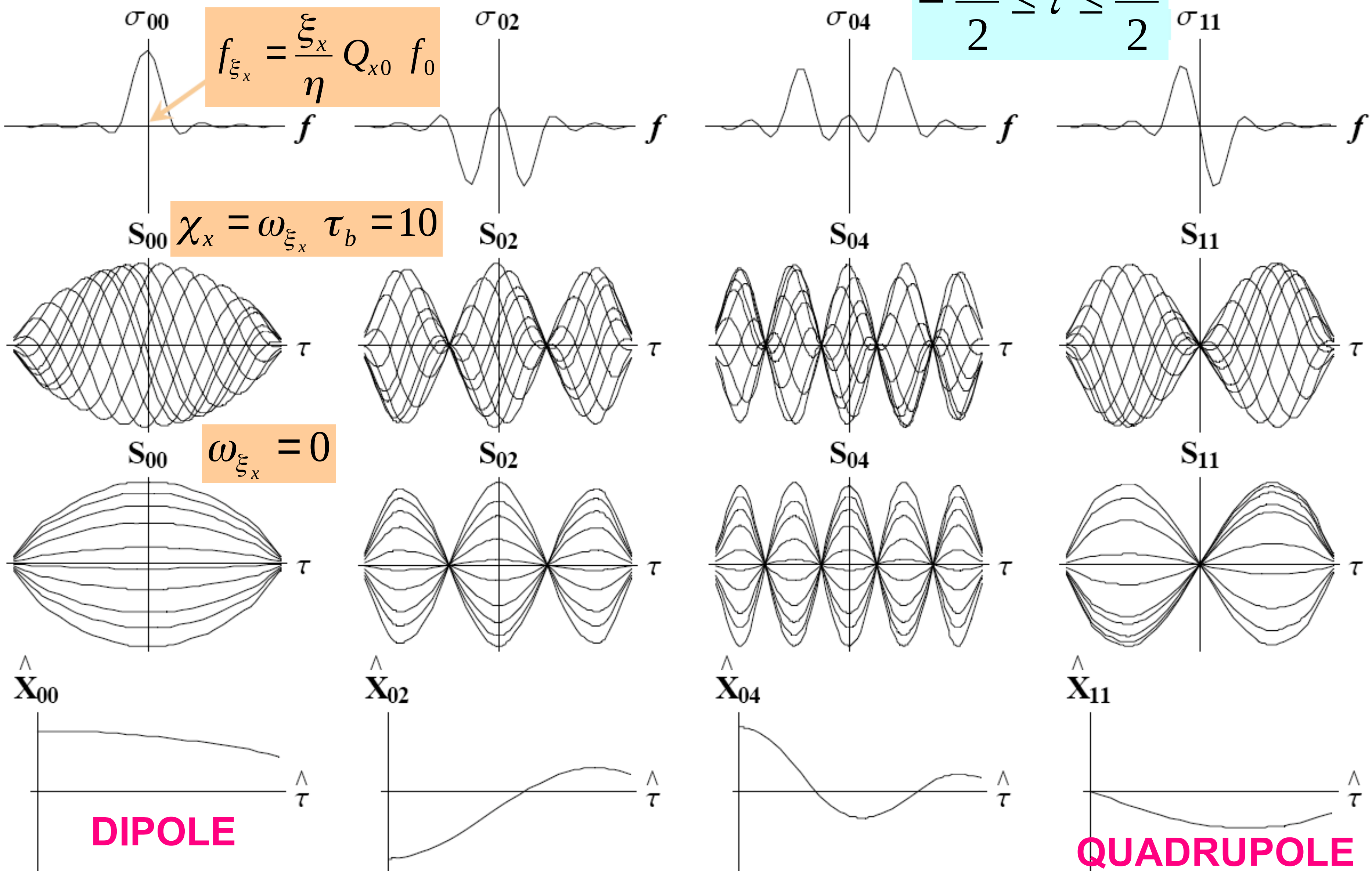
$$-\frac{\tau_b}{2} \le \tau \le \frac{\tau_b}{2}$$

$$Q_{x0} = \text{x}.13$$

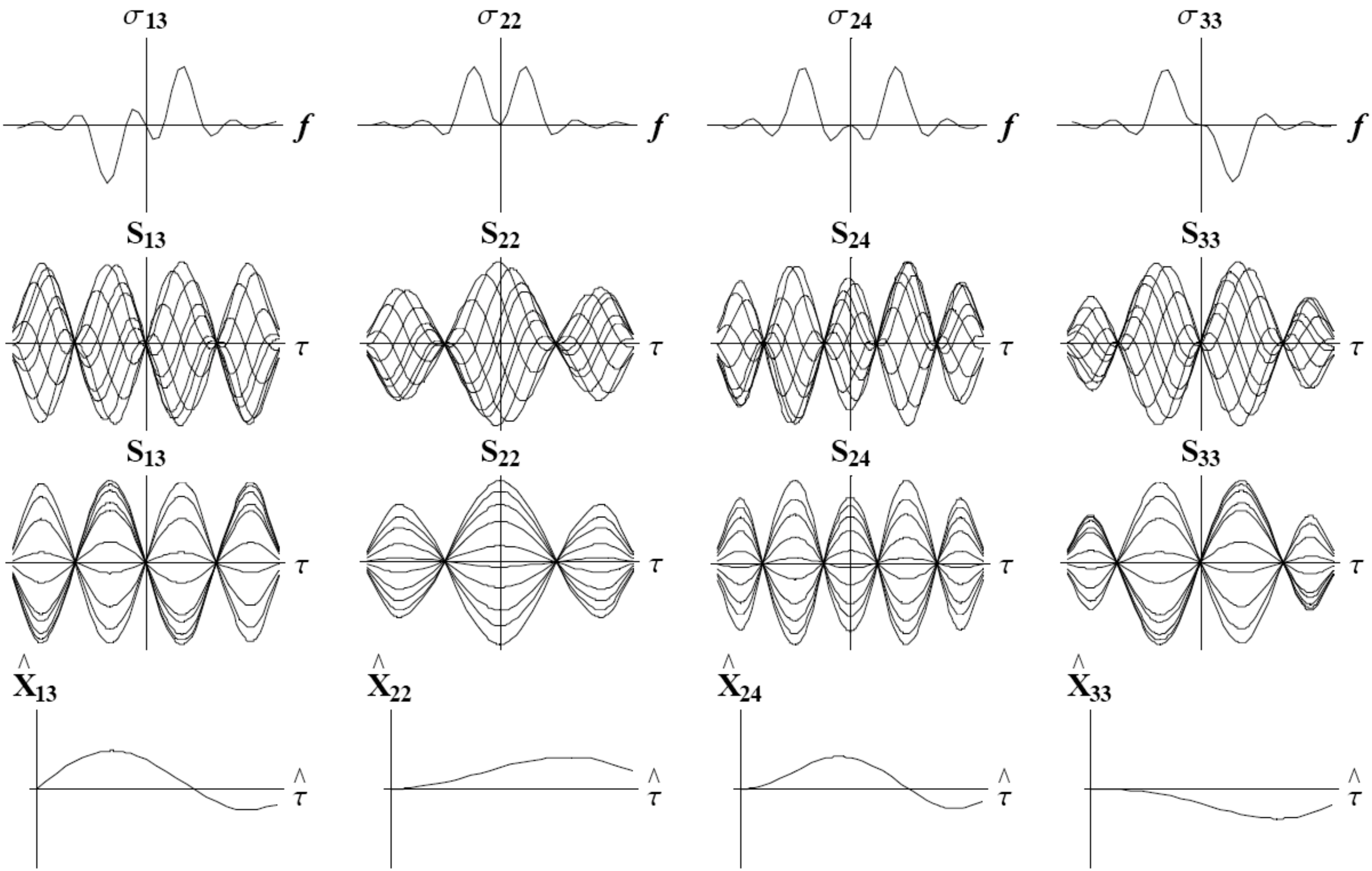
$$f_{\xi_x} = \frac{\xi_x}{\eta} Q_{x0} \ f_0$$

$$\chi_x = \omega_{\xi_x} \ \tau_b = 10$$

$$\omega_{\xi_x} = 0$$

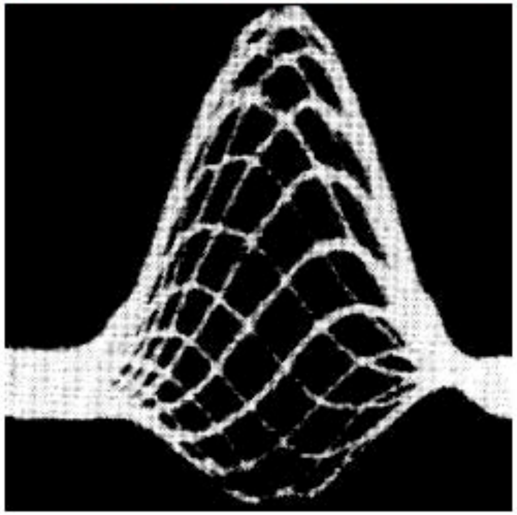


EFFECT OF THE PERTURBATION (9/10)

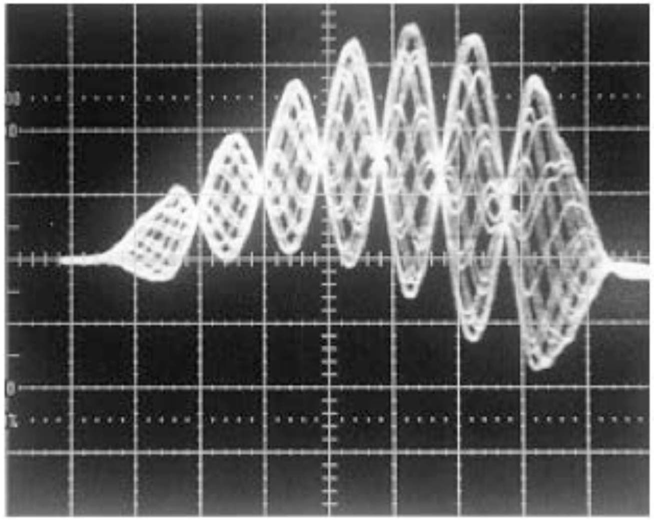


EFFECT OF THE PERTURBATION (10/10)

Observation in the CERN PSB in ~1974
(J. Gareyte and F. Sacherer)

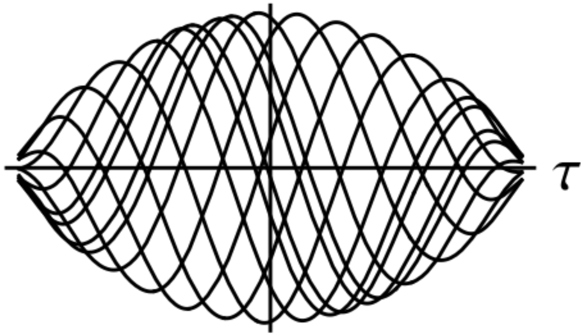


Observation in the CERN PS in 1999

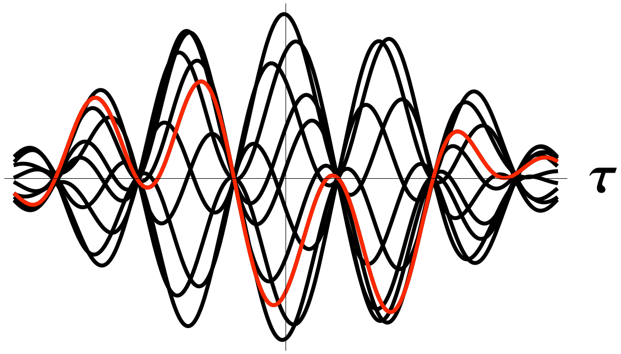


(Laclare's) theory

S_{00}



S_{66}



REMINDER OF THE PROCEDURE: BOTH L & T

- Start with the single particle motion => Harmonic oscillator + beam-induced electromagnetic force (L or T)
- Look at the single particle signal => Line spectrum with an infinite number of synchrotron satellites *m* (centered at 0 for L and at the chromatic frequency for T)
- Consider a distribution of particles (particle density in phase space) and express it as a sum of a stationary distribution + a perturbation
- The beam-induced electromagnetic force can be expressed through the impedance (complex function of frequency) for both L and T
- Study the effect of the impedance on the stationary distribution (for L) => A new fixed point is obtained, with a dependency on the bunch intensity of the synchronous phase, incoherent frequency, effective (total) voltage and bunch length
- Around the new fixed point (for L), write the perturbation => Coherent with respect to the satellite number *m*
- Apply the Vlasov equation to first order **=> One ends up with an eigen-value system to solve**
- The result is an infinite number of modes of oscillation ***mq***

azimuthal

radial

APPROXIMATE FORMULAE: SACHERER FORMULAE (1/4)

- Finding the eigenvalues and eigenvectors of a complex matrix by computer can be difficult in some cases, and a simple approximate formula for the eigenvalues is useful in practice to have a rough estimate
  - **Assuming sinusoidal modes**

$$\boxed{p_m(t) = \begin{cases} \cos\left[\left(|m|+1\right)\pi\, t/\tau_b\right], & m \text{ even} \\ \sin\left[\left(|m|+1\right)\pi\, t/\tau_b\right], & m \text{ odd} \end{cases}}$$

the difference signal from a beam position monitor has the form

$$\Delta - \text{signal} \propto p_m(t)\, e^{\,j\left(\chi_x\, t/\tau_b + 2\pi\, k\, Q_{x0}\right)}$$

For the kth revolution

$$\chi_x = \omega_{\xi_x}\, \tau_b$$

Total phase shift between head and tail

APPROXIMATE FORMULAE: SACHERER FORMULAE (2/4)

- The function $h_{m,m}\left(\omega-\omega_{\xi_x}\right)=\left|\,p_m\left(\omega-\omega_{\xi_x}\right)\right|^2$,

where $p_m\left(\omega-\omega_{\xi_x}\right)$ is the Fourier transform of $p_m\left(t\right)e^{j\omega_{\xi_x}t}$

is a good approximation of the power spectrum $\boxed{h_{m,m}\left(\omega\right)\approx\left|\,\sigma_{mm}\left(\omega\right)\right|^2}$

$$h_{m,m}\left(\omega\right)=\frac{\tau_b^2}{2\pi^4}\left(\left|m\right|+1\right)^2\frac{1+\left(-1\right)^{\left|m\right|}\cos\left(\omega\,\tau_b\right)}{\left[\left(\omega\tau_b/\pi\right)^2-\left(\left|m\right|+1\right)^2\right]^2}$$

APPROXIMATE FORMULAE: SACHERER FORMULAE (3/4)

- Making this approximation, it can be shown that the **Sacherer formulae** are obtained

  - **In longitudinal**

$$\boxed{\Delta\omega^{l}_{m,m} = \frac{|m|}{|m|+1} \times \frac{j\, I_b\, \omega_s}{3 B^3 \hat{V}_T\, h \cos\phi_s} \times \left[\frac{Z_l(p)}{p}\right]^{eff}_{m,m}}$$

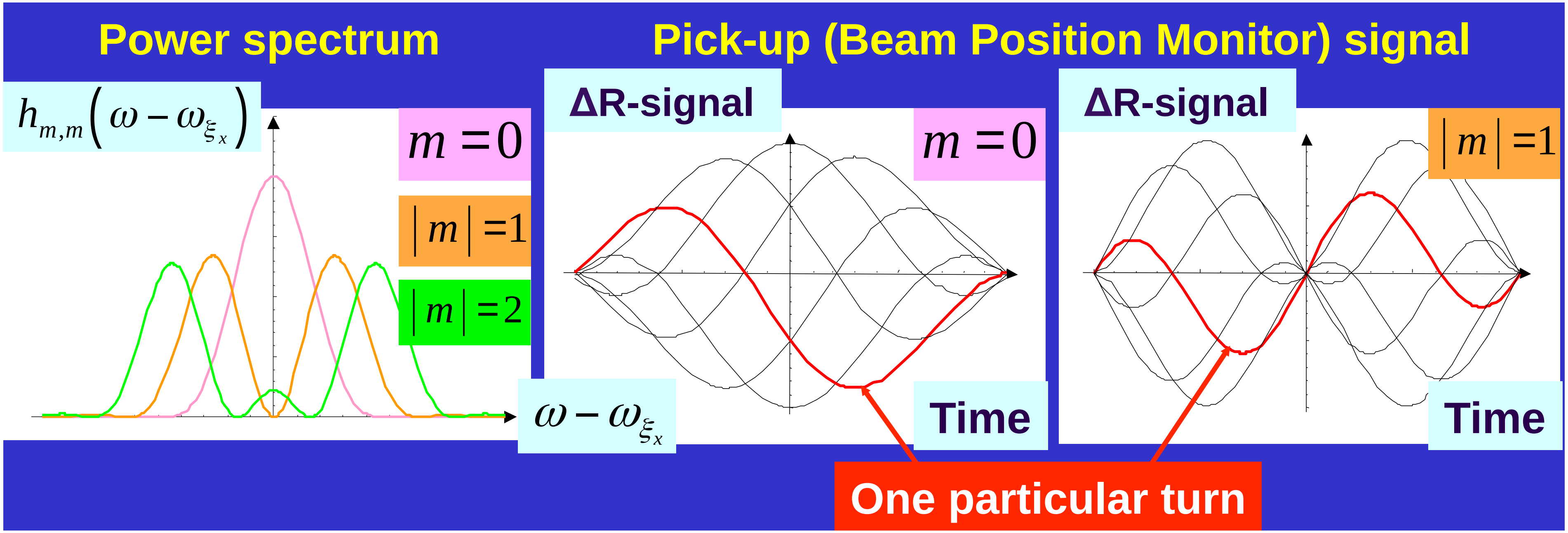
$$\left[\frac{Z_l(p)}{p}\right]^{eff}_{m,m} = \frac{\sum\limits_{p=-\infty}^{p=+\infty} \frac{Z_l\left(\omega^l_p\right)}{p} h_{m,m}\left(\omega^l_p\right)}{\sum\limits_{p=-\infty}^{p=+\infty} h_{m,m}\left(\omega^l_p\right)} \qquad \omega^l_p = p\,\Omega_0 + m\,\omega_s$$

  - **In transverse**

$$\boxed{\Delta\omega^{x,y}_{m,m} = \left(|m|+1\right)^{-1} \frac{j\, e\, \beta\, I_b}{4\pi R B m_0 \gamma Q_{x0,y0} \Omega_0} \left(Z^{eff}_{x,y}\right)_{m,m}}$$

$$\left(Z^{eff}_{x,y}\right)_{m,m} = \frac{\sum\limits_{p=-\infty}^{p=+\infty} Z_{x,y}\left(\omega^{x,y}_p\right) h_{m,m}\left(\omega^{x,y}_p - \omega_{\xi_{x,y}}\right)}{\sum\limits_{p=-\infty}^{p=+\infty} h_{m,m}\left(\omega^{x,y}_p - \omega_{\xi_{x,y}}\right)} \qquad \omega^{x,y}_p = \left(p + Q_{x0,y0}\right)\Omega_0 + m\,\omega_s$$

APPROXIMATE FORMULAE: SACHERER FORMULAE (4/4)



COUPLED-BUNCH INSTABILITIES: BOTH L & T

- In the case of ***M* equi-populated equi-spaced bunches**
  - *M* possible **coupled-bunch modes *n* (from 0 to *M* – 1)**
  - Mode *n* corresponds to a phase shift between 2 adjacent bunches of

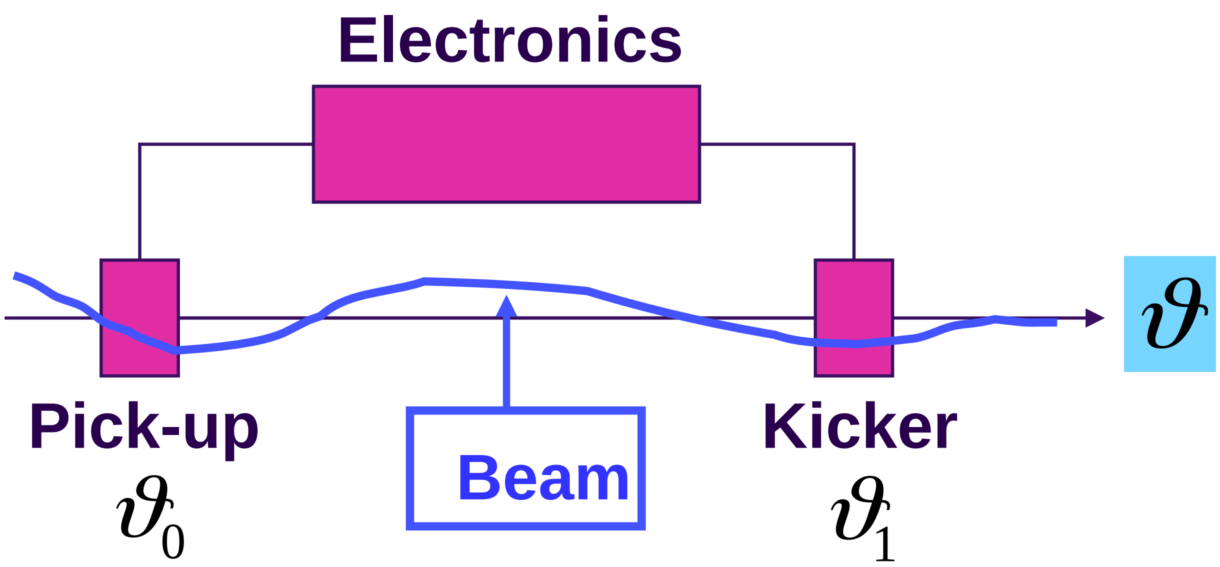
$$2\,\pi\,\frac{n}{M}$$

  - **The single-bunch eigenvalue is extended to the coupled-bunch regime by making the following modifications**
    - I_b => $M\,I_b$
    - l => $n + l\,M$
    - p => $n + p\,M$

- As concerns Sacherer formulae => Only change: sum over the coupled-bunch mode spectrum (see above, instead of the single-bunch spectrum)

MITIGATIONS (1/6)

◆ **Electronic dampers**

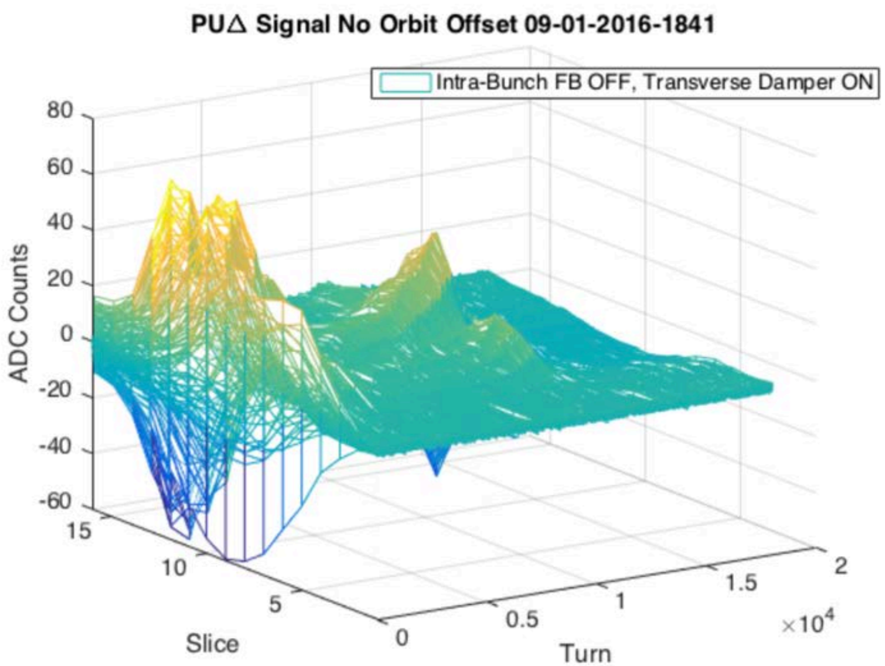


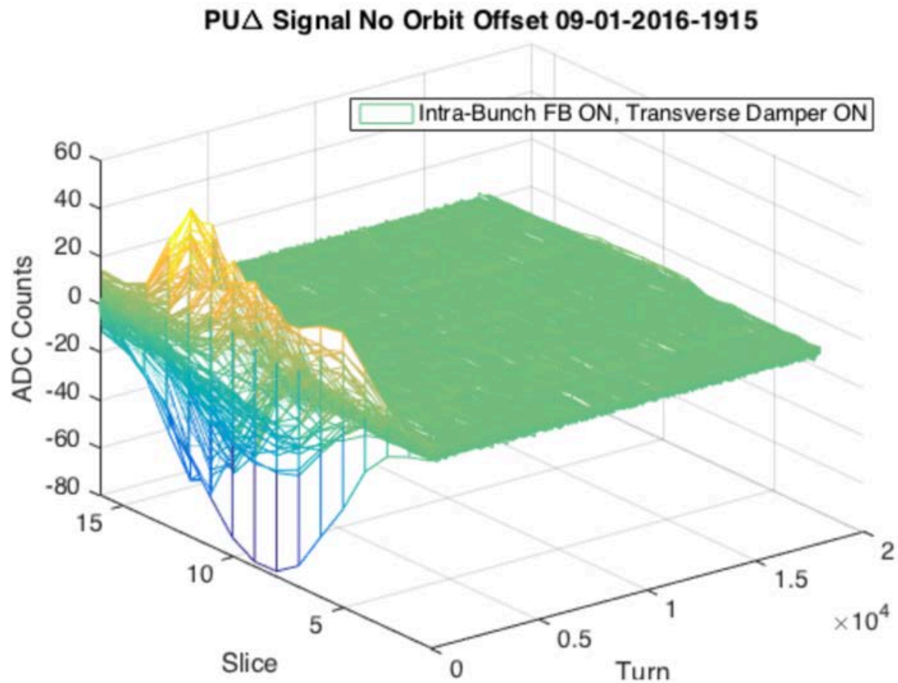
- Used for coupled-bunch instabilities (both L & T) and intra-bunch instabilities with long bunches => Work very well

- Not used yet for intra-bunch instabilities with short bunches => Bandwidth issue. Intense studies since several years to develop a transverse wide-band damper in SPS and promising results have been reached

MITIGATIONS (2/6)

- Example of studies in the SPS in 2016





Courtesy of J.D. Fox et al.

=> All these instabilities will / should be cured in the future with dampers!

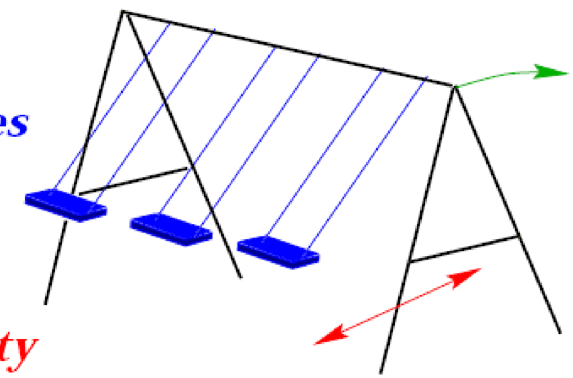
MITIGATIONS (3/6)

- **Landau damping** => Generate a (controlled) tune spread such that the coherent tune shift remains inside the spread (in fact **inside a stability diagram**)

Landau Damping

- ***Three Coupled Oscillators:***

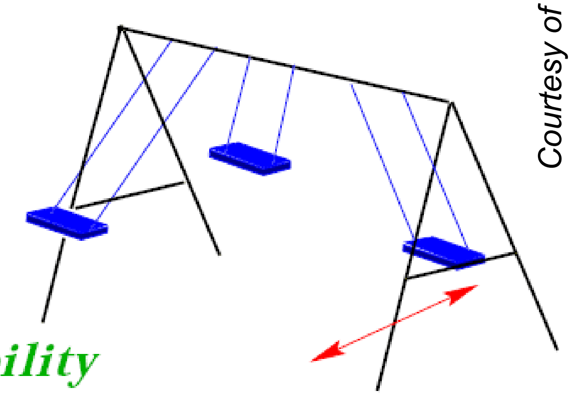
  equal frequencies



  → *instability*

- ***Three Coupled Oscillators:***

  different frequencies



  → *no instability*

- ***Limit:***

  frequency spread (tune spread)

  single particle resonances

Courtesy of O. Bruning

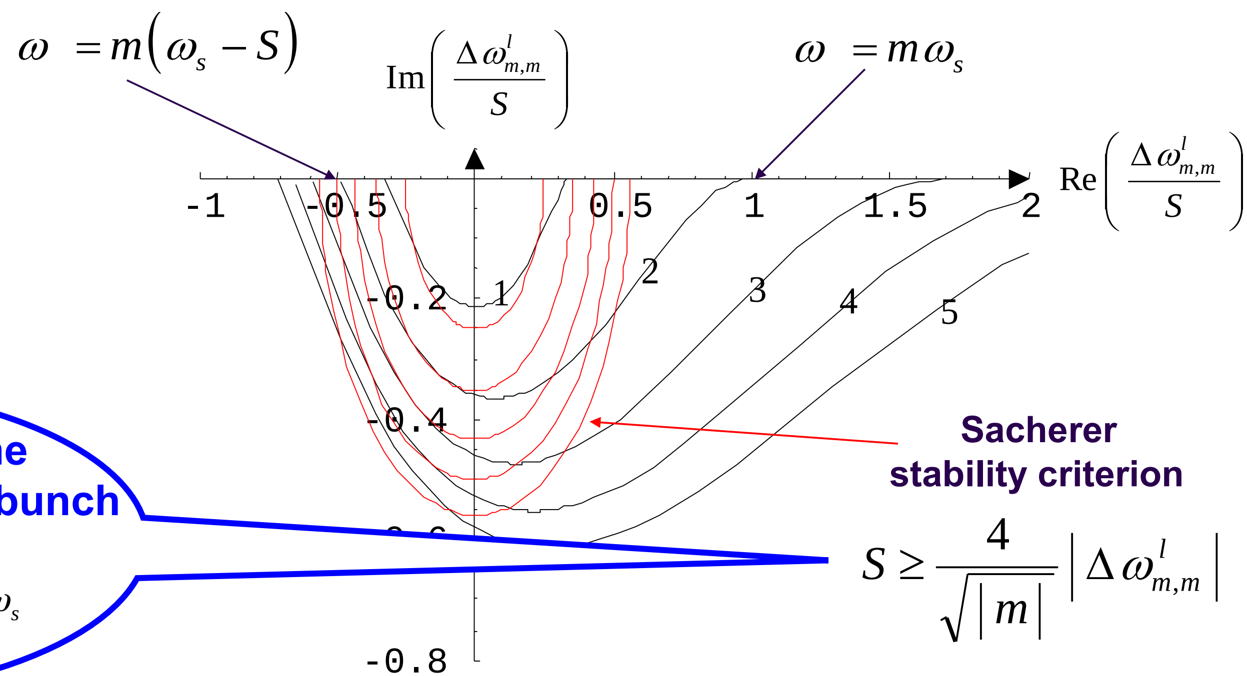
MITIGATIONS (4/6)

- **In longitudinal**: use the nonlinearity of the RF bucket => It can be shown that

$$\boxed{I_m^{-1}(\omega) = \Delta\omega_{m,m}^{l}}$$

$$I_m(\omega) = \frac{\displaystyle\int_0^{\infty} \frac{\hat{\tau}^{2m}}{\omega - m\,\omega_s(\hat{\tau})} \frac{d g_0(\hat{\tau})}{d\hat{\tau}} d\hat{\tau}}{\displaystyle\int_0^{\infty} \hat{\tau}^{2m} \frac{d g_0(\hat{\tau})}{d\hat{\tau}} d\hat{\tau}}$$

$$g_0(\hat{\tau}) \propto (1-\hat{\tau}^2)^2$$



Full spread between the centre and the edge of the bunch

$$S = \left(1 + \frac{5}{3}\tan^2\phi_s\right)\frac{\pi^2}{16}(h\,B)^2\,\omega_s$$

Sacherer stability criterion

$$S \geq \frac{4}{\sqrt{|m|}}\left|\Delta\omega_{m,m}^{l}\right|$$

MITIGATIONS (5/6)

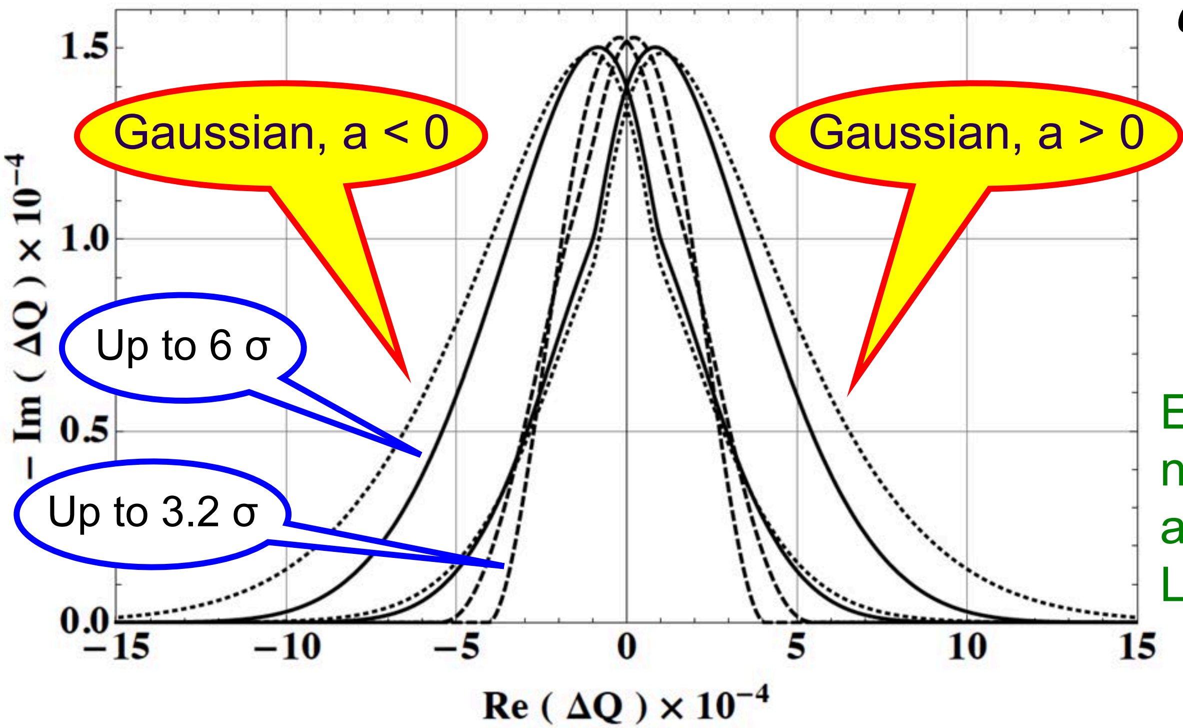
- **In transverse**: use controlled nonlinearities (e.g. Landau octupoles) => It can be shown that (e.g. in the horizontal plane)

$$\boxed{I_m^{-1}(\omega) = \Delta\omega_{m,m}^{x}}$$

$$I_m(\omega) = -\int_{J_x=0}^{+\infty} dJ_x \int_{J_y=0}^{+\infty} dJ_y \frac{J_x \dfrac{\partial f(J_x, J_y)}{\partial J_x}}{\omega - \omega_x(J_x, J_y) - m\,\omega_s}$$

$$\omega_x(J_x, J_y) = \omega_0 + a\,J_x + b\,J_y$$

Transverse actions

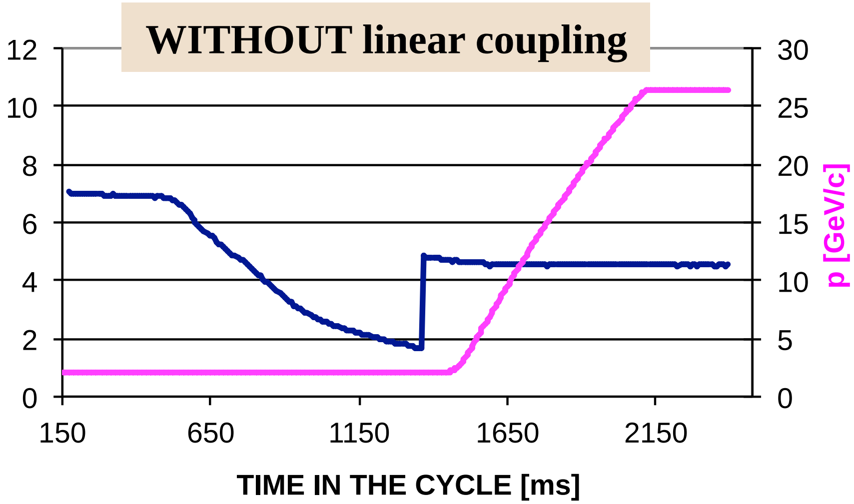


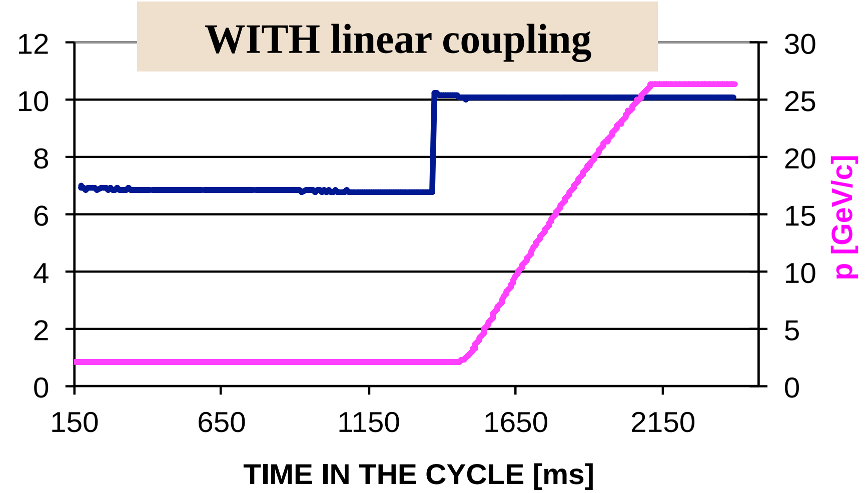
Example of the LHC at 7 TeV with nominal transverse emittance and maximum current in the Landau octupoles

MITIGATIONS (6/6)

- **Linear coupling**
  - Can have a beneficial effect if asymmetries between the 2 transverse planes (different impedances, chromaticities, etc.) => Sharing of the instability growth rates and frequency spreads
  - Detrimental effect in case there is nothing to gain from one plane (two identical planes) and coupling is too strong => Loss of Landau damping (coherent tune outside of tune spread)
  - **Stabilization of the PS low-energy instability by linear coupling**

INTENSITY [10^{12} protons/pulse]





TRANSVERSE:
HIGH-INTENSITY

- Reminder: general equation of coherent motion considering the contributions from all the modes *m*

$$j\,\Delta\omega_{cm}^{x}\,\hat{x}_m(\hat{\tau}) = -\frac{e\,I_b}{2\,\gamma\,m_0\,c\,Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p)\,\sigma_x(p)\,j^m\,J_{m,x}(p,\hat{\tau})$$

- Multiplying both sides by $j^{-m}\,J_{m,x}(l,\hat{\tau})\,g_0(\hat{\tau})\,\hat{\tau}$ and integrating over $\hat{\tau}$

$$\Rightarrow \qquad \Delta\omega_{cm}^{x}\,\sigma_{x,m}(l) = \sum_{p=-\infty}^{p=+\infty} K_{lp}^{x,m}\,\sigma_x(p)$$

- Dividing both sides by $\Delta\omega_{cm}^{x}$ and summing over *m*

$$\boxed{\sigma_x(l) = \varepsilon_x \sum_{p=-\infty}^{p=+\infty} \left[\,j\,Z_x(p)\right] M_{lp}^{x}\,\sigma_x(p)}$$

with

$$\boxed{M_{lp}^{x} = 2\,B \sum_{m} \frac{1}{\frac{\omega_c}{\omega_s} - m} \int_{0}^{1} J_{m,\,x}\left(l,\ \frac{\tau_b}{2}\,u \right) J_{m,\,x}\left(p,\ \frac{\tau_b}{2}\,u \right) u\,du}$$

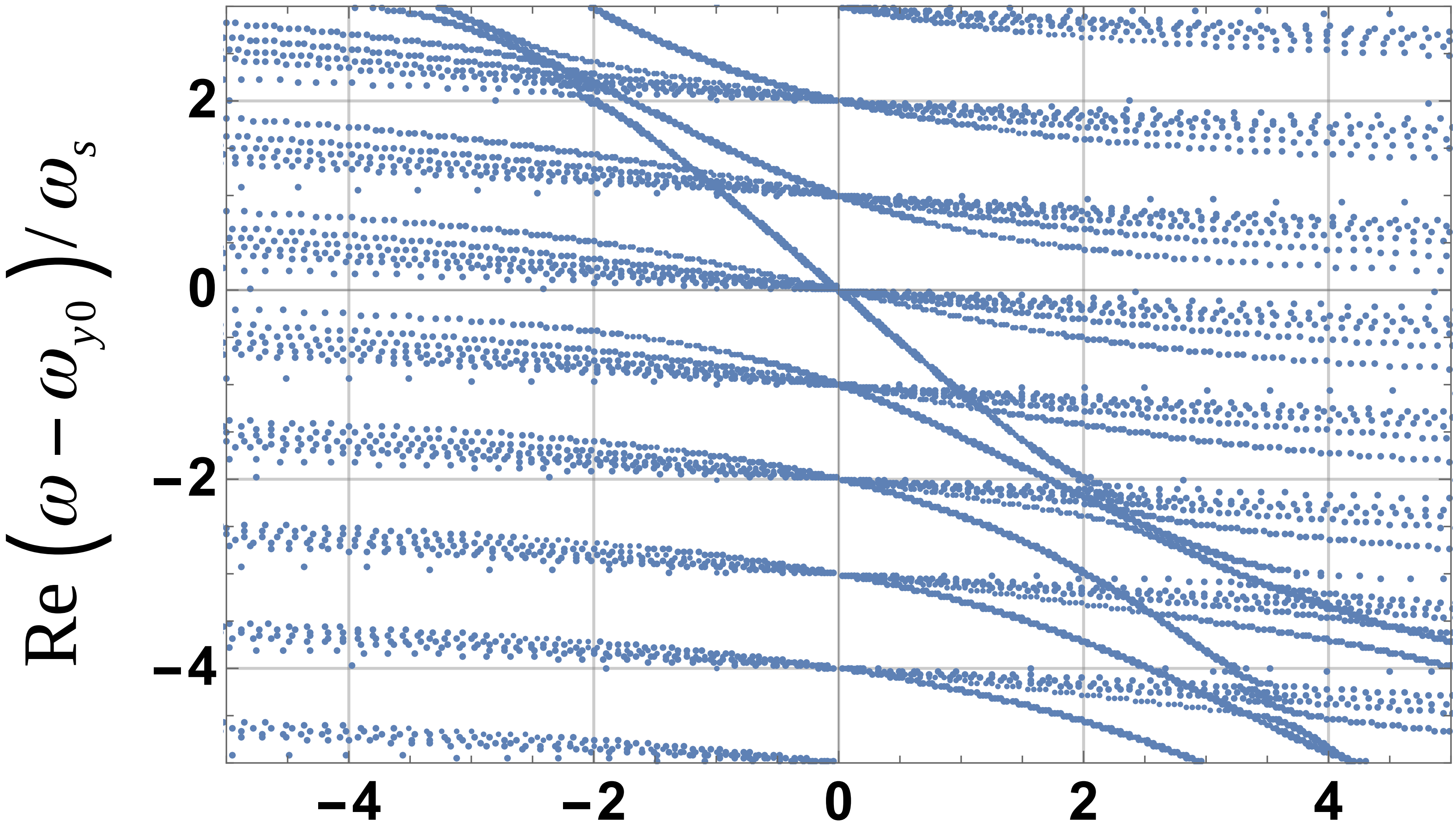
(assuming a water-bag for the stationary distribution, as before)

and

$$\boxed{\varepsilon_x = \frac{e\,I_b}{4\,\pi\,\gamma\,m_0\,c\,Q_{x0}\,B\,\omega_s}}$$

- **Method to solve this equation**
  - Assume a real coherent betatron frequency shift measured in incoherent synchrotron frequency unit $\omega_c / \omega_s = \left(\omega - \omega_{x0}\right) / \omega_s$
  - Look for the eigenvalues of the matrix $\left[\, j\,Z_x\left(p\right) M_{lp}^{x} \,\right]$
  - Scale the intensity parameter ε_x in order to adjust the eigenvalue to unity

◆ Case of a **constant (vertical) inductive impedance**



Also noted sometimes – j Z ε later

$$\boxed{x = -\, j\, Z_y\left(0\right)\varepsilon_y}$$

$$x = \frac{\mathrm{Im}[\,Z_y(0)]\; e\, I_b}{4\,\pi\,\gamma\, m_0 c\, Q_{y0} B\, \omega_s}$$

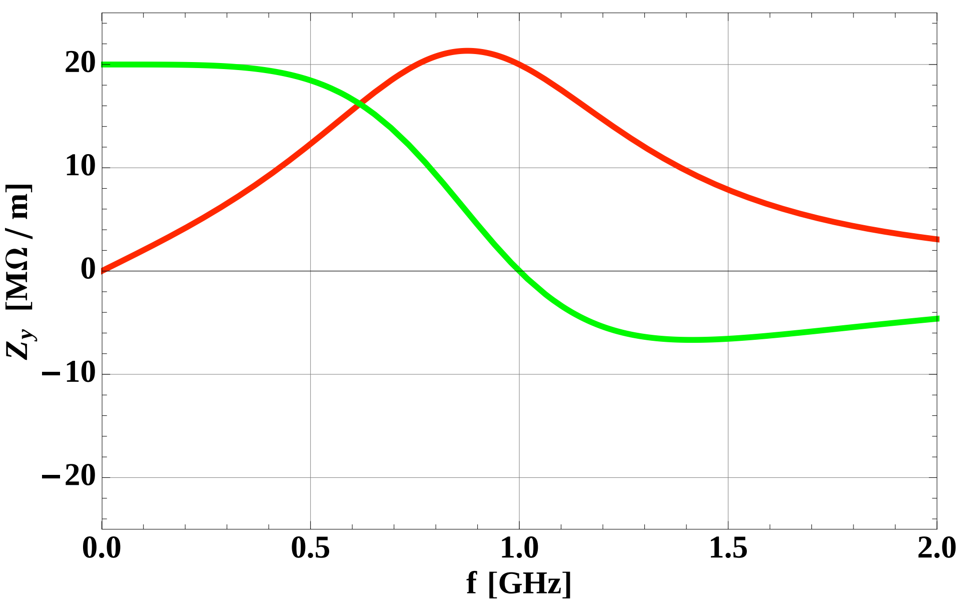
◆ Case of a **Broad-Band resonator impedance**

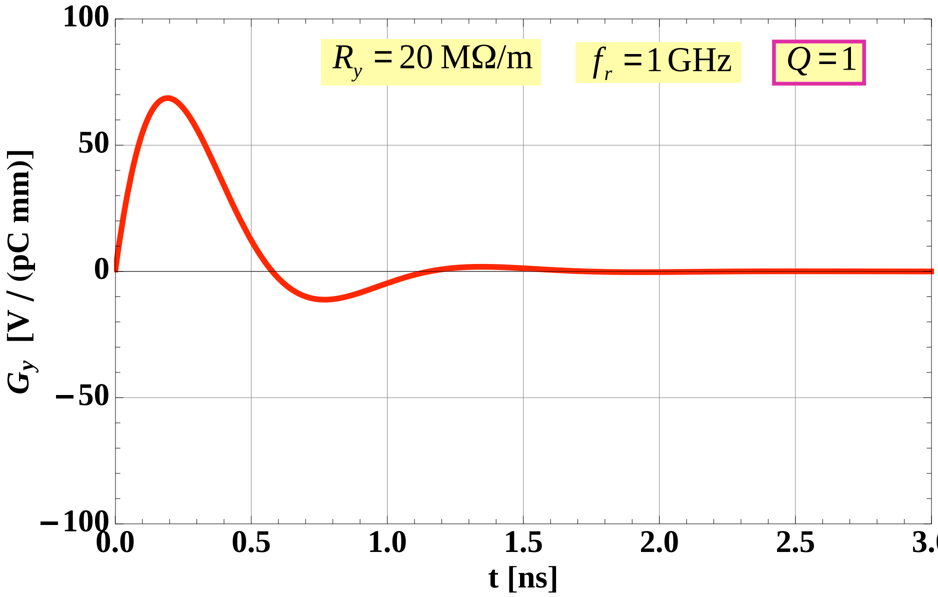
$$\bar{\omega}_r = \omega_r \sqrt{1 - \frac{1}{4\,Q^2}}$$

$$\alpha = \frac{\omega_r}{2\,Q}$$

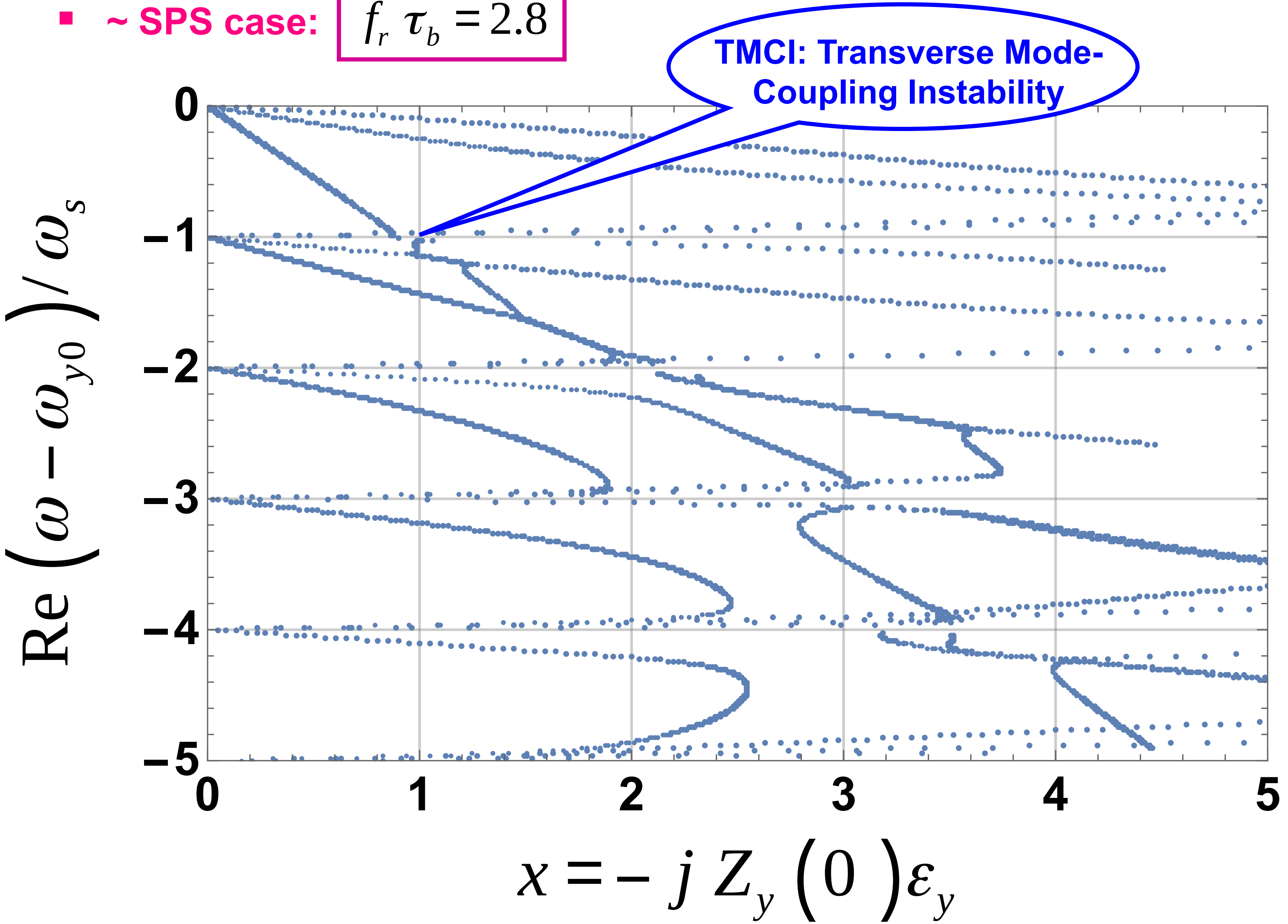
$$\boxed{Z_y(\omega) = \frac{\omega_r}{\omega} \frac{R_y}{1 + j\,Q\left(\dfrac{\omega}{\omega_r} - \dfrac{\omega_r}{\omega}\right)}}$$

$$\boxed{G_y(t) = \frac{\omega_r^2\,R_y}{Q\,\bar{\omega}_r}\,e^{-\alpha t}\,\sin\left(\bar{\omega}_r\,t\right)}$$





- ~ SPS case: $\boxed{f_r \, \tau_b = 2.8}$



- Another possibility to solve this problem is to **use a decomposition on the low-intensity eigenvectors (as proposed by Garnier in 1987)**

- Using this formalism, the effect of a transverse damper was recently added

- **Remark: 2 other codes (Vlasov solvers) including the transverse damper were developed in the recent years**
    - **A. Burov developed a Nested Head-Tail Vlasov Solver (NHTVS) with transverse damper in 2014**
    - **N. Mounet solved Sacherer integral equation with transverse damper, using a decomposition over Laguerre polynomials of the radial functions (DELPHI code, 2015)**

** Sacherer integral equation was also solved using a decomposition over Laguerre polynomials of the radial functions by Besnier in 1974 and Y.H. Chin in 1985 in the code **MOSES***

Without transverse damper

◆ Damping time of a transverse damper

- The damper gain G is defined by $G = \dfrac{\Delta \vartheta}{x} \beta_x$, where $\Delta \vartheta$ is the change of the slope produced by a measured displacement x , assuming the same $\beta_x - \text{value}$ at the PU and the kicker. After one turn, the displacement has been corrected by $\Delta x = \beta_x \, \Delta \vartheta = G \, x$

=> $\dfrac{dx}{dt} = \dfrac{G\,x}{T_0} = G\, f_0 \, x$ => Damping time: $\tau_{damper} = \dfrac{1}{G\, f_0}$

- Averaging over all the possible betatron phases at the PU position (as the tune cannot be an integer):

$$\boxed{\tau_{damper} = \frac{2}{G\, f_0} = \frac{n_d}{f_0}}$$

Damping time in # of turns

- Decomposition on the low-intensity modes (following Garnier1987) + adding a (perfect) transverse damper => GALACTIC

$$\sigma_x(l) = \sum_{i,j} a_{ij}\, \sigma_{x,ij}(l)$$

$$\frac{\omega_c}{\omega_s} a_{kl} = H^x\, a_{ij}$$

$$H^x_{kl,ij} = k\, \delta_{ki}\, \delta_{lj} + \Delta\omega_{ckl} \sum_{p=-\infty}^{p=+\infty} \sigma^*_{x,kl}(p) \left[-j\, \frac{Z_x(p)}{\left|Z_x(0)\right|} + \delta_{p0}\, \frac{F_{damper}\, f_0}{n_d\, \omega_s\, (-x\, 2\pi\, \omega_s\, B)} \right] \sigma_{x,ij}(p)$$

Complex conjugate

- F_{damper} = 0 => No damper
- F_{damper} = + j => Resistive damper
- F_{damper} = + 1 => Reactive damper
- n_d = Damper damping time (in # of turns)

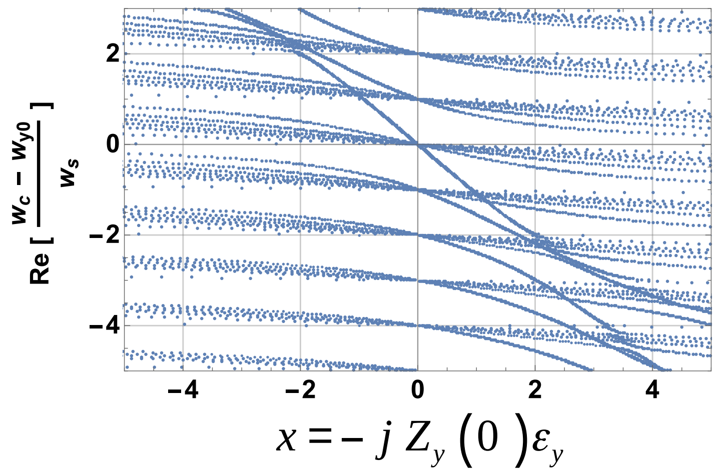
Eigenvalue solution of the low-intensity eigenvalue problem with constant inductive impedance

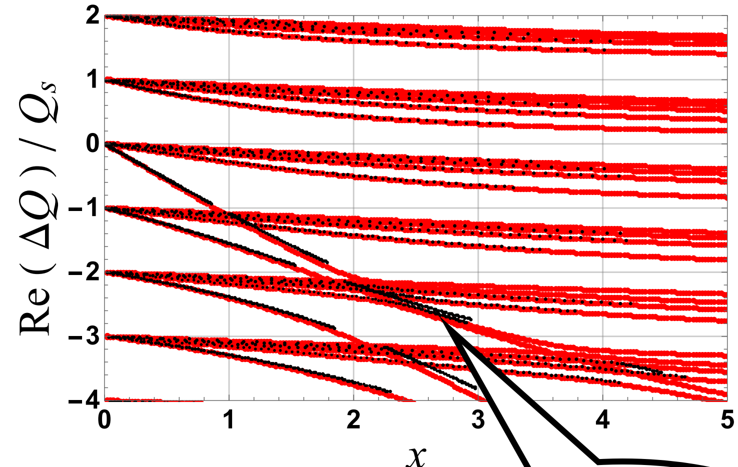
Eigenvector solution of the low-intensity eigenvalue problem with constant inductive impedance

- Check between the 2 methods => **1) Constant inductive impedance**

Laclare1987: Eigenvalue problem without decomposition (without damper)

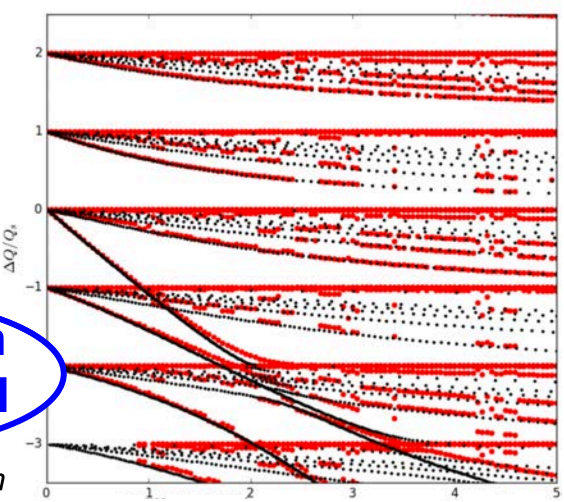
Decomposition on the low-intensity Eigenvectors following Garnier1987 formalism (without damper)



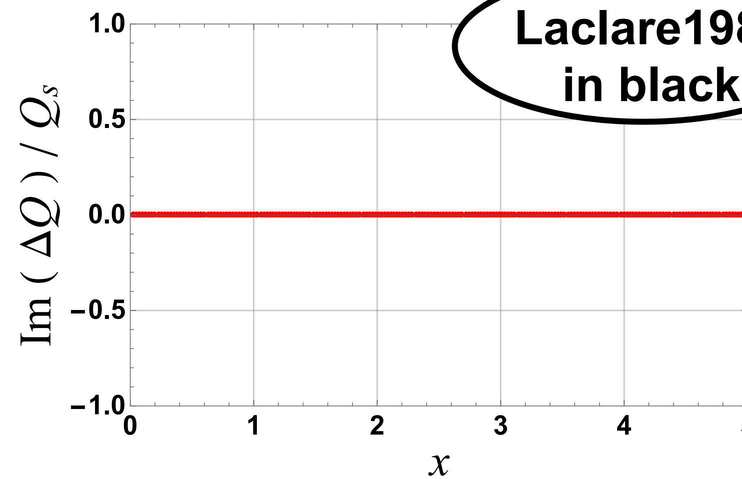


Red: DELPHI results, uniform distribution

Comparison with DELPHI



Courtesy of D. Amorim



Laclare1987 in black

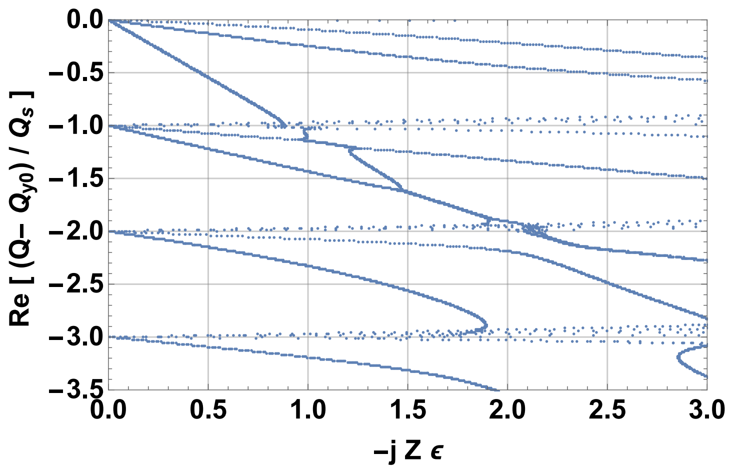
See also IPAC19 paper (https://ipac2019.vrws.de/papers/mopgw087.pdf)

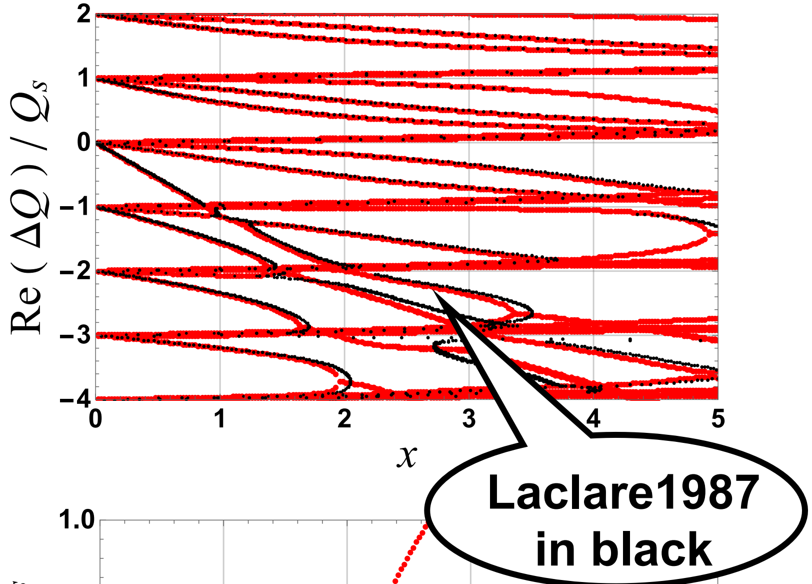
2) Broad-Band impedance $f_r \, \tau_b = 2.8$

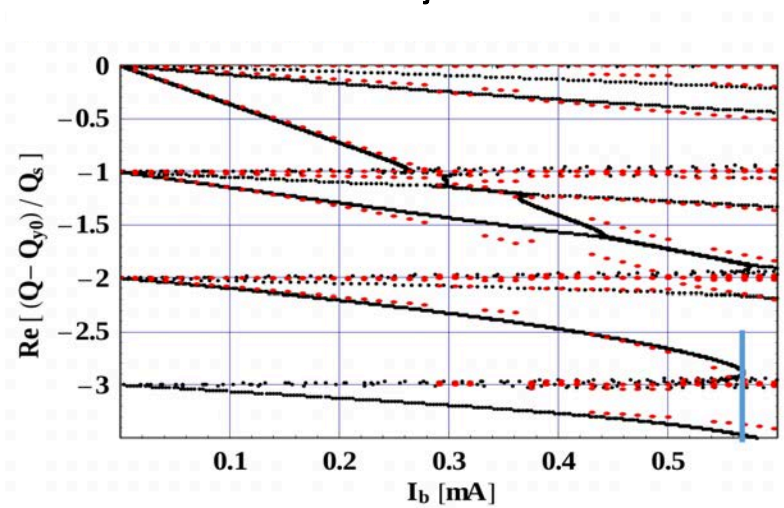
Q' = 0

Laclare1987: Eigenvalue problem without decomposition (without damper)

Decomposition on the low-intensity Eigenvectors following Garnier1987 formalism (without damper)

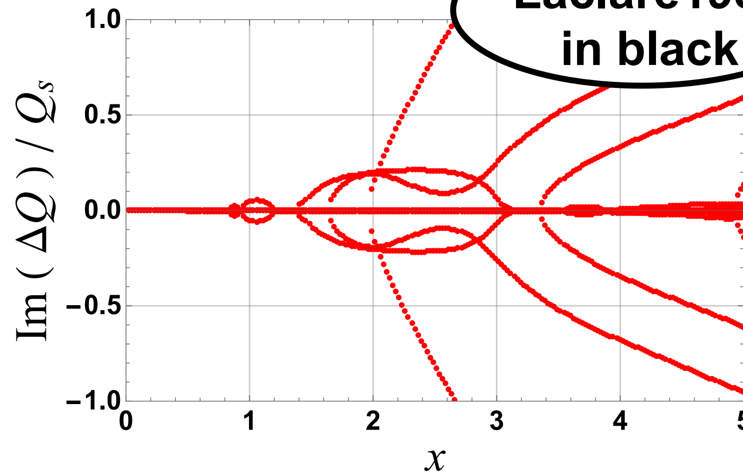






Black: Laclare's problem
Red: DELPHI

Courtesy of D. Amorim



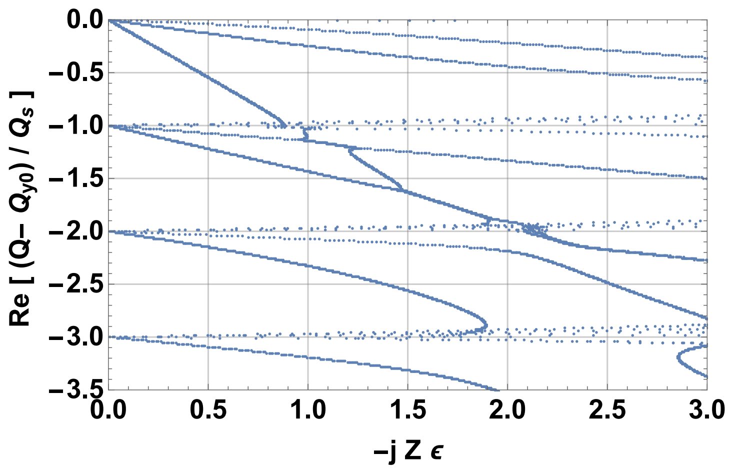
See also IPAC19 paper (https://ipac2019.vrws.de/papers/mopgw087.pdf)

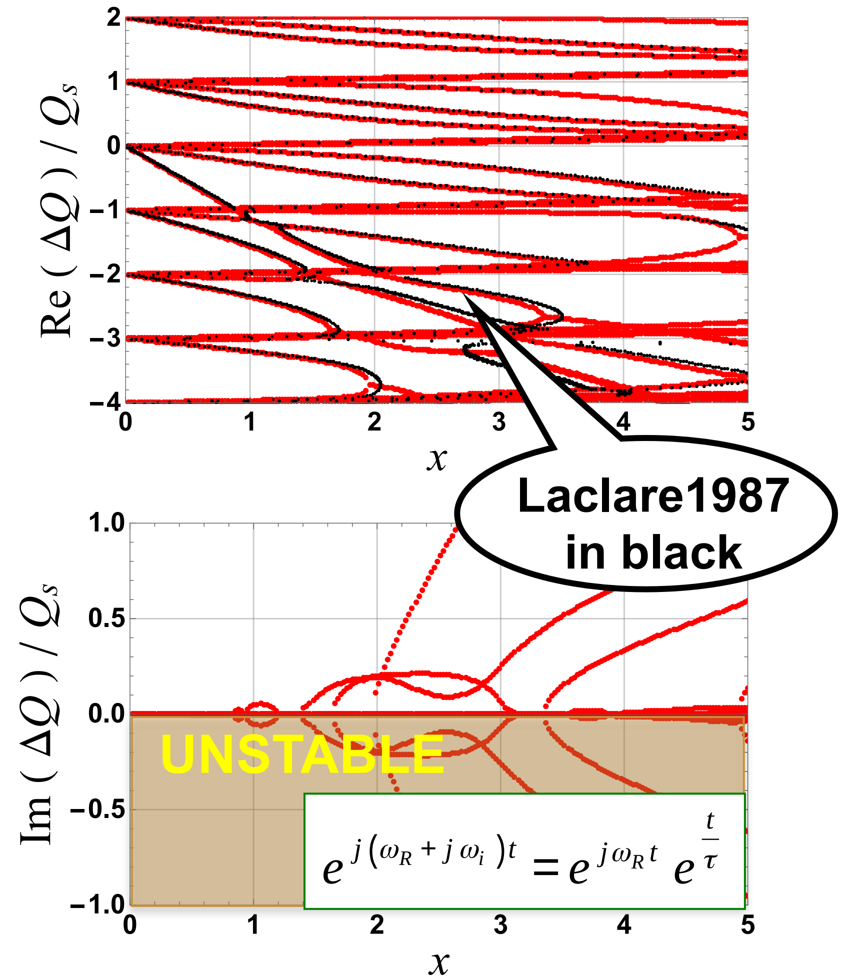
2) Broad-Band impedance $f_r \tau_b = 2.8$

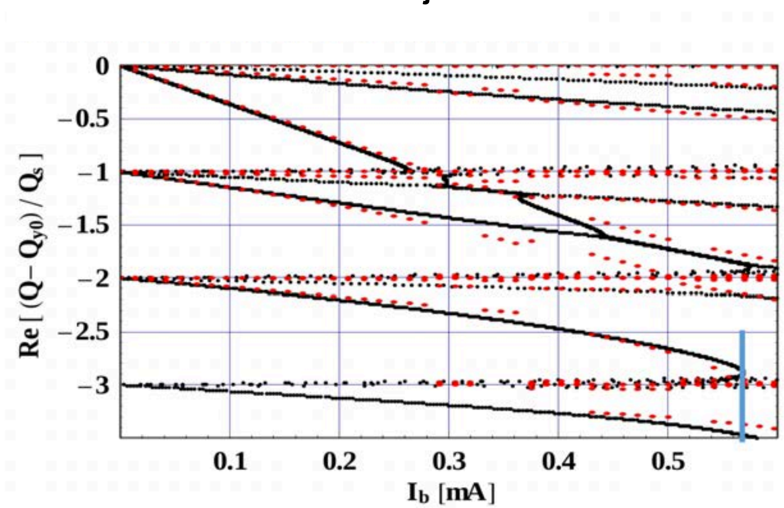
Q' = 0

Laclare1987:
Eigenvalue problem without decomposition (without damper)

Decomposition on the low-intensity Eigenvectors following Garnier1987 formalism (without damper)





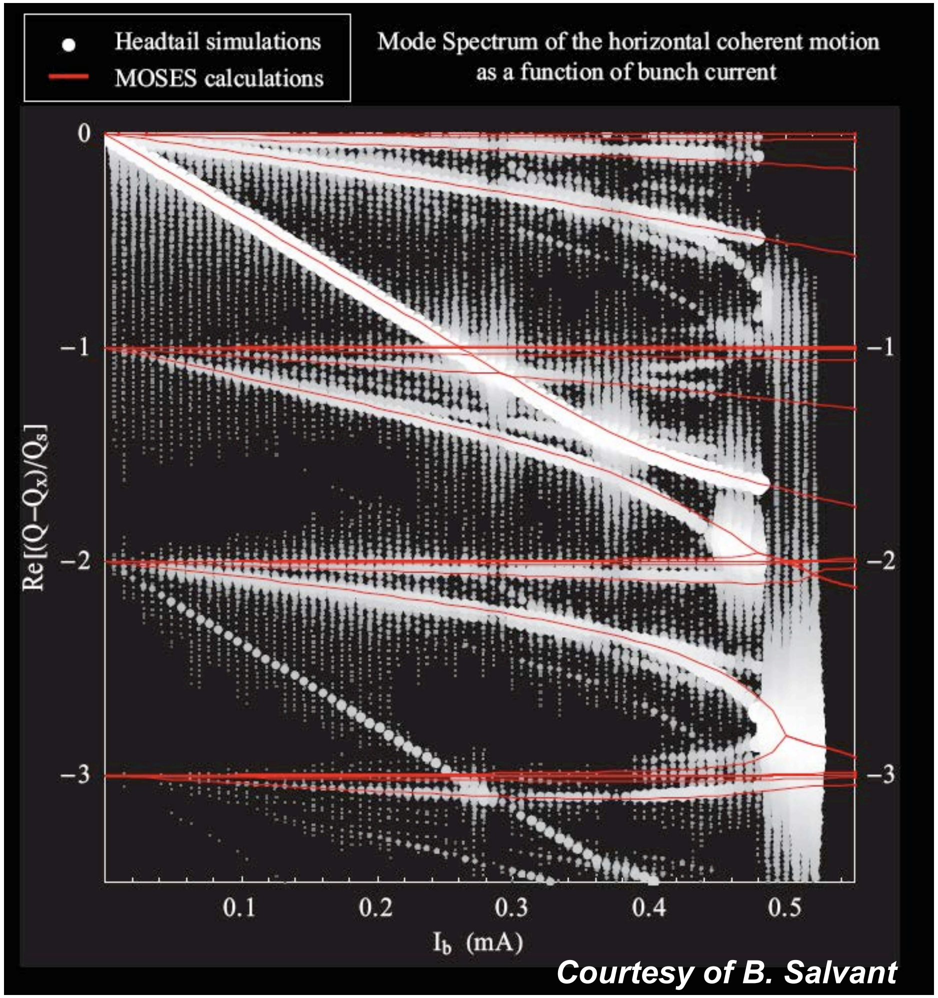


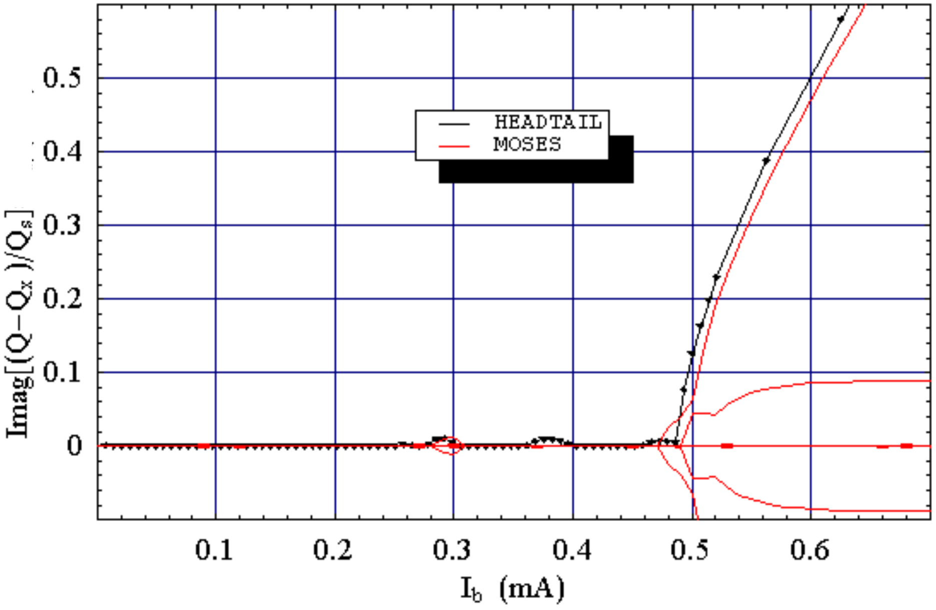
Black: Laclare's problem
Red: DELPHI

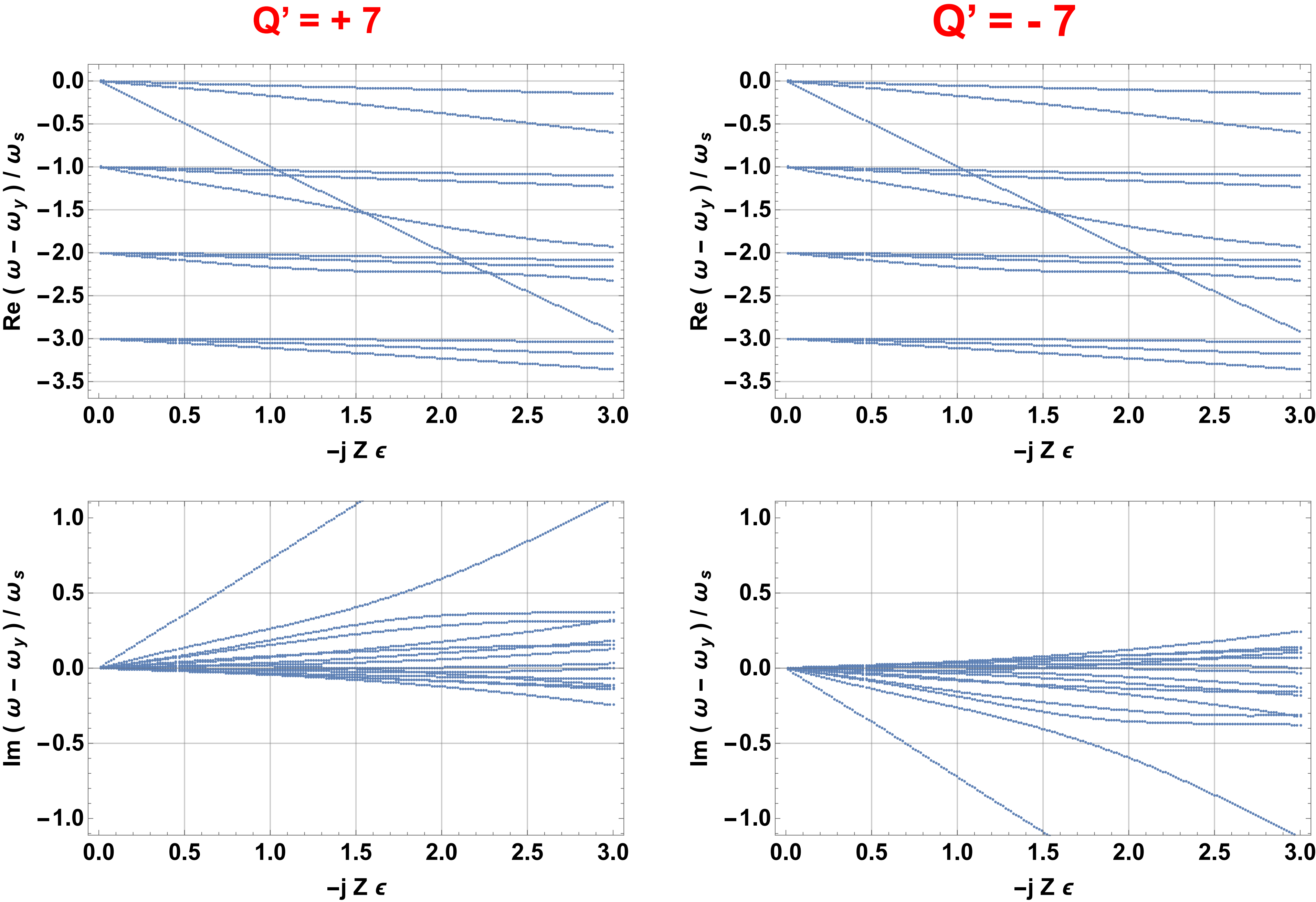
Courtesy of D. Amorim

See also IPAC19 paper (https://ipac2019.vrws.de/papers/mopgw087.pdf)

Past comparison between MOSES and HEADTAIL simulations
(for a Gaussian longitudinal distribution)





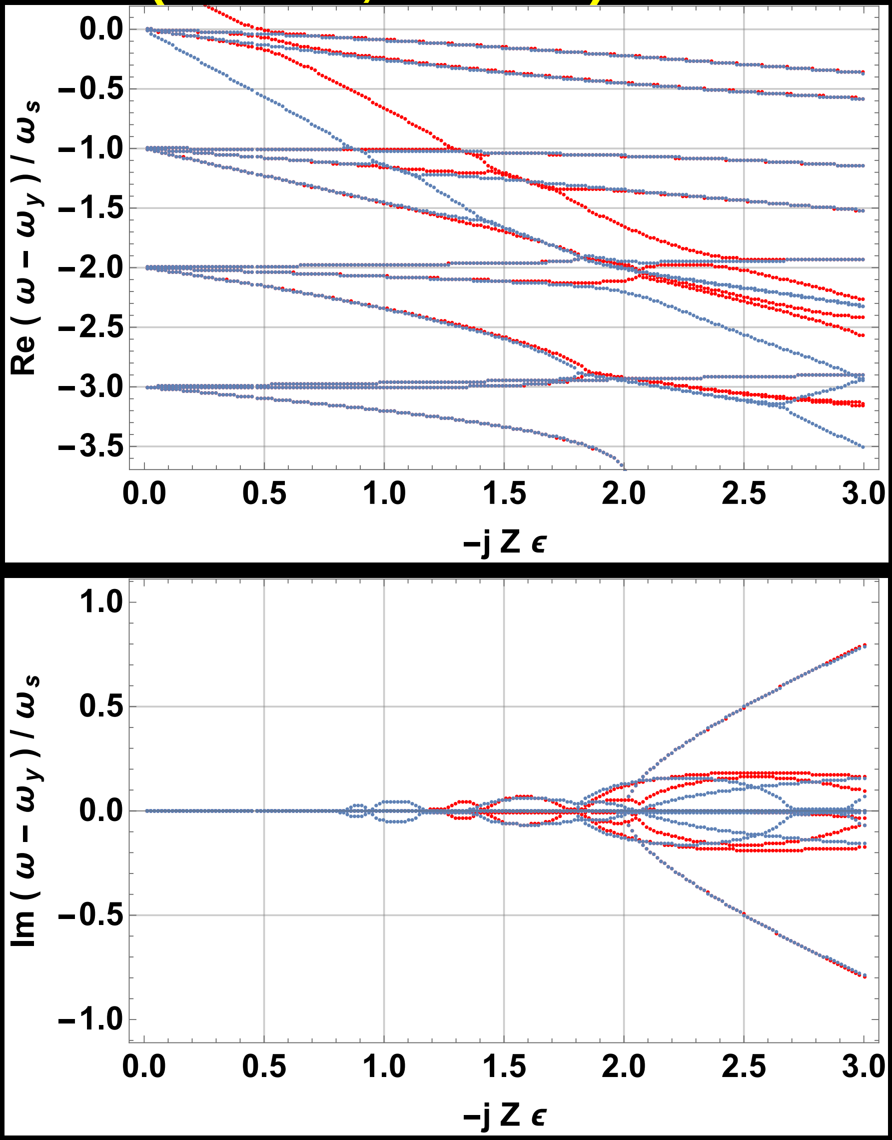


Q' = + 7
Q' = - 7
Re (ω − ω_y) / ω_s
Im (ω − ω_y) / ω_s
−j Z ϵ

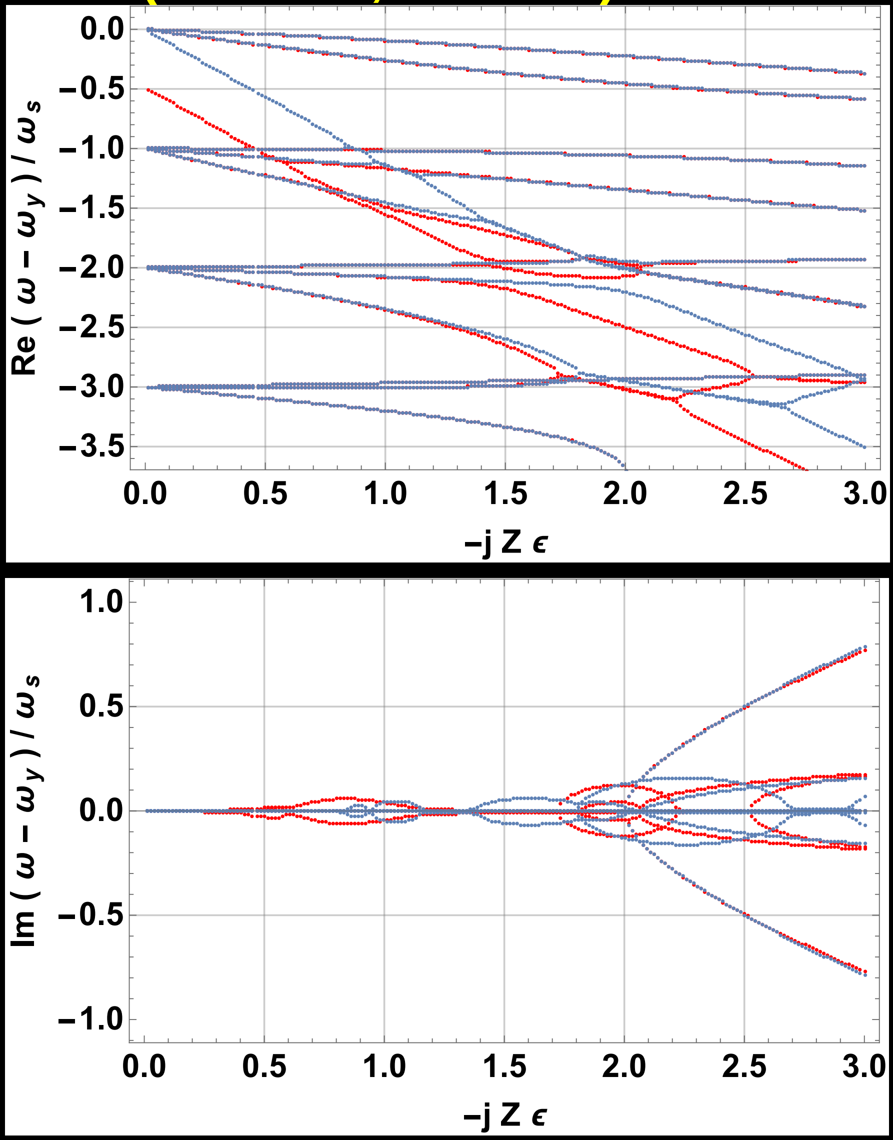
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE OF THE (BROAD-BAND) SPS

Q’ = 0

With transverse damper (reactive, 50 turns) in red



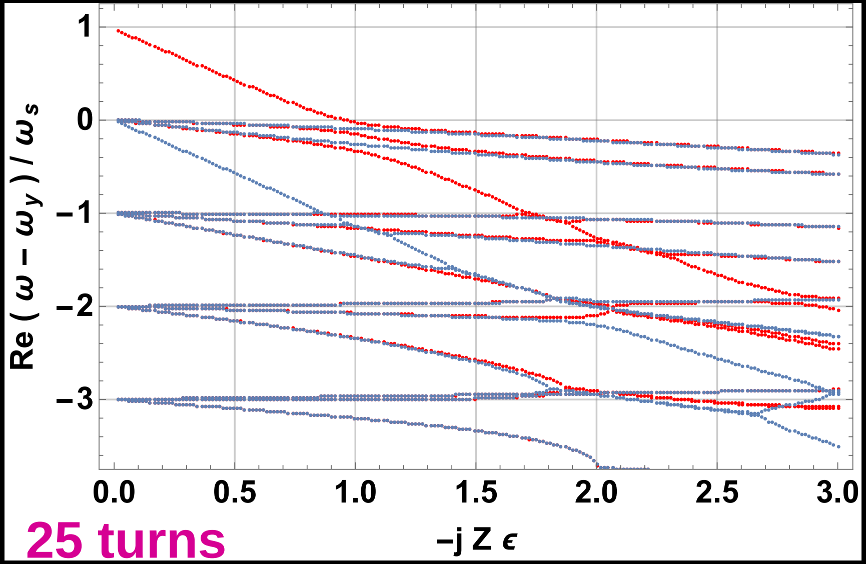
With transverse damper (-reactive, 50 turns) in red

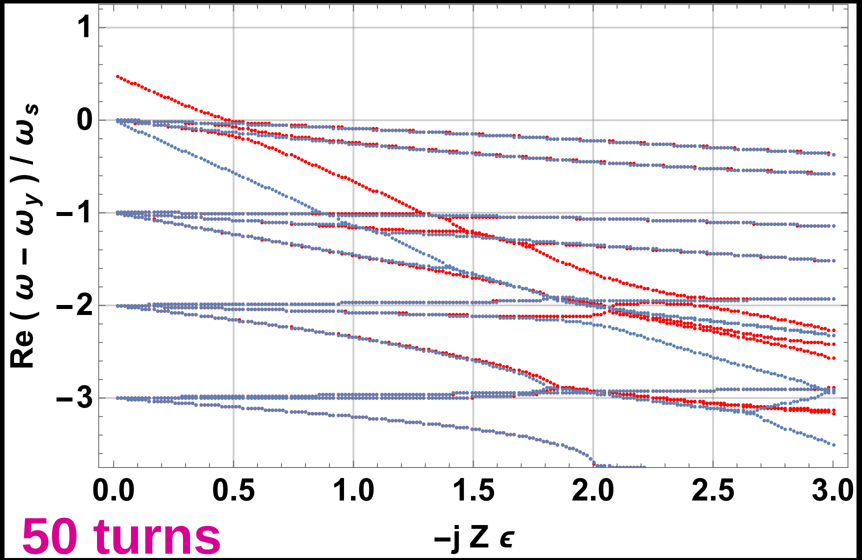


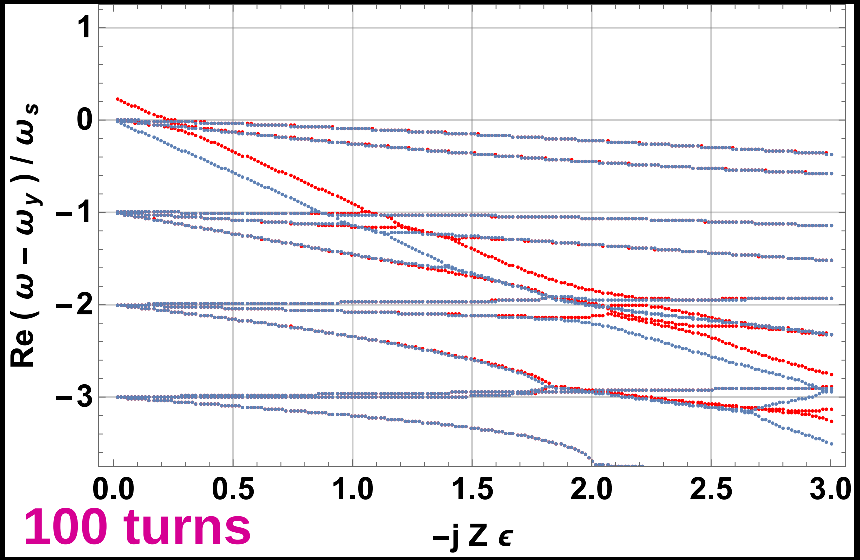
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE OF THE (BROAD-BAND) SPS

Q’ = 0

With transverse damper (reactive) in red: 25, 50 and 100 turns



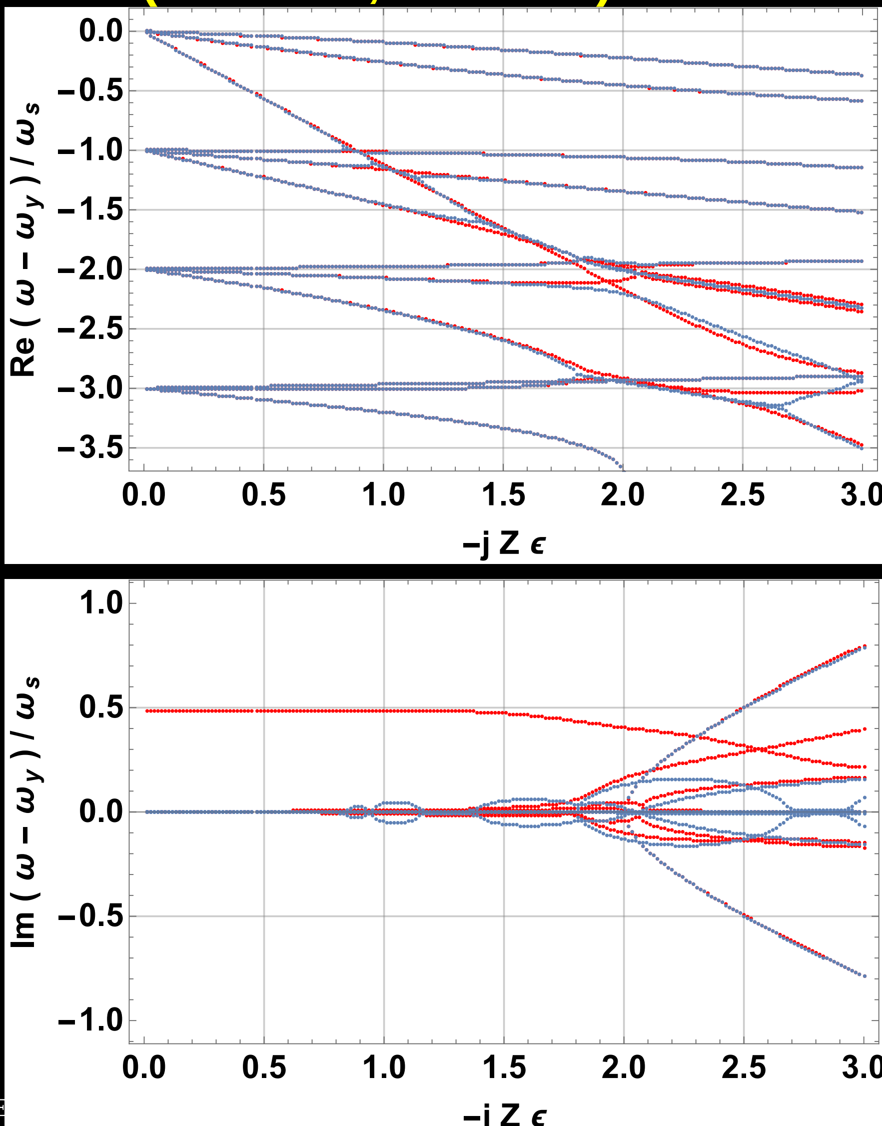




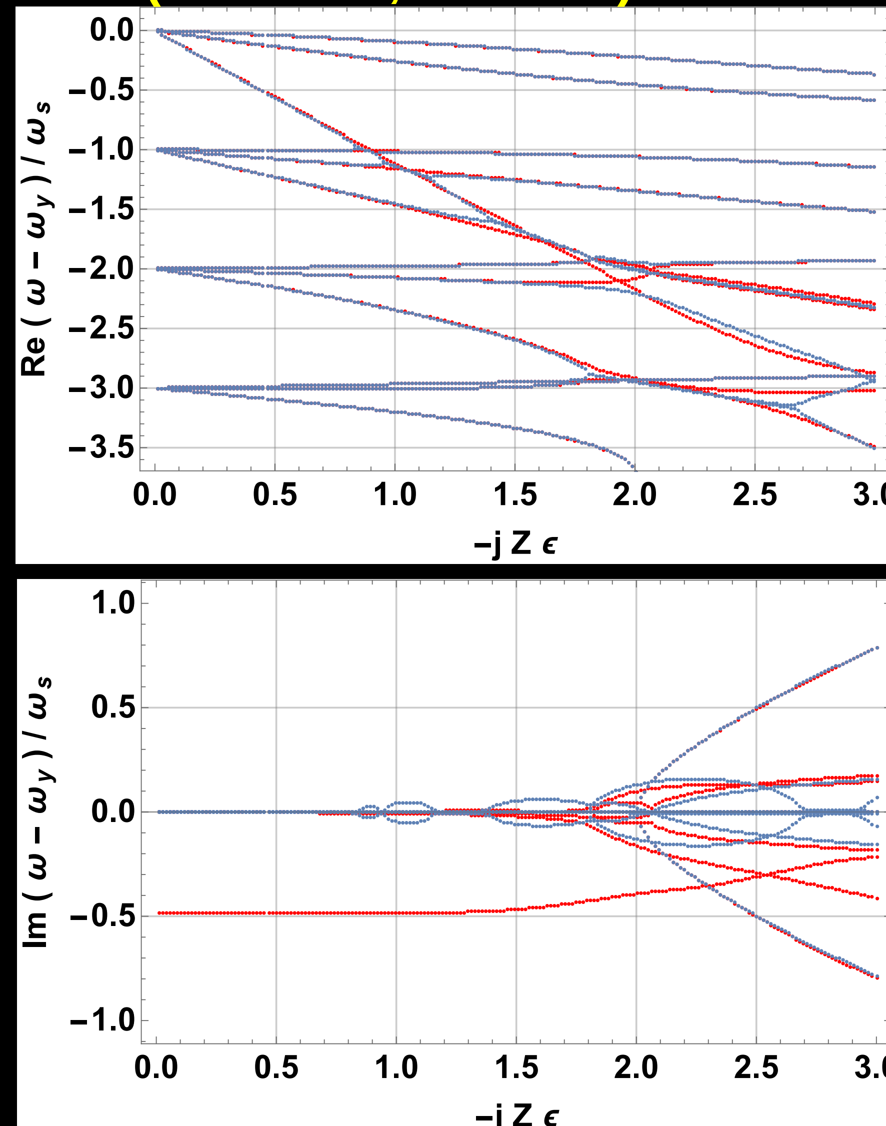
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE OF THE (BROAD-BAND) SPS

Q’ = 0

With transverse damper (resistive, 50 turns) in red



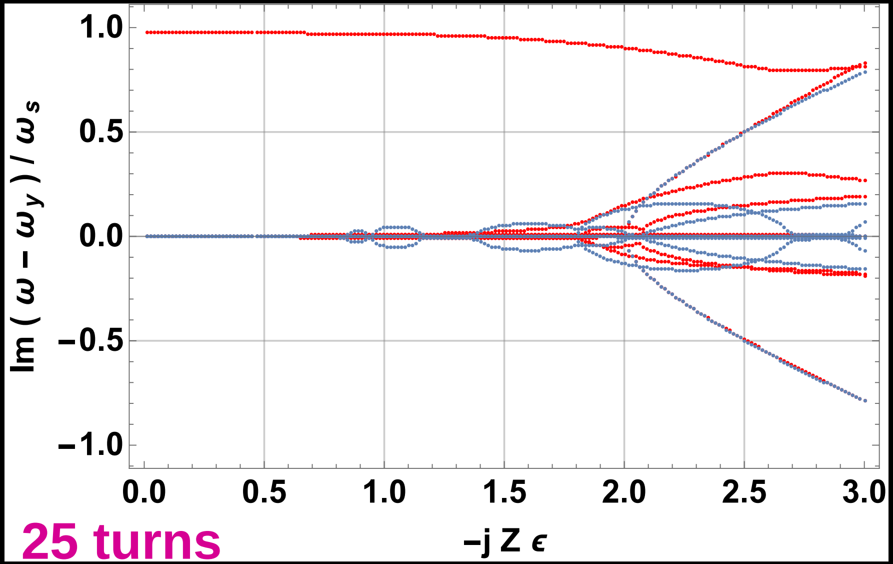
With transverse damper (-resistive, 50 turns) in red

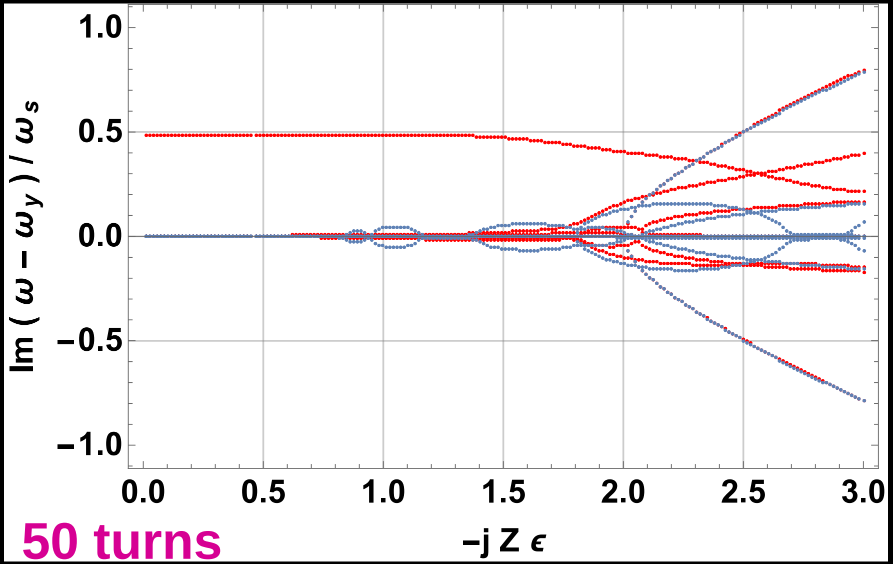


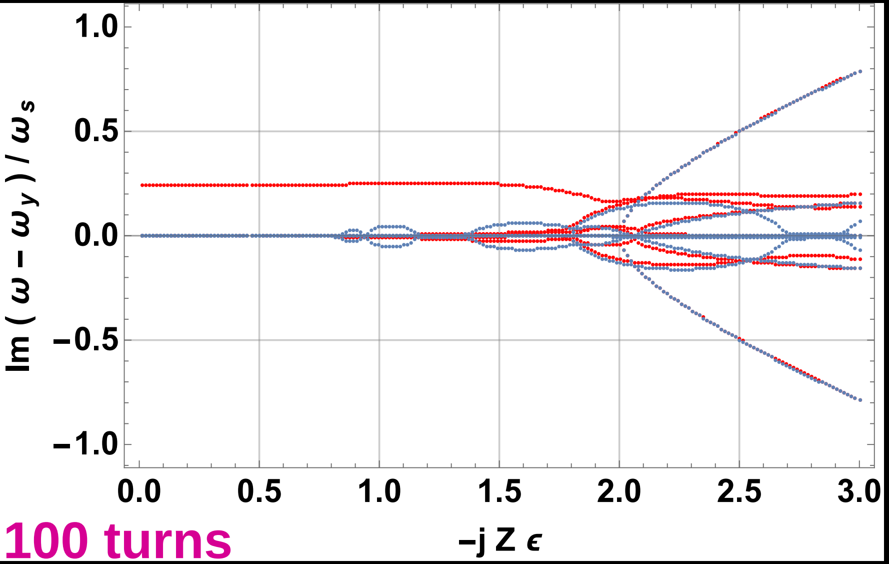
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE OF THE (BROAD-BAND) SPS

Q' = 0

With transverse damper (resistive) in red: 25, 50 and 100 turns

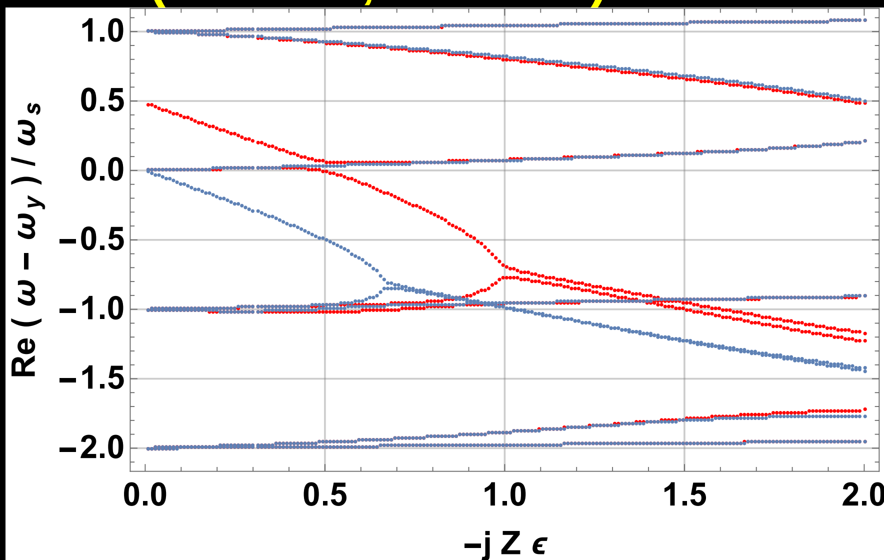


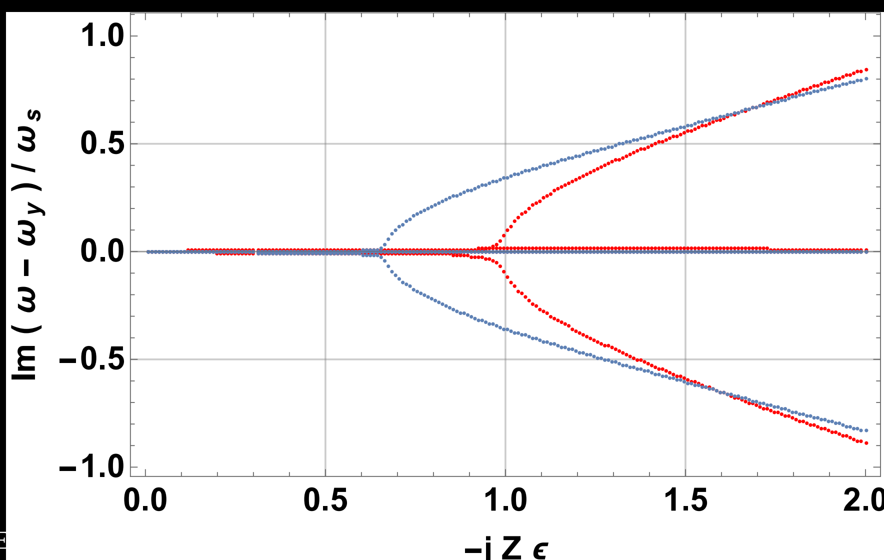




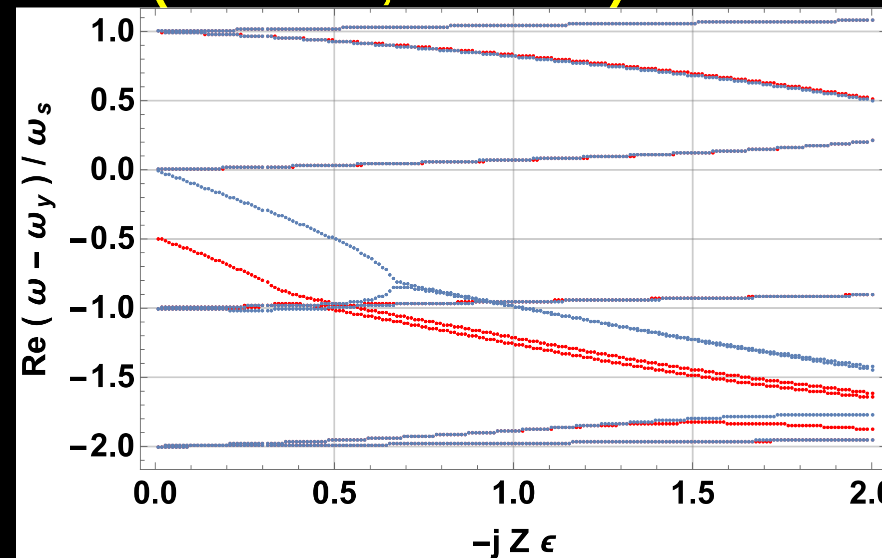
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE WITH fr × taub = 0.8 (instead of 2.8 before) Q' = 0

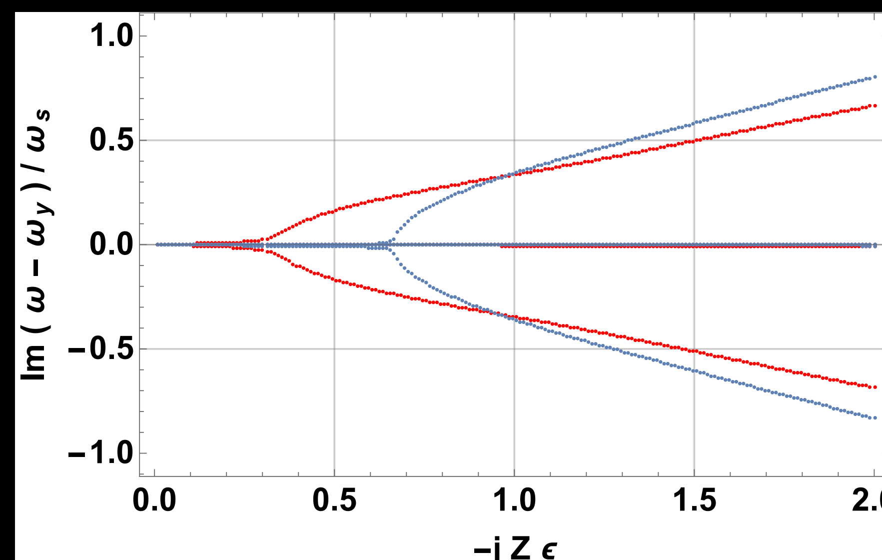
With transverse damper (reactive, 50 turns) in red





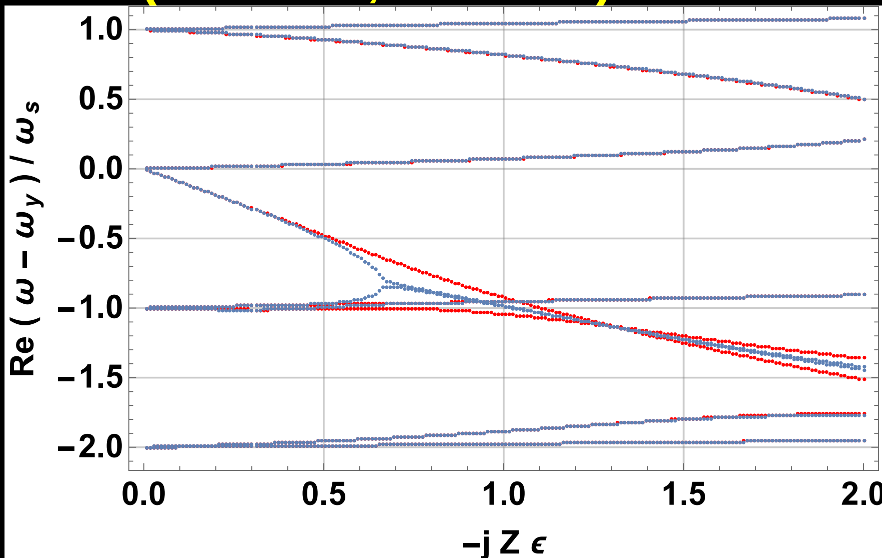
With transverse damper (-reactive, 50 turns) in red

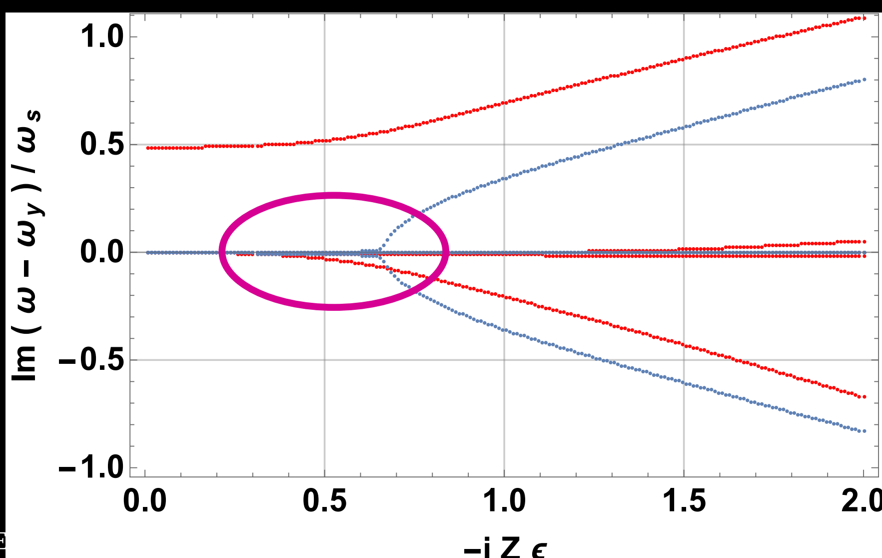




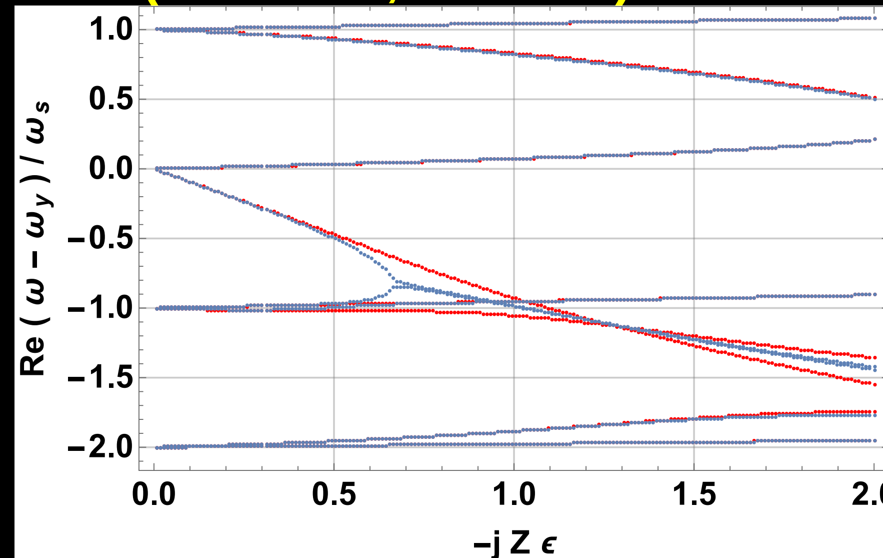
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE WITH fr × taub = 0.8 (instead of 2.8 before) Q’ = 0

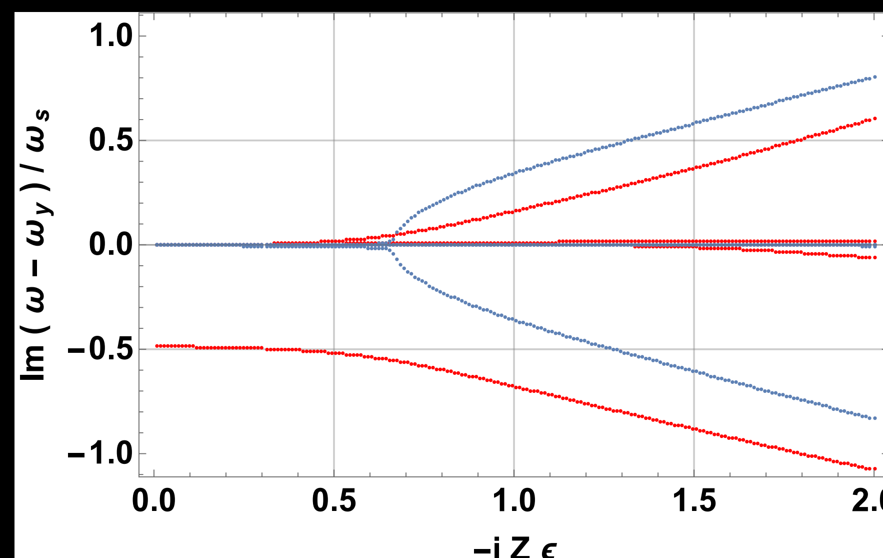
With transverse damper (resistive, 50 turns) in red





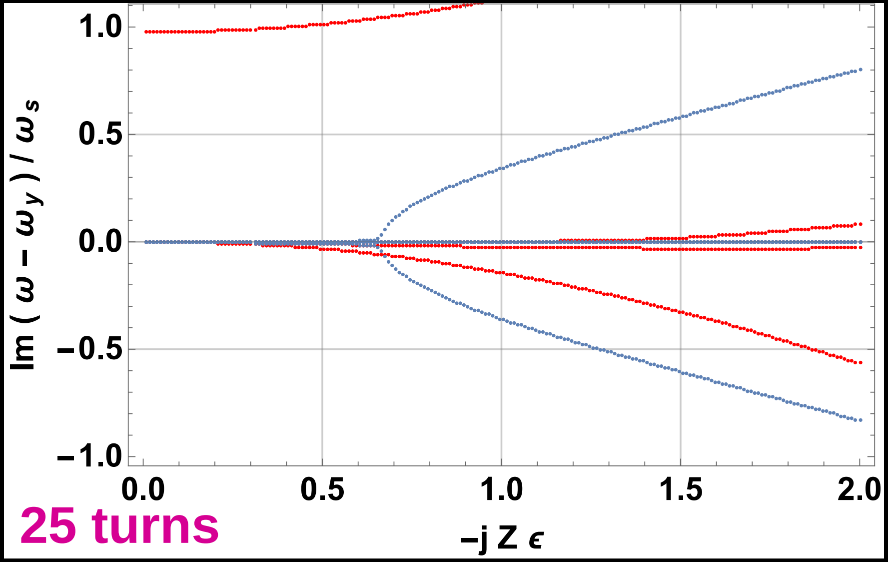
With transverse damper (-resistive, 50 turns) in red

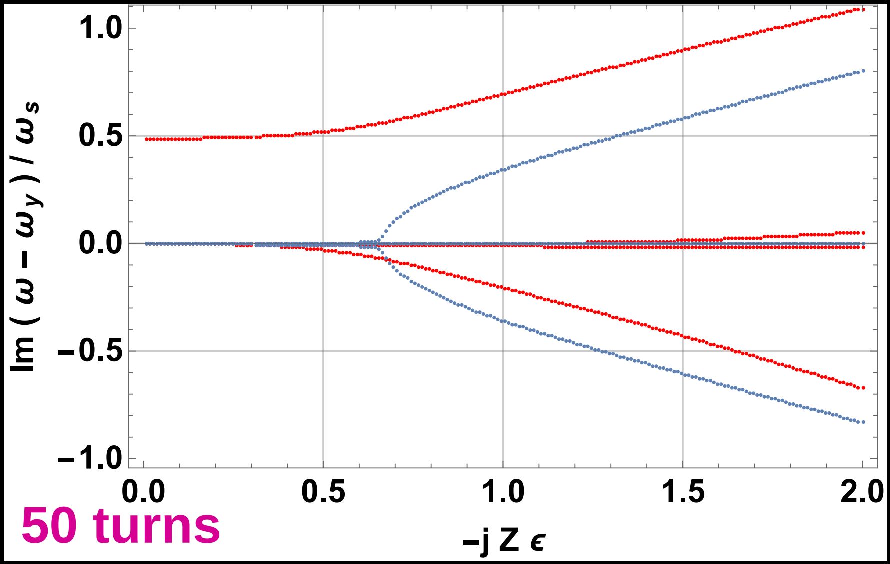


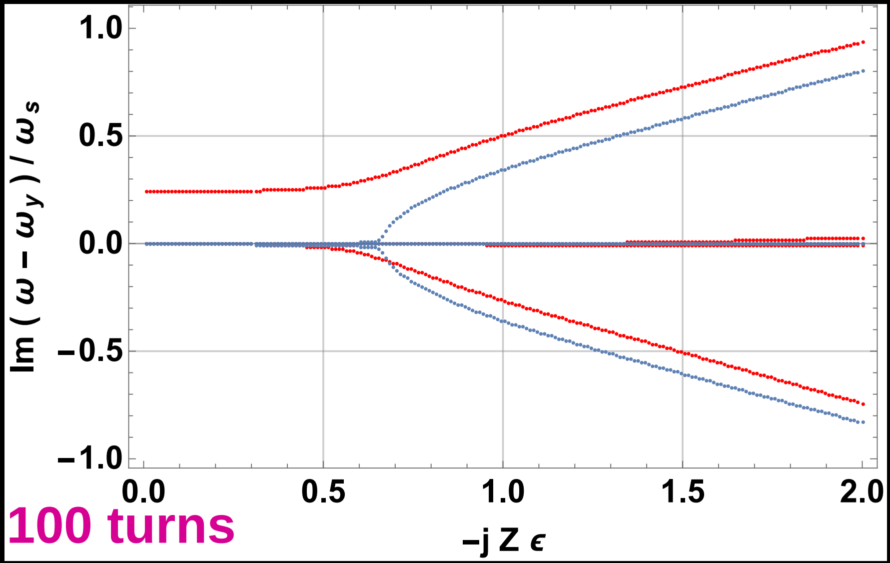


RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE WITH fr × taub = 0.8 (instead of 2.8 before) Q' = 0

With transverse damper (resistive) in red: 25, 50 and 100 turns

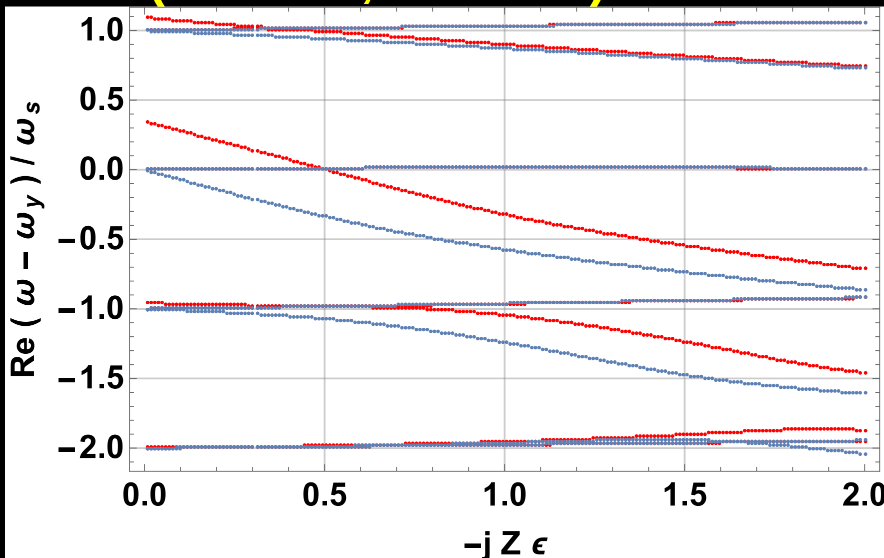


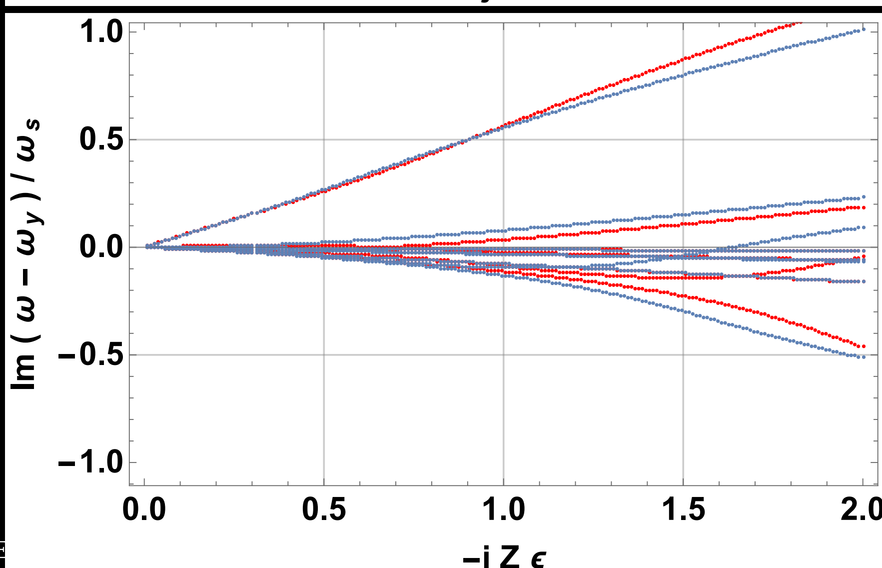




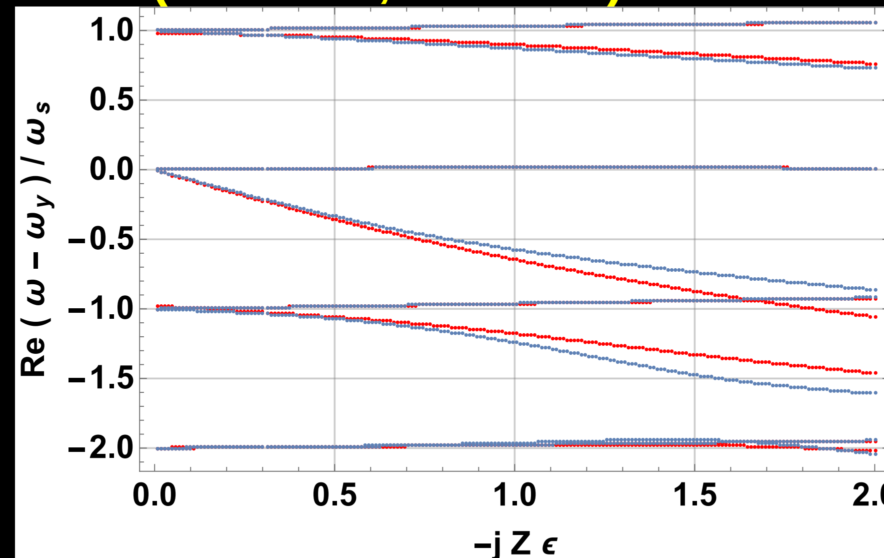
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE WITH fr × taub = 0.8 (instead of 2.8 before) & Q' = + 7

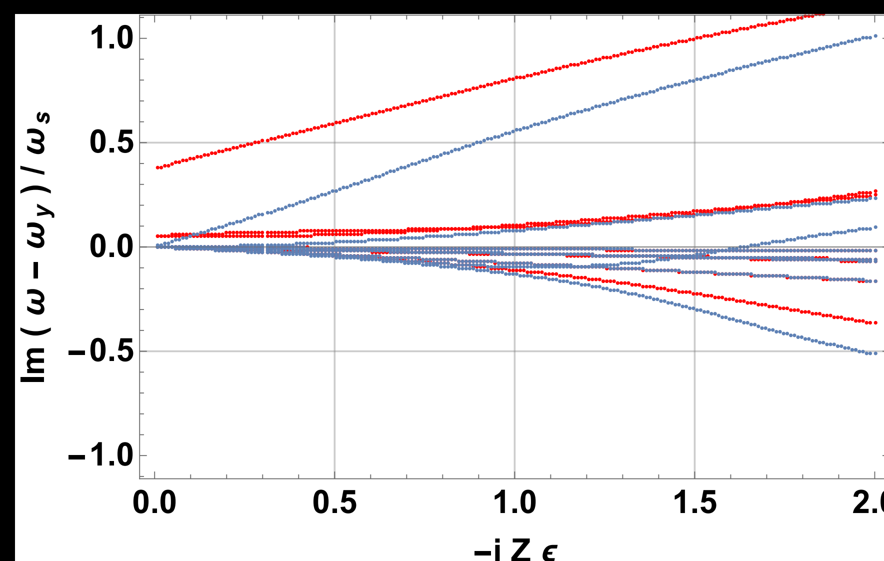
With transverse damper (reactive, 50 turns) in red





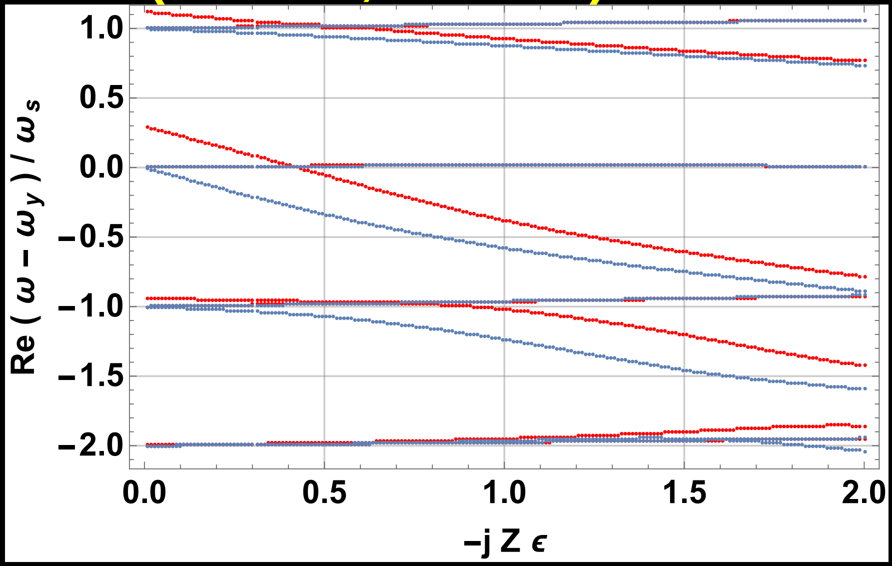
With transverse damper (resistive, 50 turns) in red

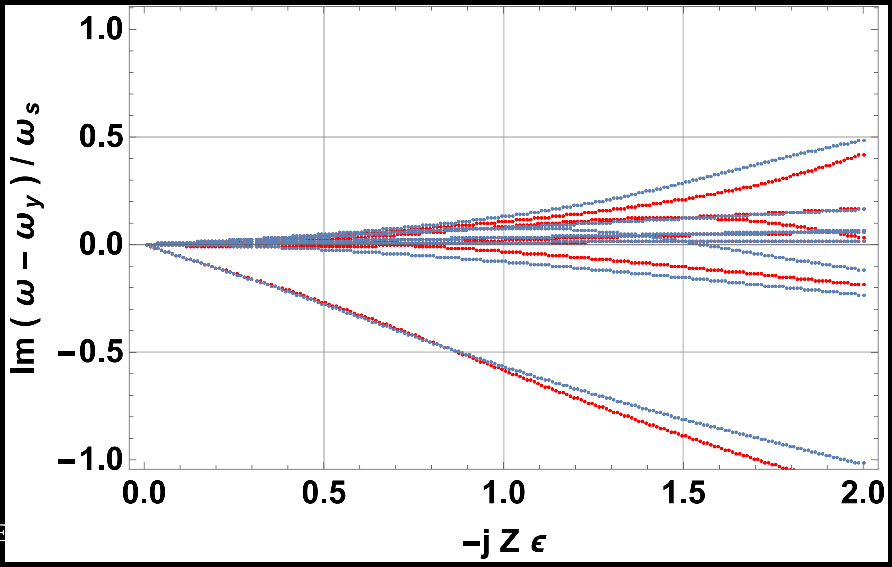




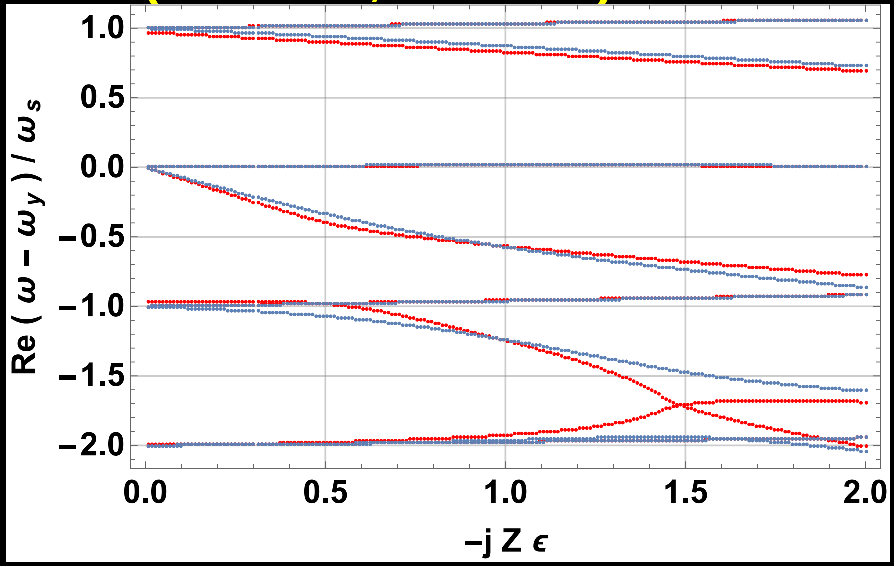
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE WITH fr × taub = 0.8 (instead of 2.8 before) & Q' = - 7

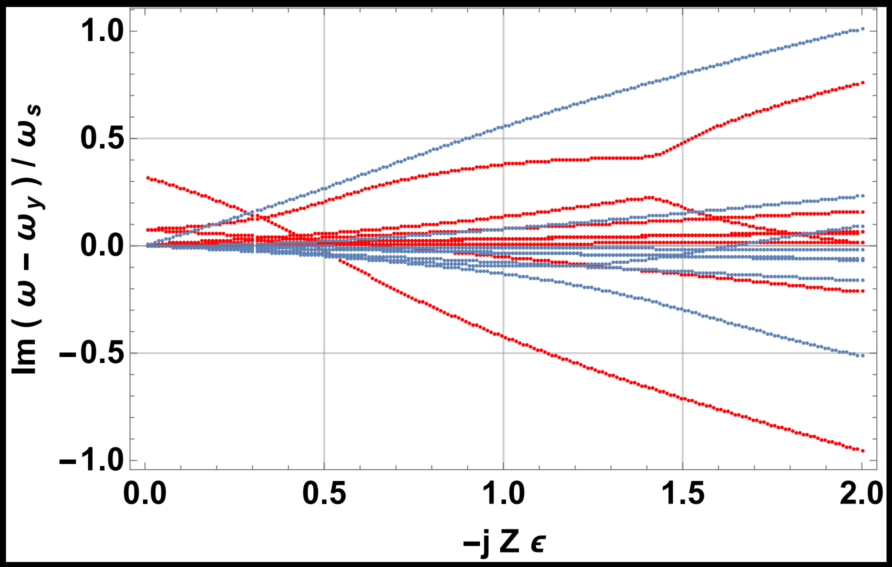
With transverse damper (reactive, 50 turns) in red



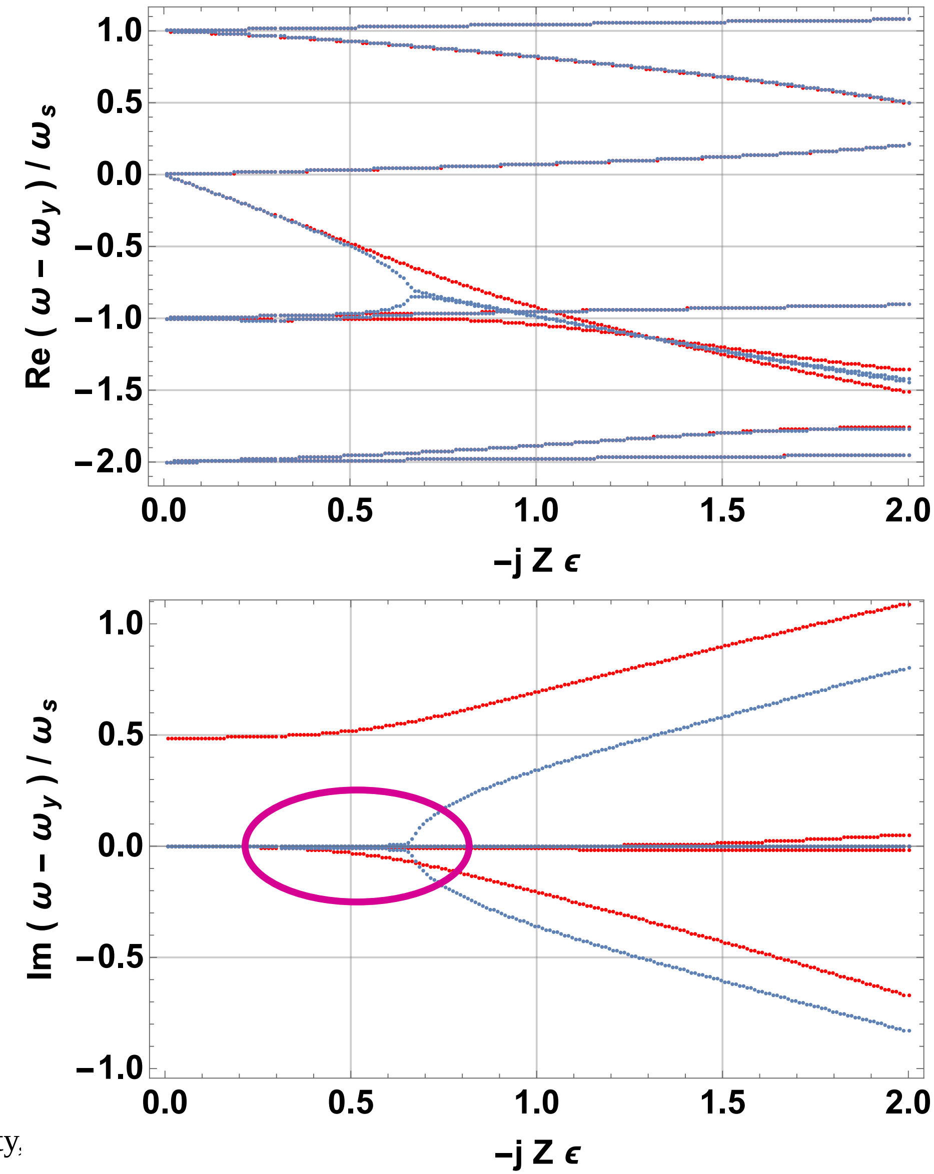


With transverse damper (resistive, 50 turns) in red



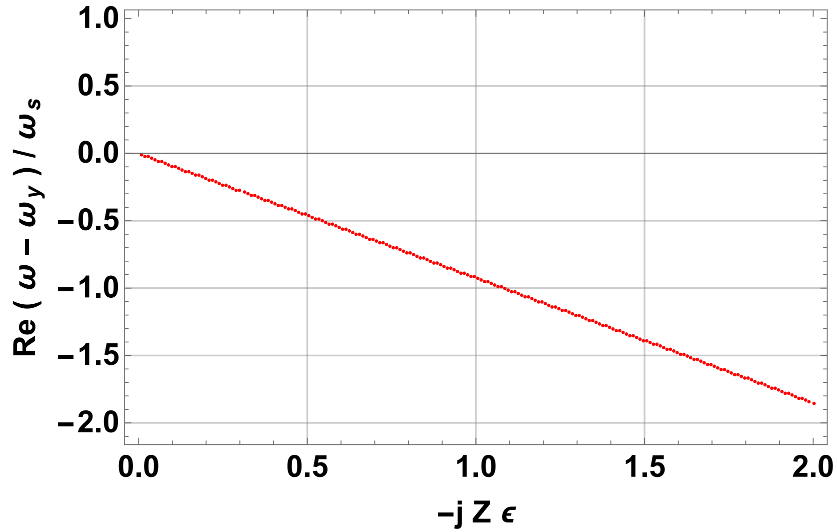


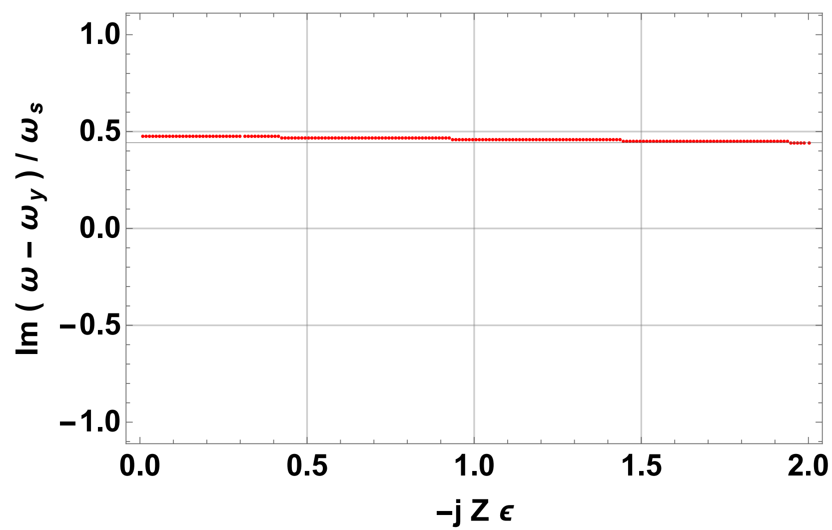
◆ **Destabilising effect of the resistive transverse damper** (e.g. with 50 turns) **for Q' = 0** => Where does the instability come from?



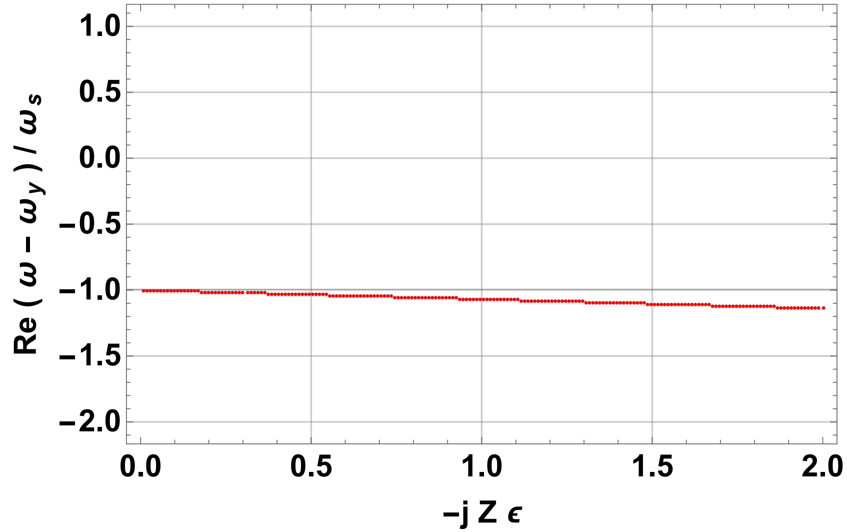
See also IPAC18 paper (https://espace.cern.ch/be-dep-workspace/abp/HSC/Meetings/THPAF048.pdf)

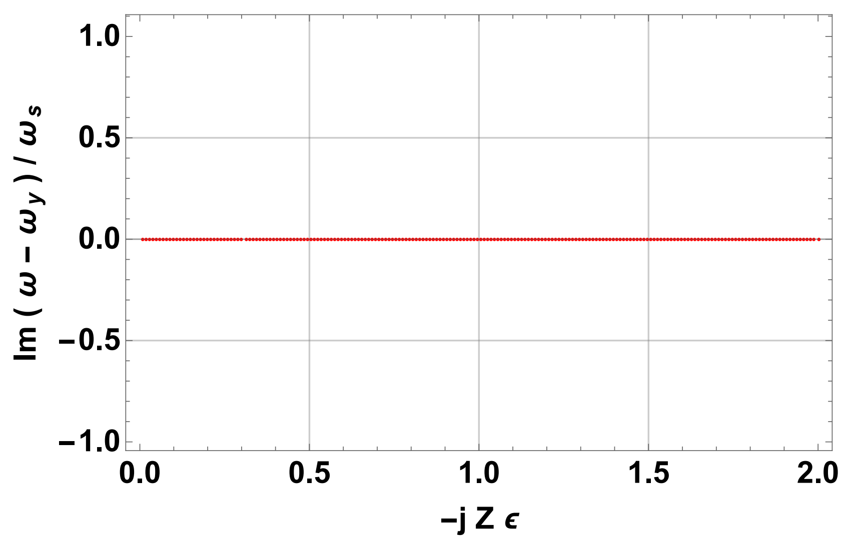
- Mode 0 (1st radial mode) only => Stable





- Mode -1 (1st radial mode) only => Stable

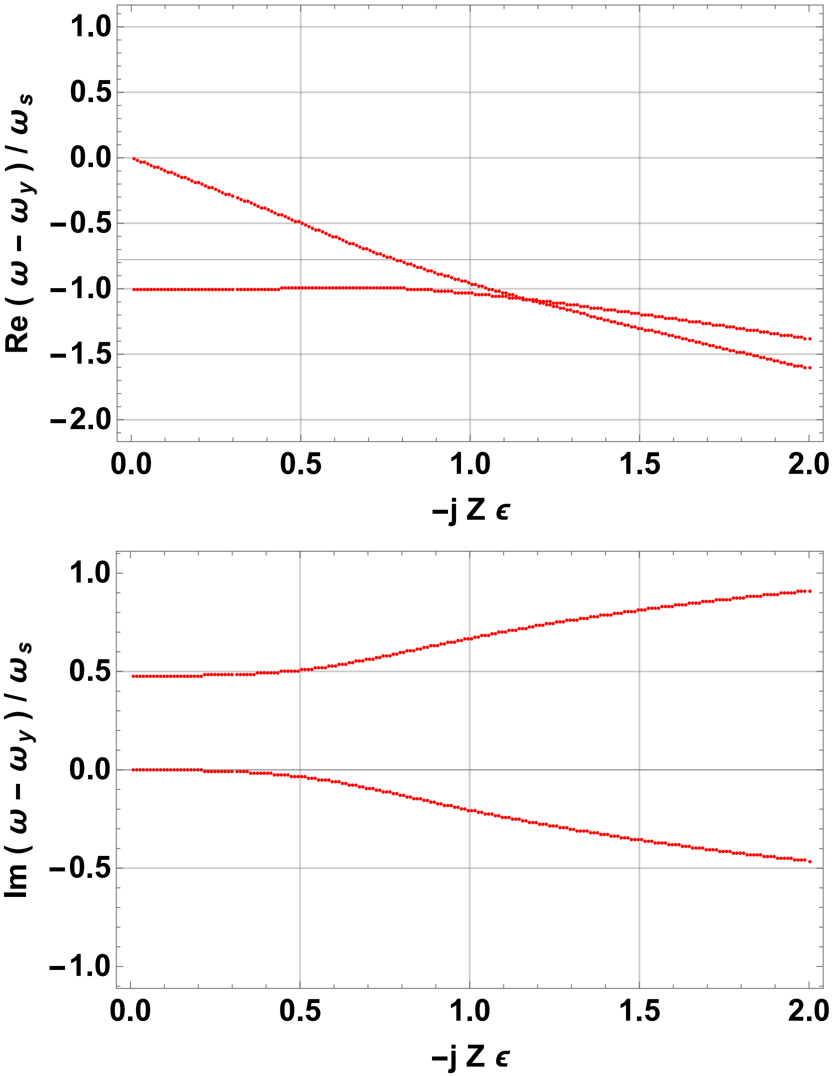




- Instability appears when both modes -1 and 0 (with only 1st radial mode) are considered

- => This is the interaction between modes -1 and 0 through the damper which creates the instability

- The "coupling" between the 2 modes pushes apart the instability growth rates and as the lowest one is 0, it becomes negative



- If one looks at the matrix to be diagonalized, it can be approximated by (with x = - j Z ε)

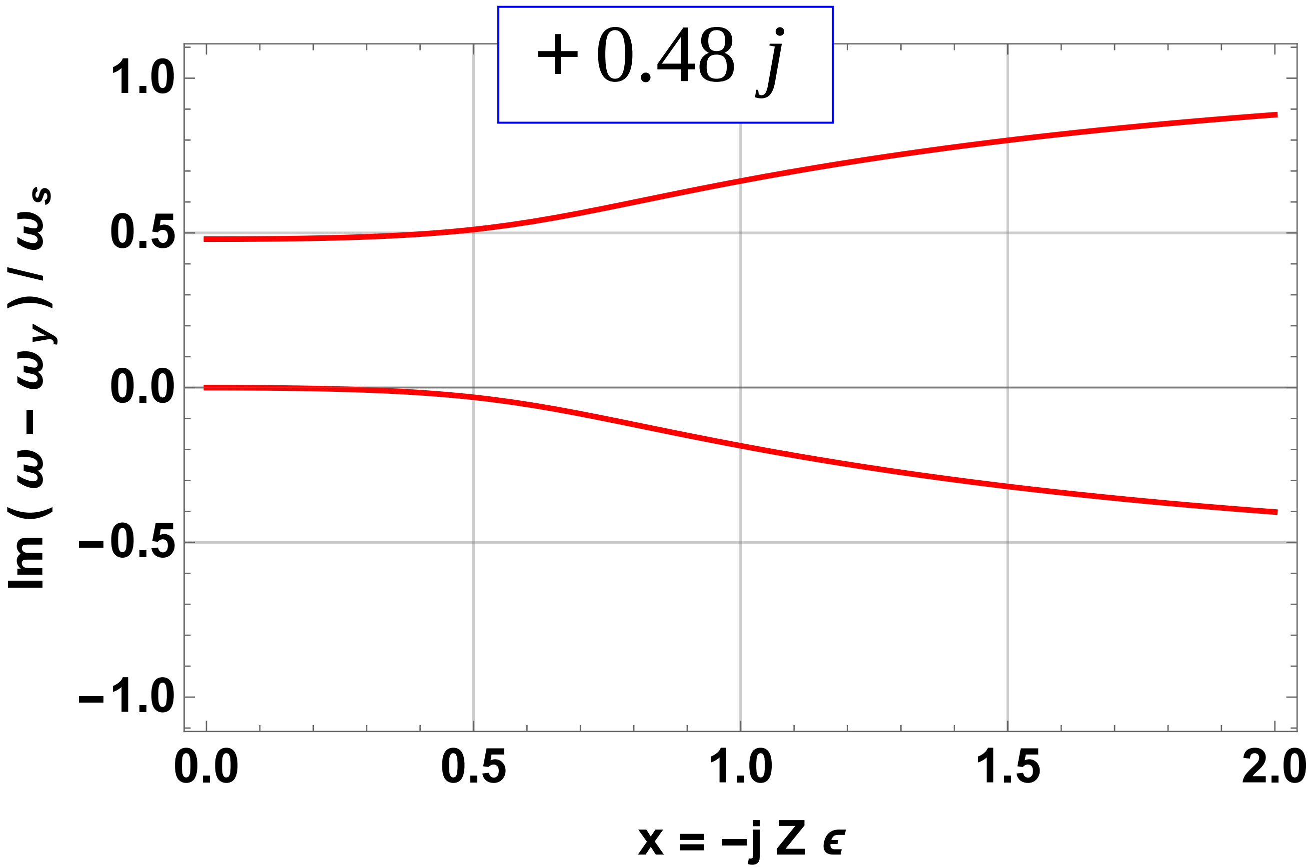
$$\begin{pmatrix} -1 & -0.23\, j\, x \\ -0.55\, j\, x & -0.92\, x + 0.48\, j \end{pmatrix}$$

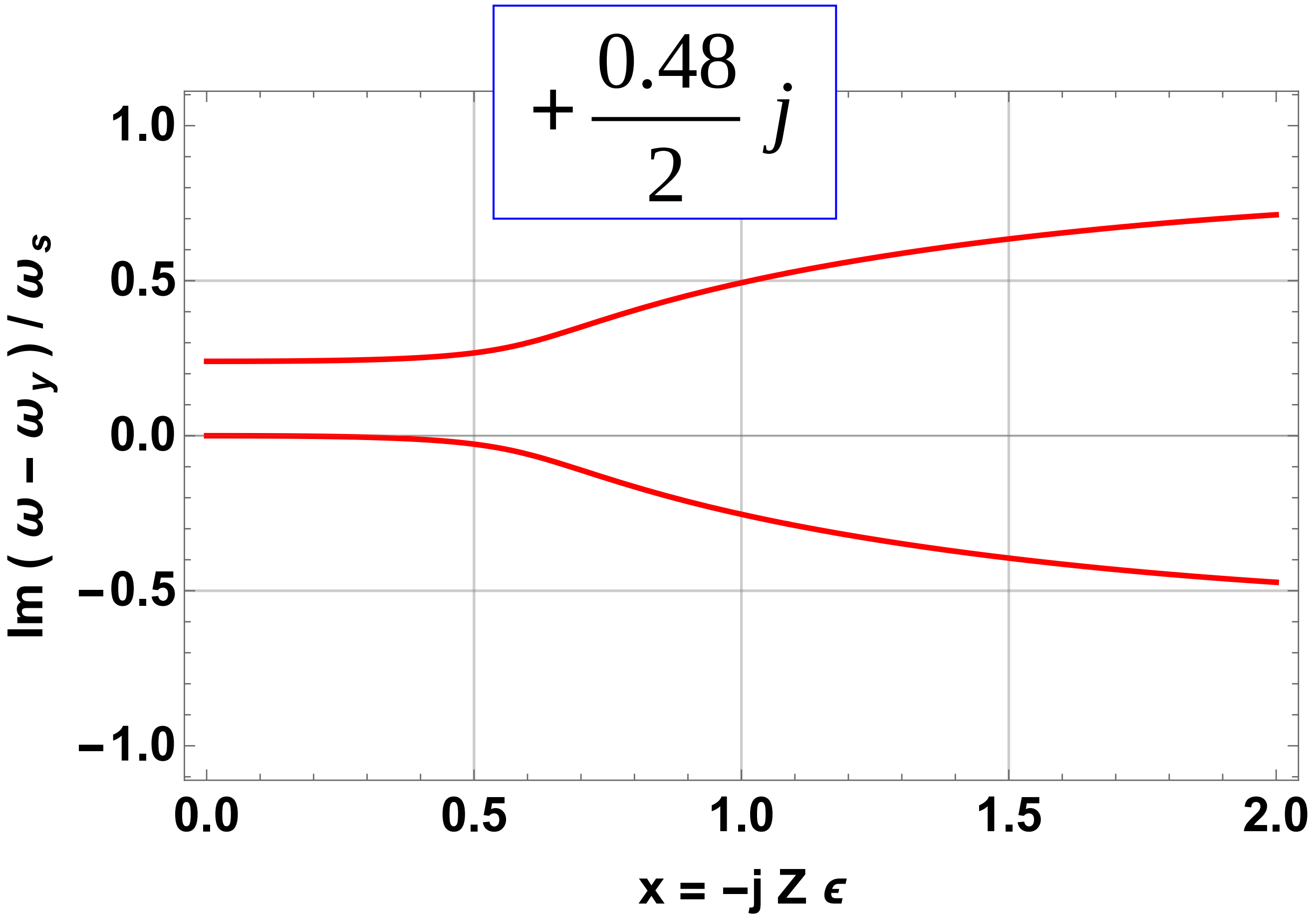
- If one looks at the matrix to be diagonalized, it can be approximated by (with x = - j Z ε)

$$\begin{pmatrix} -1 & -0.23\, j\, x \\ -0.55\, j\, x & -0.92\, x + 0.48\, j \end{pmatrix}$$

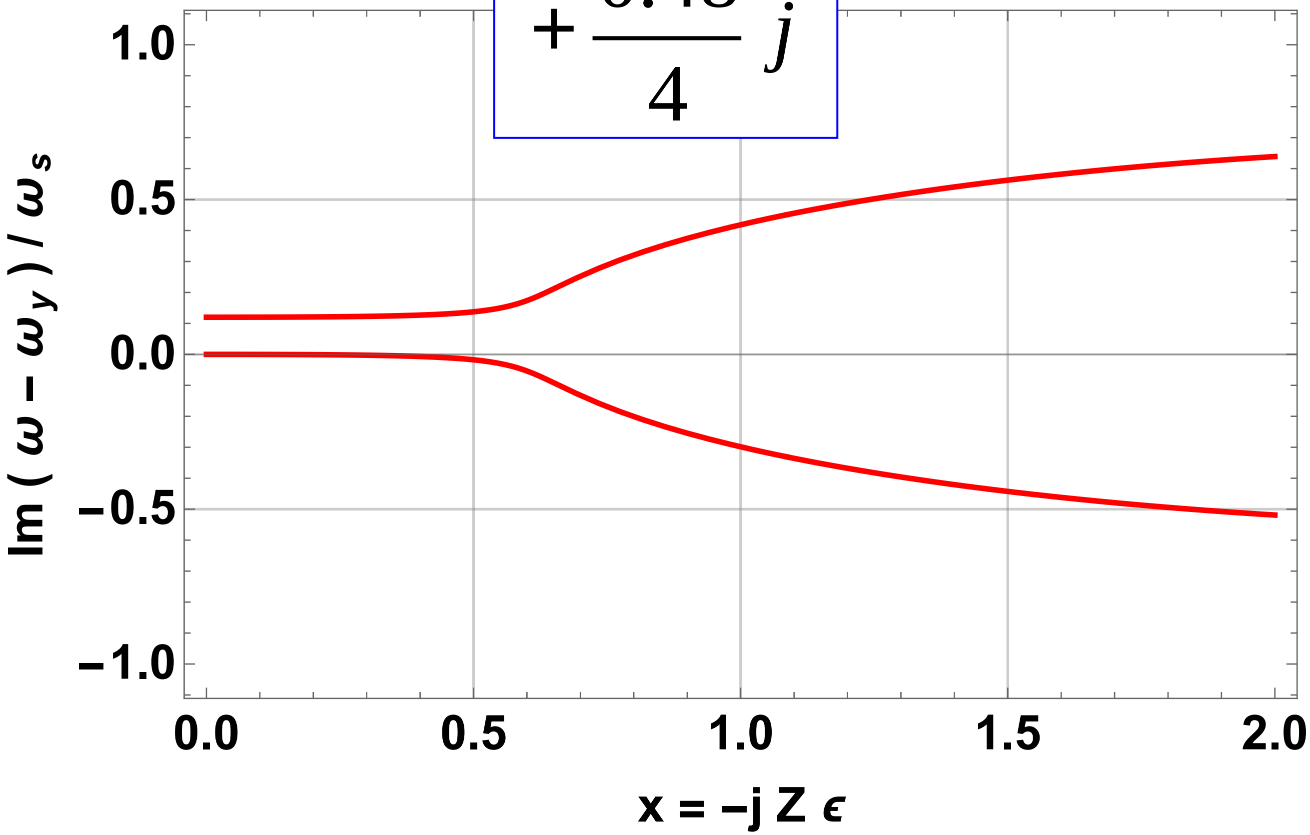
Introduced by the transverse (+ resistive) damper

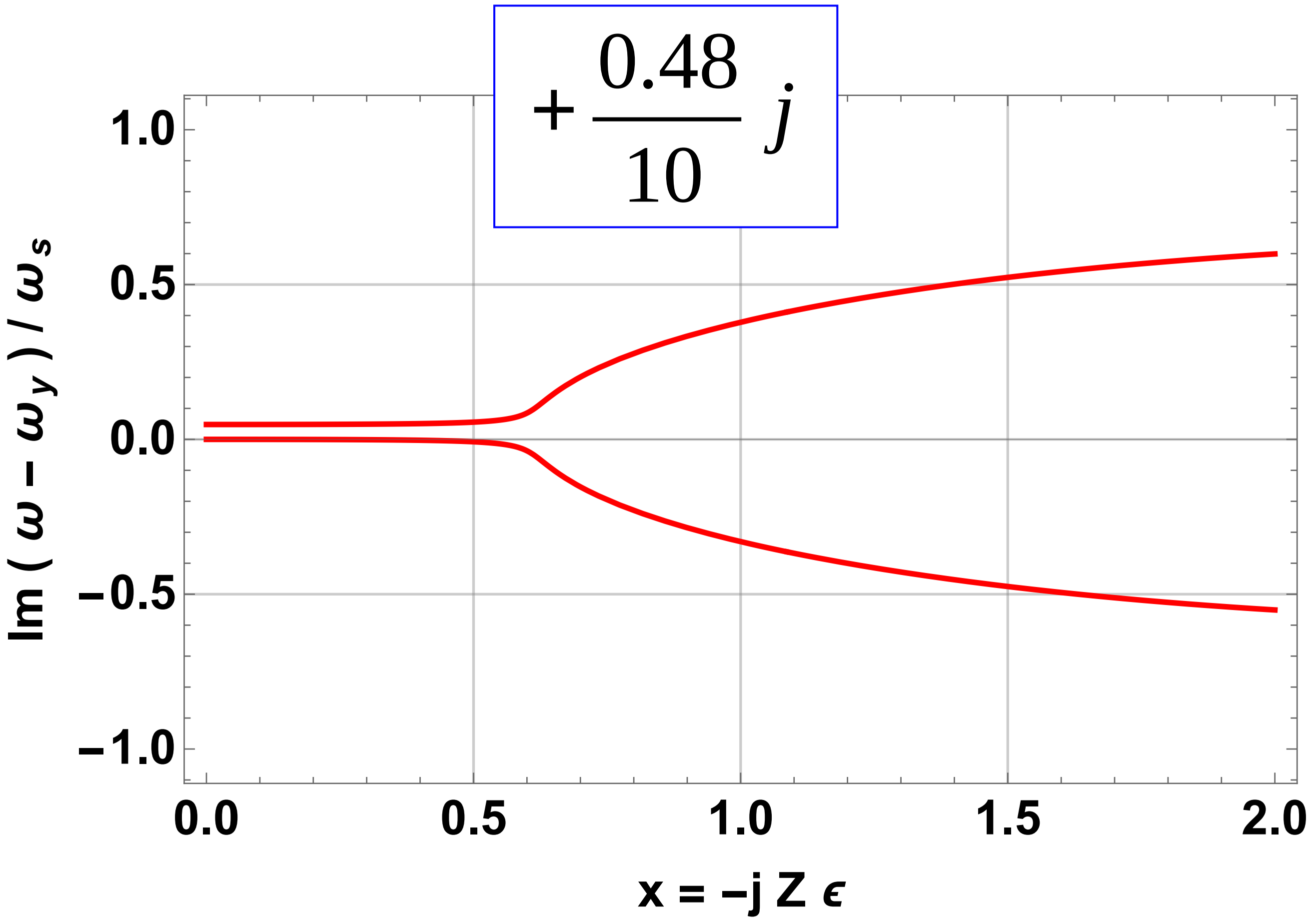
- N.B.: Would be $+0.48$ for a + reactive damper


$+0.48\ j$
1.0
0.5
0.0
−0.5
−1.0
Im (ω − ω_y) / ω_s
0.0
0.5
1.0
1.5
2.0
x = −j Z ϵ


$$+\frac{0.48}{2}\,j$$
1.0
0.5
0.0
−0.5
−1.0
Im (ω − ω_y) / ω_s
0.0
0.5
1.0
1.5
2.0
x = −j Z ϵ

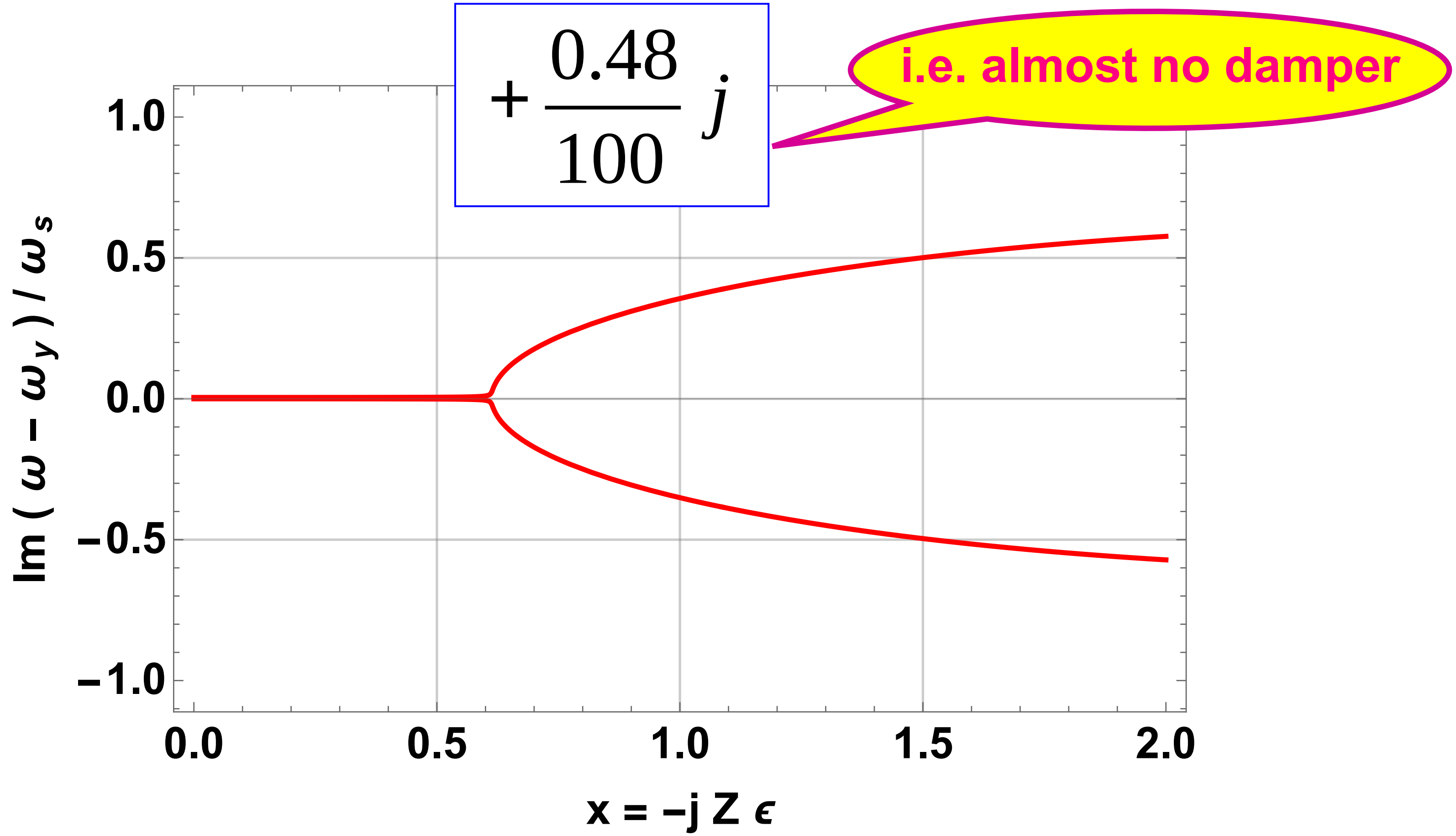
$$\boxed{+\frac{0.48}{4}\,j}$$

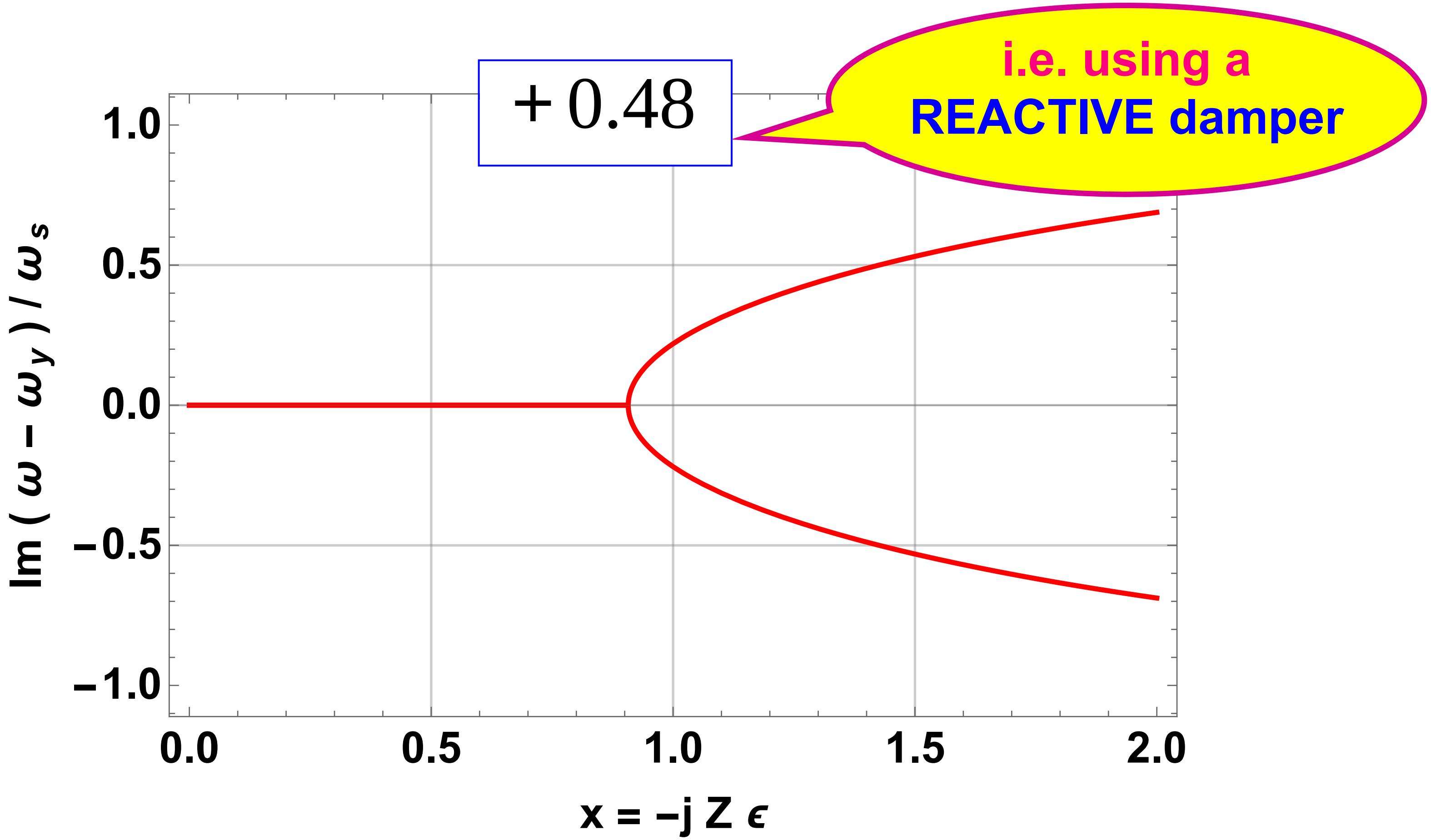


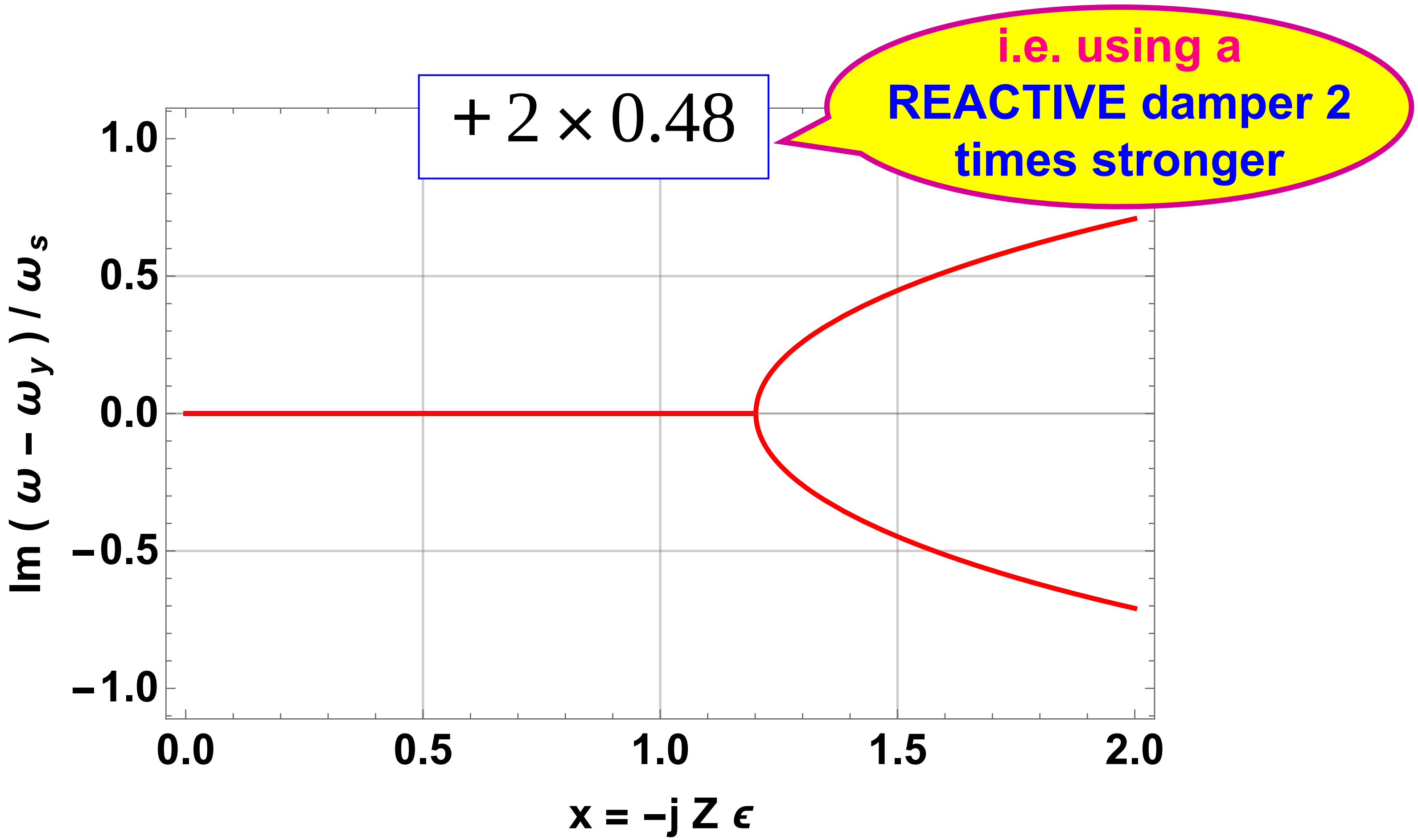


$+\frac{0.48}{10}\,j$
1.0
0.5
0.0
−0.5
−1.0
Im (ω − ω_y) / ω_s
0.0
0.5
1.0
1.5
2.0
x = −j Z ϵ

$$+\frac{0.48}{100}\,j$$




+ 0.48
i.e. using a
REACTIVE damper
1.0
0.5
0.0
−0.5
−1.0
Im (ω − ω_y) / ω_s
0.0
0.5
1.0
1.5
2.0
x = −j Z ϵ

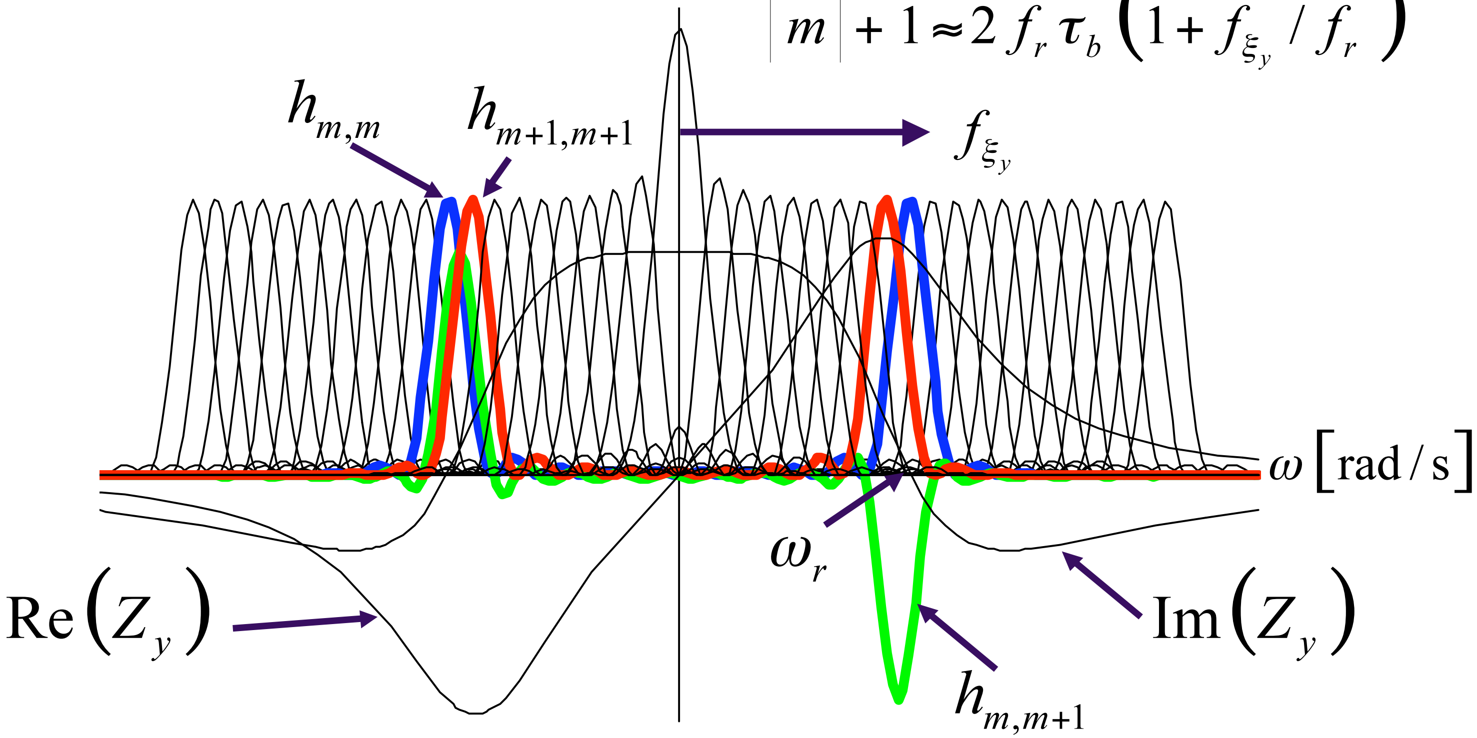

$+2 \times 0.48$
i.e. using a
REACTIVE damper 2
times stronger
1.0
0.5
0.0
−0.5
−1.0
Im (ω − ω_y) / ω_s
0.0
0.5
1.0
1.5
2.0
x = −j Z ϵ

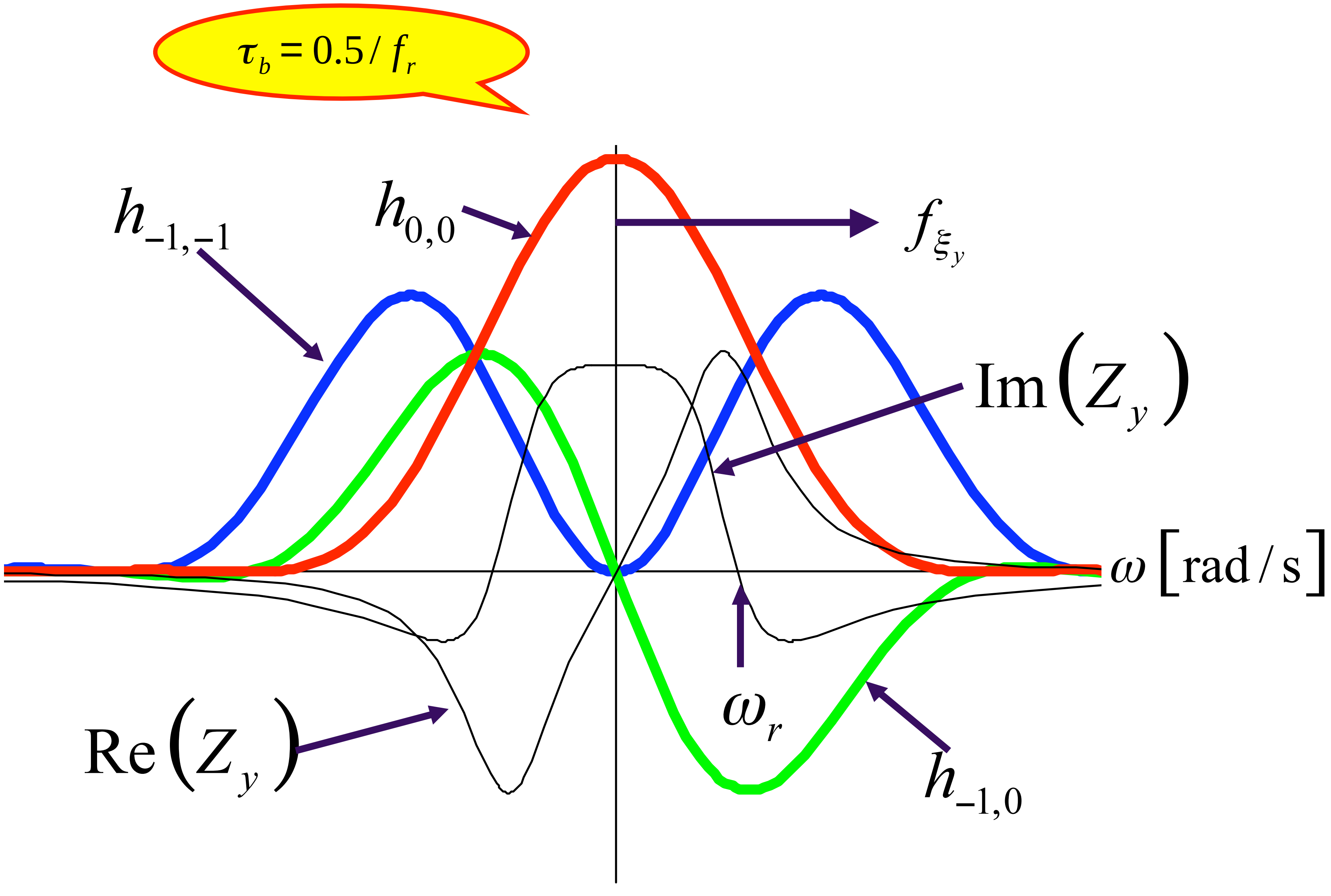
- **Simple formula for the intensity threshold** in the case of a bunch interacting with a Broad-Band impedance in the long-bunch regime (as for the SPS case before), considering only the mode-coupling between the 2 adjacent modes overlapping the maximum of the resonator impedance

"Long-bunch" regime:

$\tau_b >> 0.5 / f_r$

$$|m| + 1 \approx 2 f_r \tau_b \left(1 + f_{\xi_y} / f_r\right)$$




$\tau_b = 0.5 / f_r$
$h_{-1,-1}$
$h_{0,0}$
f_{ξ_y}
$\mathrm{Im}\left(Z_y\right)$
$\omega\left[\mathrm{rad/s}\right]$
ω_r
$\mathrm{Re}\left(Z_y\right)$
$h_{-1,0}$

- Using $h_{m,n}\left(\omega-\omega_{\xi_x}\right)=p_m^*\left(\omega-\omega_{\xi_x}\right)p_n\left(\omega-\omega_{\xi_x}\right)$

=>

$$h_{m,n}(\omega)=\frac{\tau_b^2}{\pi^4}\left(|m|+1\right)\times\left(|n|+1\right)\times F_m^n$$
$$\times\left\{\left(\omega\tau_b/\pi\right)^2-\left(|m|+1\right)^2\right\}^{-1}\times\left\{\left(\omega\tau_b/\pi\right)^2-\left(|n|+1\right)^2\right\}^{-1}$$

$$F_{m\,even}^{n\,even}=(-1)^{\left(|m|+|n|\right)/2}\times\cos^2\left[\omega\tau_b/2\right]$$

$$F_{m\,even}^{n\,odd}=\frac{(-1)^{\left(|m|+|n|+3\right)/2}}{2j}\times\sin\left[\omega\tau_b\right]$$

$$F_{m\,odd}^{n\,even}=\frac{(-1)^{\left(|m|+|n|+1\right)/2}}{2j}\times\sin\left[\omega\tau_b\right]$$

$$F_{m\,odd}^{n\,odd}=(-1)^{\left(|m|+|n|+2\right)/2}\times\sin^2\left[\omega\tau_b/2\right]$$

- **In longitudinal**

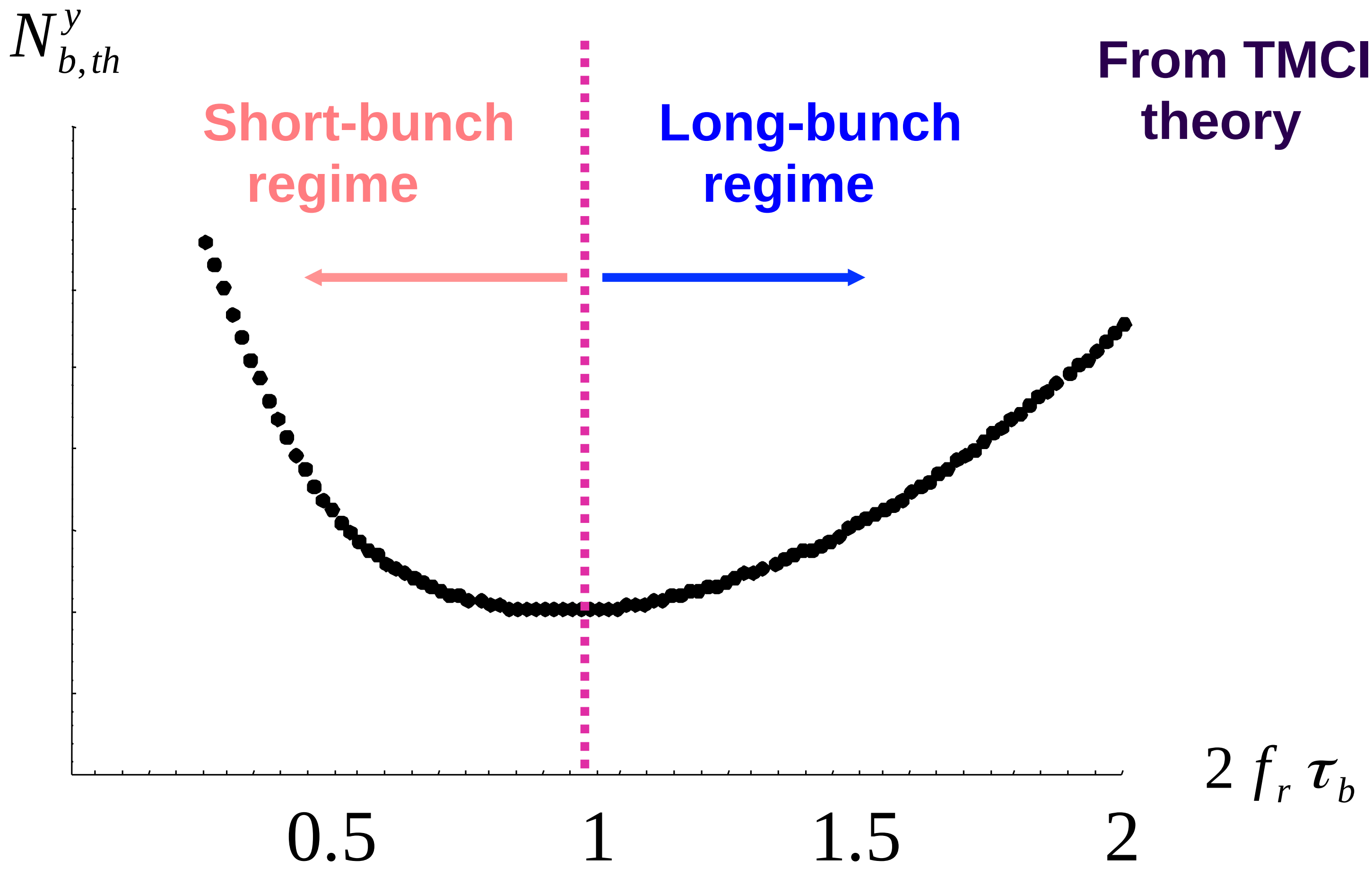
$$\boxed{\Delta\omega_{m,n}^{l}=\frac{|m|}{|m|+1}\times\frac{j\,I_b\,\omega_s}{3B^3\,\hat{V}_T\,h\cos\phi_s}\times\left[\frac{Z_l\left(p\right)}{p}\right]_{m,n}^{eff}}$$

$$\left[\frac{Z_l\left(p\right)}{p}\right]_{m,n}^{eff}=\frac{\sum_{p=-\infty}^{p=+\infty}\frac{Z_l\left(\omega_p^l\right)}{p}h_{m,n}\left(\omega_p^l\right)}{\sum_{p=-\infty}^{p=+\infty}h_{m,m}\left(\omega_p^l\right)} \qquad \omega_p^l=p\,\Omega_0+m\,\omega_s$$

- **In transverse**

$$\boxed{\Delta\omega_{m,n}^{x,y}=\left(|m|+1\right)^{-1}\frac{j\,e\,\beta\,I_b}{4\pi\,R\,B\,m_0\,\gamma\,Q_{x0,y0}\,\Omega_0}\left(Z_{x,y}^{eff}\right)_{m,n}}$$

$$\left(Z_{x,y}^{eff}\right)_{m,n}=\frac{\sum_{p=-\infty}^{p=+\infty}Z_{x,y}\left(\omega_p^{x,y}\right)h_{m,n}\left(\omega_p^{x,y}-\omega_{\xi_{x,y}}\right)}{\sum_{p=-\infty}^{p=+\infty}h_{m,m}\left(\omega_p^{x,y}-\omega_{\xi_{x,y}}\right)} \qquad \omega_p^{x,y}=\left(p+Q_{x0,y0}\right)\Omega_0+m\,\omega_s$$


$N_{b,th}^{y}$
Short-bunch regime
Long-bunch regime
From TMCI theory
0.5
1
1.5
2
$2 f_r \tau_b$

- Simple formula for TMCI (with the 2 assumptions):

$$N_{b,th}^{y} \propto \boxed{\frac{f_r}{|Z_y|}} \boxed{|\eta| \, Q_y} \boxed{\varepsilon_L} \boxed{\left(1 + \frac{f_{\xi_y}}{f_r}\right)}$$

- Increase the chromatic frequency
- Chromaticity jump in case transition has to be crossed

Increase the beam longitudinal emittance (when possible)

Try to decrease the impedance and/or increase the resonance frequency => Impedance reduction campaign

Change the optics to increase the betatron tune (decrease the beta function at critical impedances) and/or go further away from transition => New optics needed

- Simple formula for TMCI (with the 2 assumptions):

$$N_{b,th}^{y} \propto \boxed{\frac{f_r}{\left|Z_y\right|}}\;\boxed{\left|\eta\right| Q_y}\;\boxed{\varepsilon_L}\;\boxed{\left(1+\frac{f_{\xi_y}}{f_r}\right)}$$

- Increase the chromatic frequency
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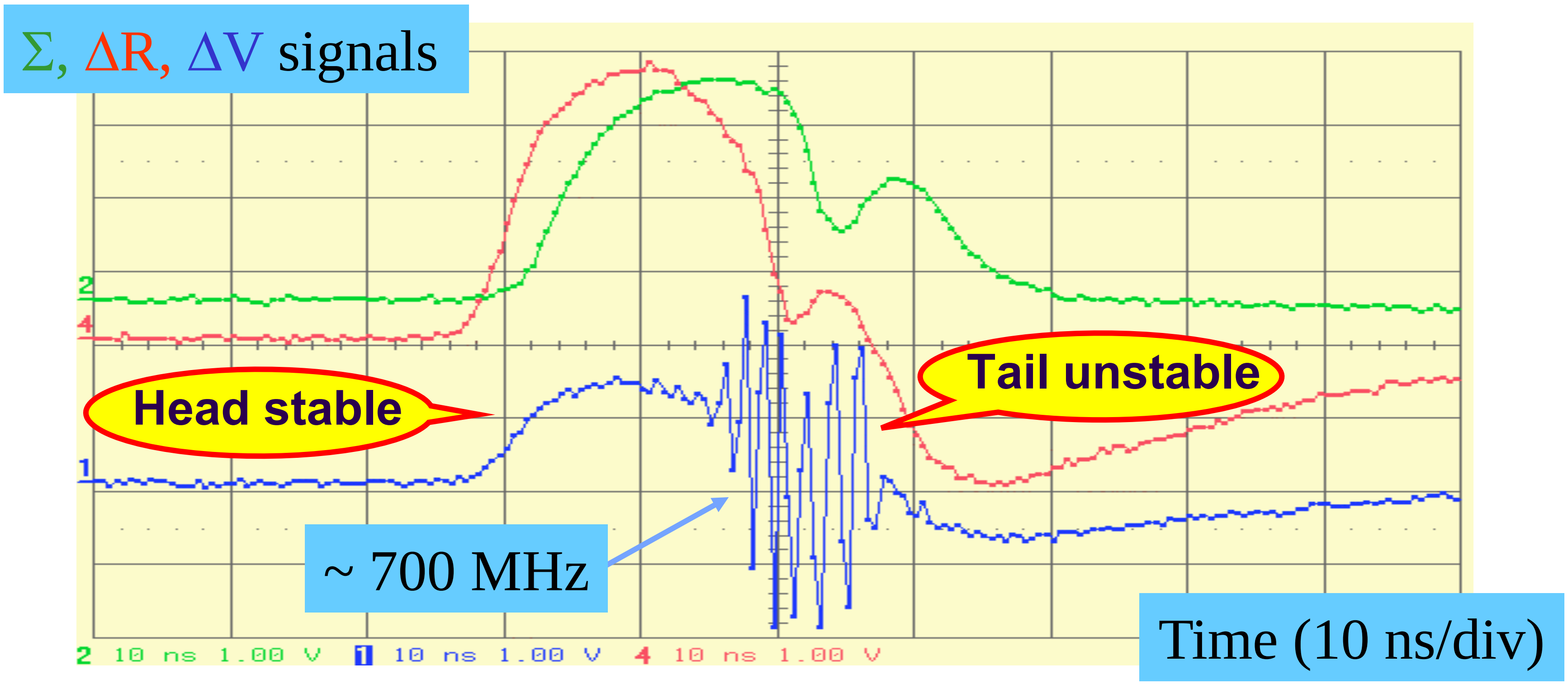
Change the optics to increase the betatron tune (decrease the beta function at critical impedances) and/or go further away from transition => New optics needed

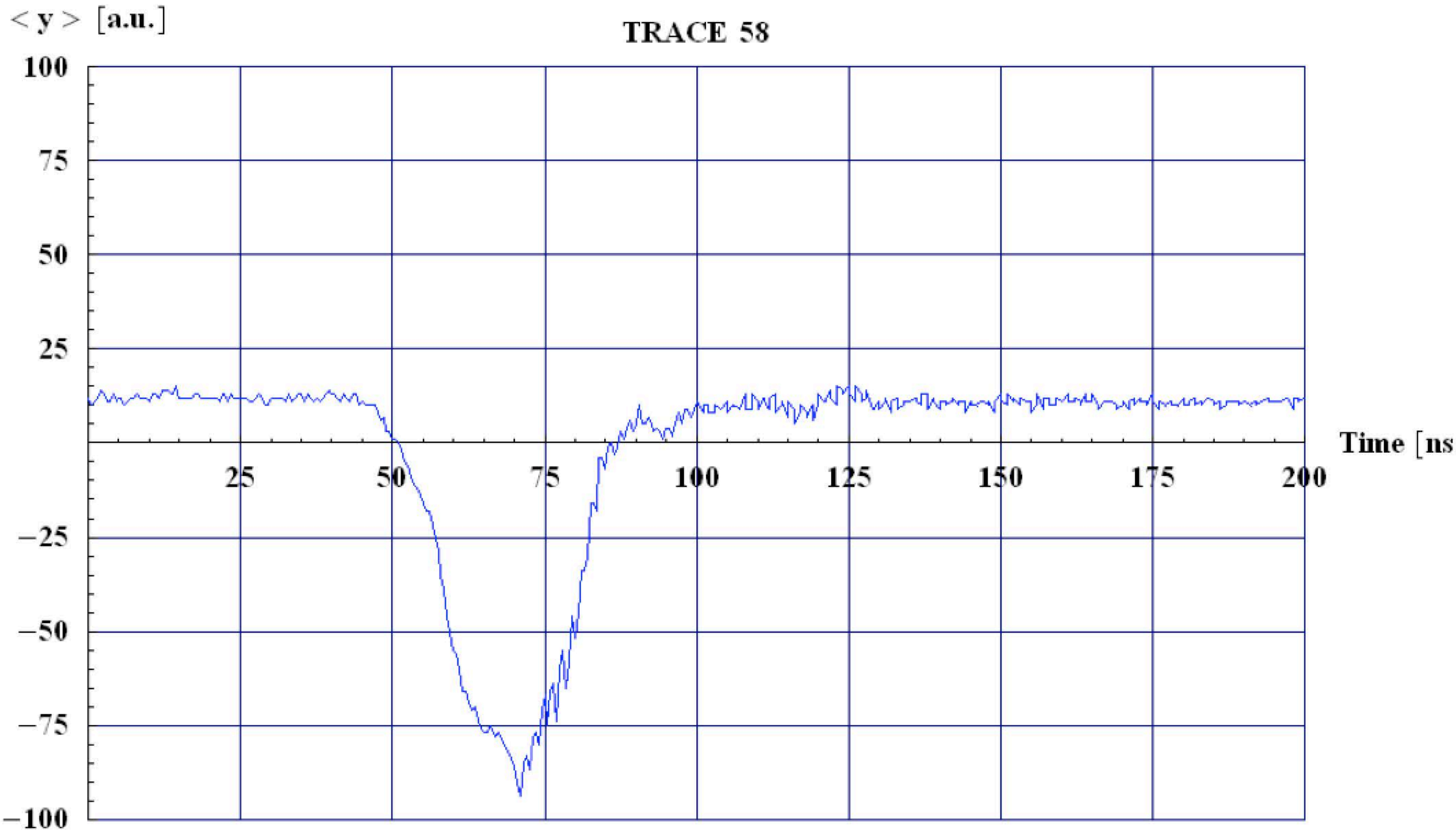
Increase the beam longitudinal emittance (when possible)

** No dependence on Q_s!*

*** It is the same formula as for coasting beams (with peak values)!*

- **Ex.1 => In the PS**: a fast vertical single-bunch instability is observed (with high-intensity bunches) when transition is crossed and when no longitudinal emittance blow-up is applied before transition

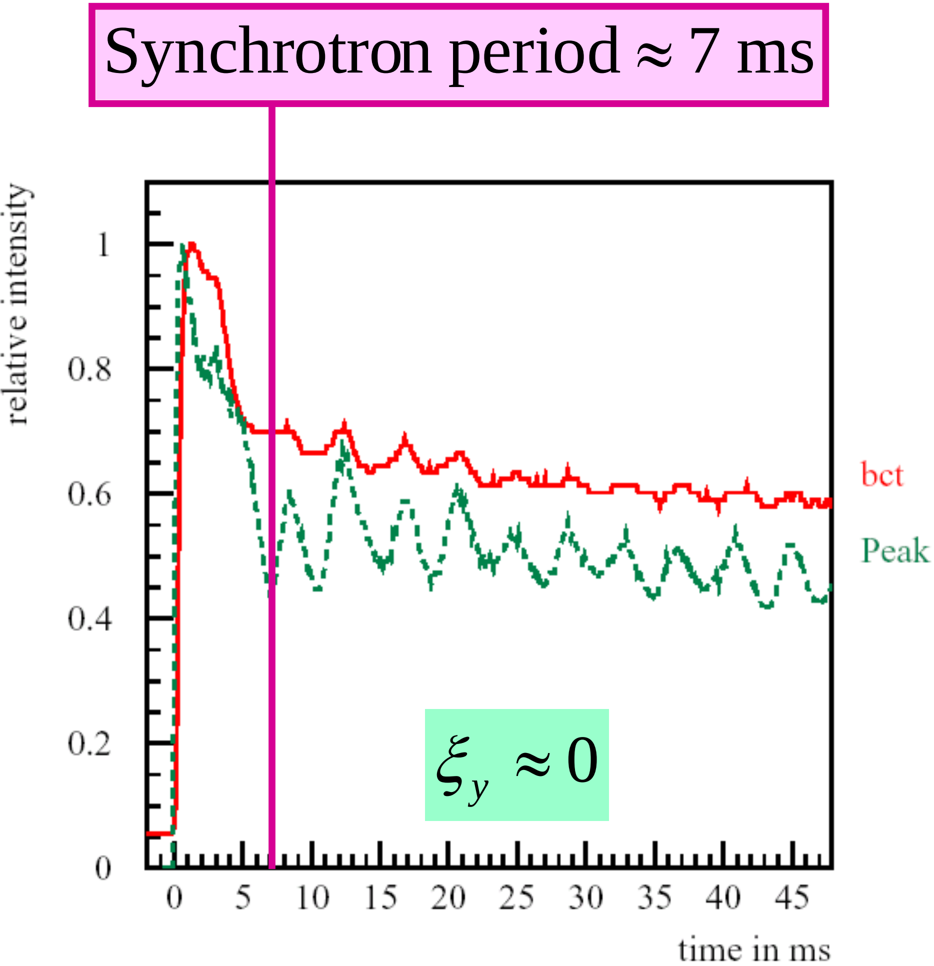


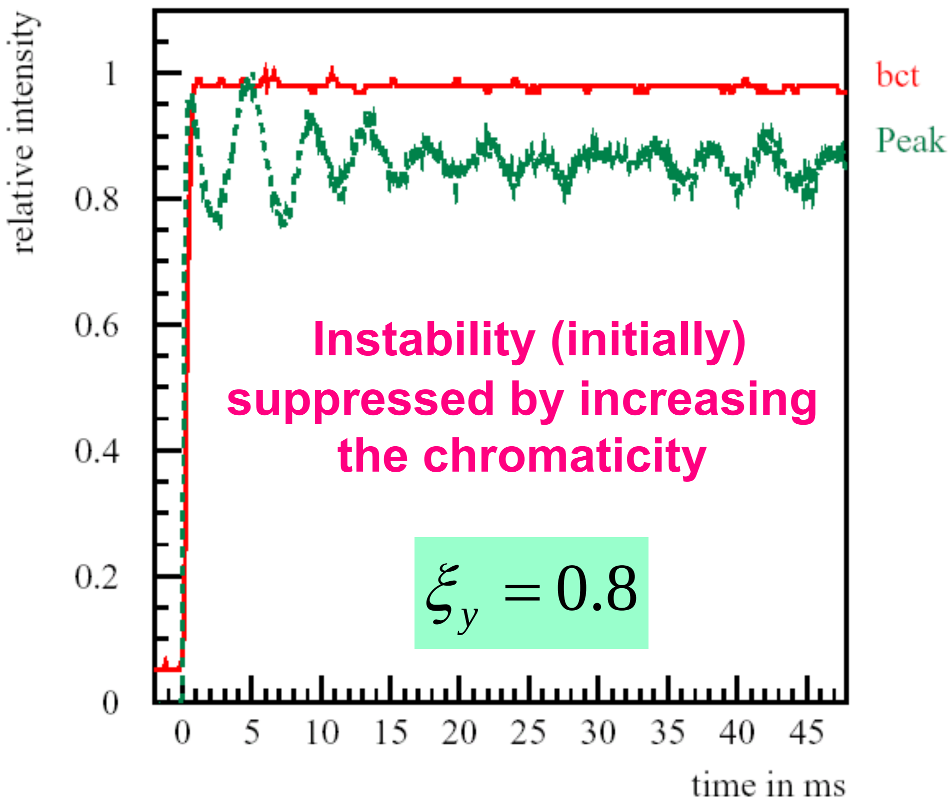


Courtesy of R. Steerenberg

=> Instability suppressed by increasing the longitudinal emittance

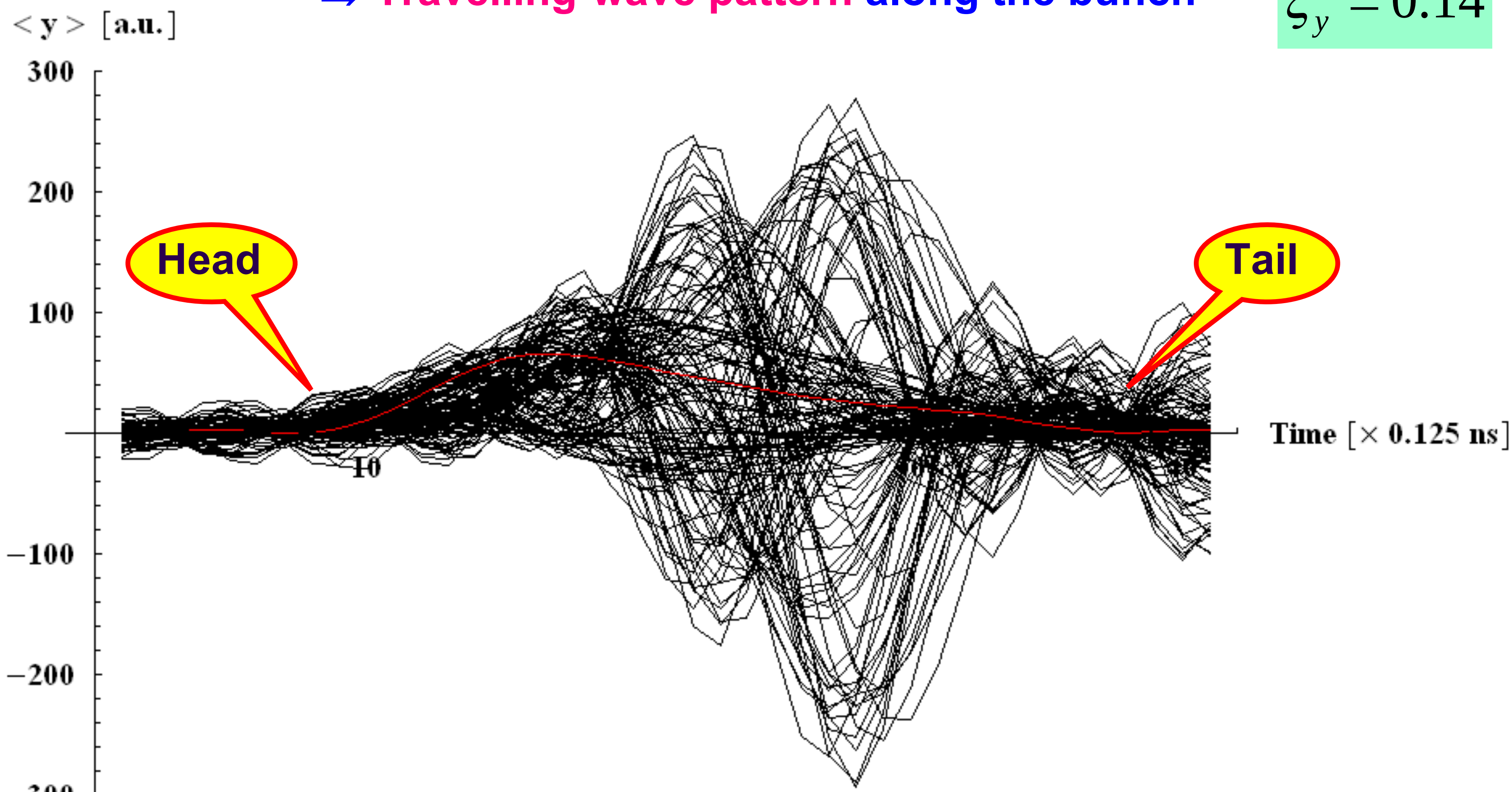
- **Ex.2 => In the SPS**





⇒ Travelling-wave pattern along the bunch

$\xi_y = 0.14$

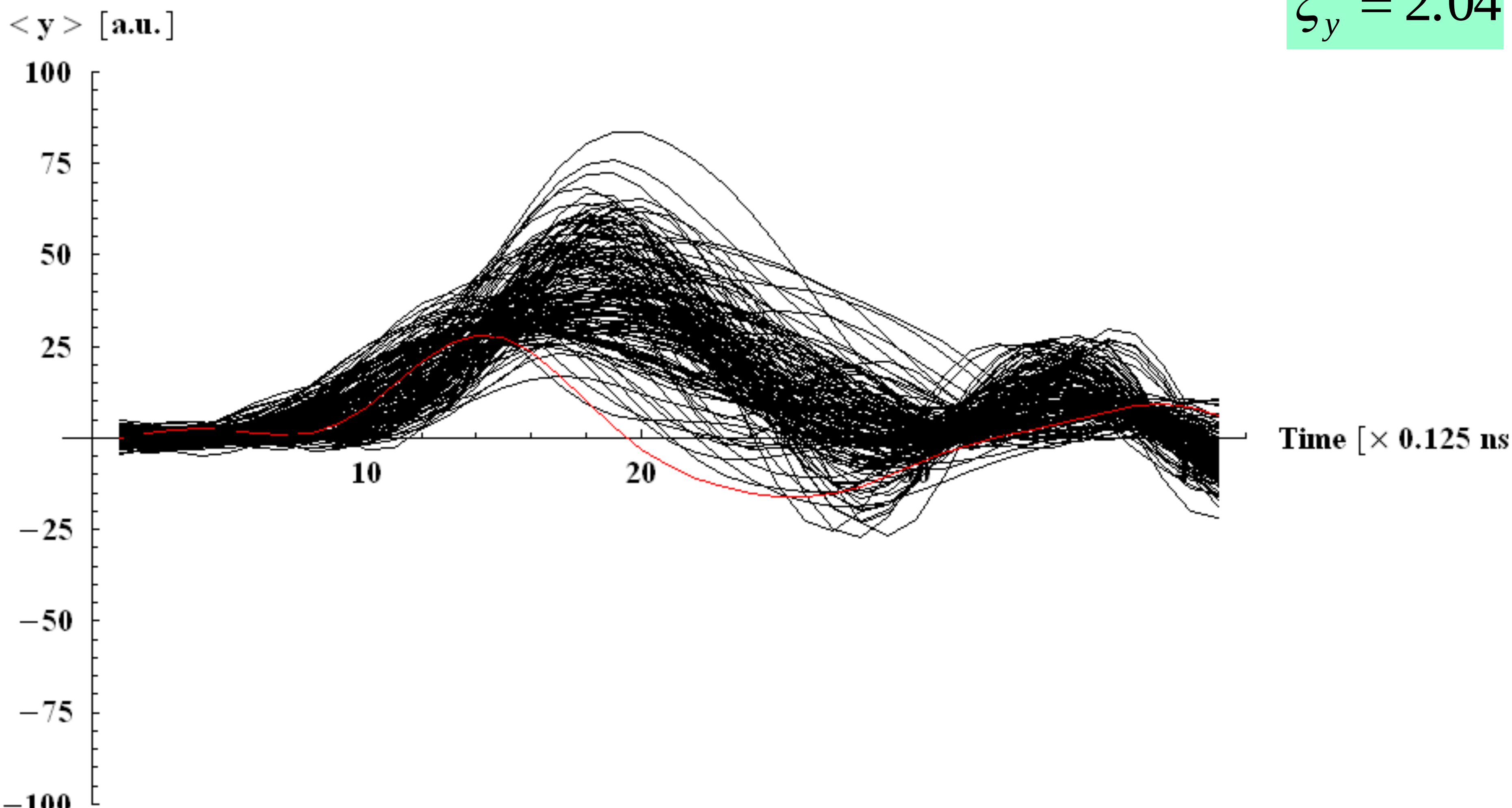


1st trace (in red) = turn 2

Last trace = turn 150

Every turn shown

$\xi_y = 2.04$



1st trace (in red) = turn 2

Last trace = turn 150

Every turn shown

- **γ_t was recently modified in the SPS to increase the TMCI intensity threshold** above the foreseen intensities for the future upgrade

- Simple rough estimate of γ_t for machines made of simple FODO cells:

  ✧ Approximating the machine radius by the bending radius, yields

$$D_x \approx \frac{\rho}{Q_x^2}$$

  ✧ Inserting this in the definition of α_p (and then expressing γ_t) yields

$$\gamma_t \approx Q_x$$

=> If one wants to modify γ_t, (increase or decrease its value) one should modify the horizontal tune

- TMCI intensity threshold with the old (Q26) optics at injection: **~ 1.7 10^{11} p/b**

- Predictions going from Q26 to the new (Q20) optics:

✧ **Q26:** $|\eta| Q_y = 0.62\, 10^{-3} \times 26.13 \approx 0.0162$ $\qquad \gamma_t = 22.8$

✧ **Q20:** $|\eta| Q_y = 1.80\, 10^{-3} \times 20.13 \approx 0.0362$ $\qquad \gamma_t = 18$

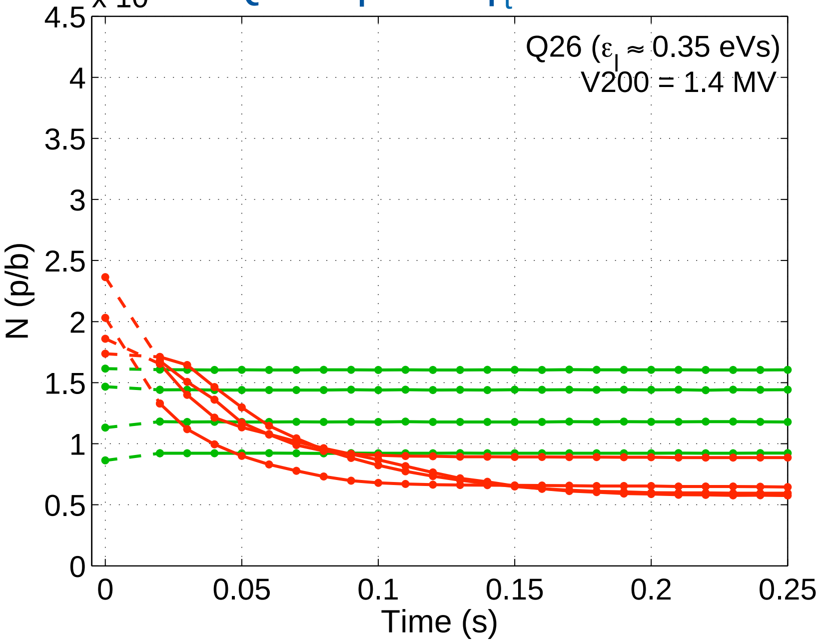
=> A **gain of a factor** 0.0362 / 0.0162 **≈ 2.2** in the intensity threshold was expected

- Measurements

=> Good agreement with simple formula

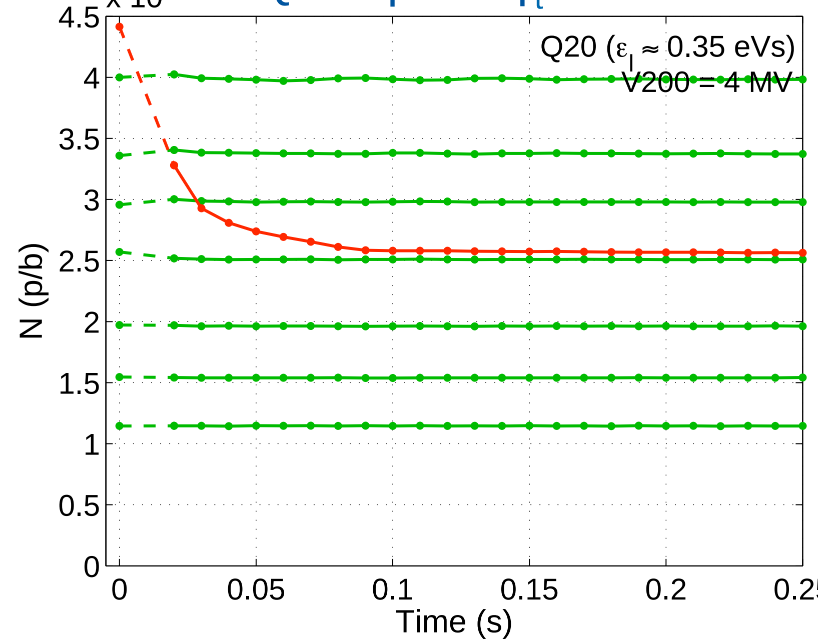
Gain of a factor
4.5 / 1.7 **≈ 2.6**

Q26 optics: $\gamma_t = 22.8$



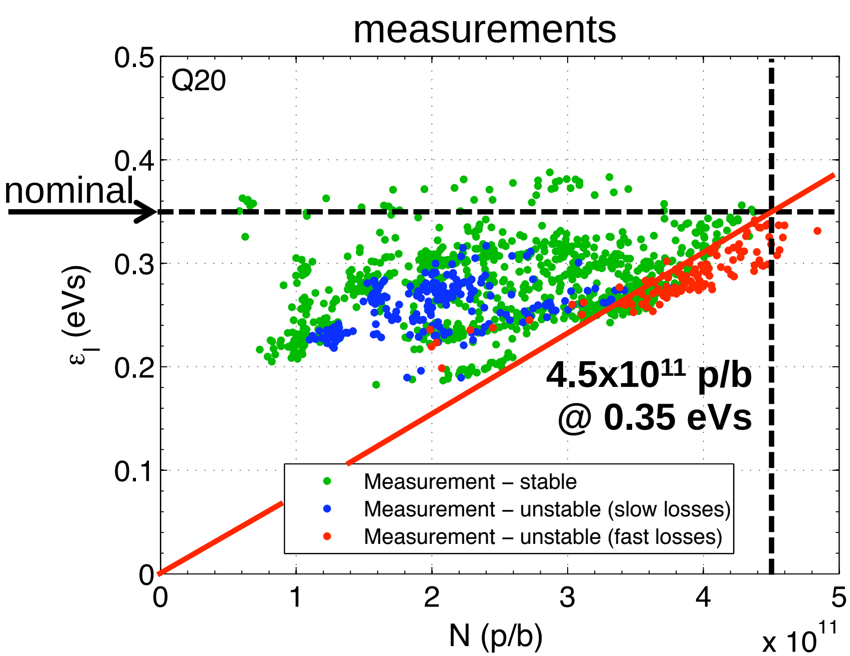
Courtesy of B. Salvant et al.

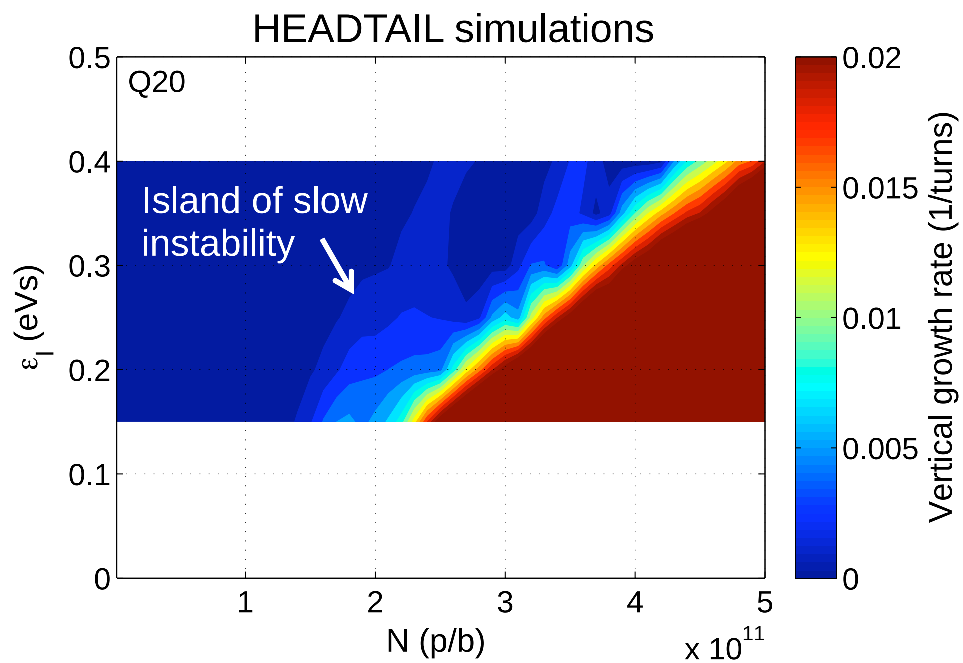
Q20 optics: $\gamma_t = 18$



Courtesy of H. Bartosik et al.

- Very good agreement between measurements and simulations





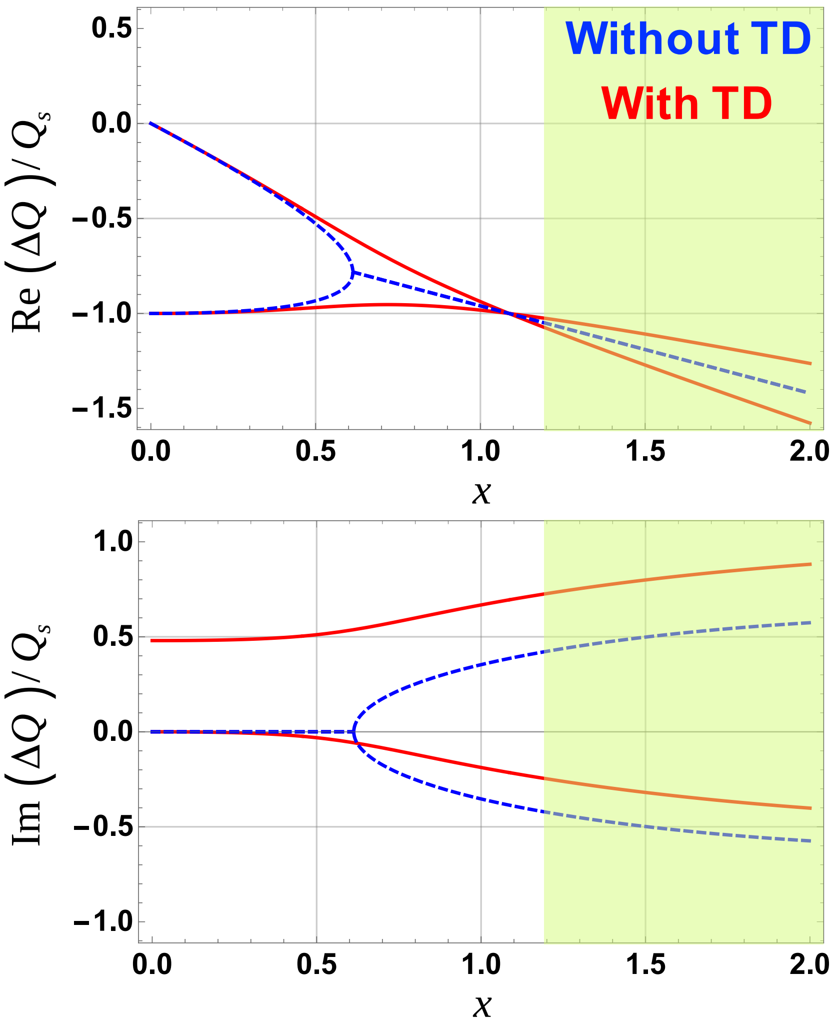
Courtesy of H. Bartosik et al.

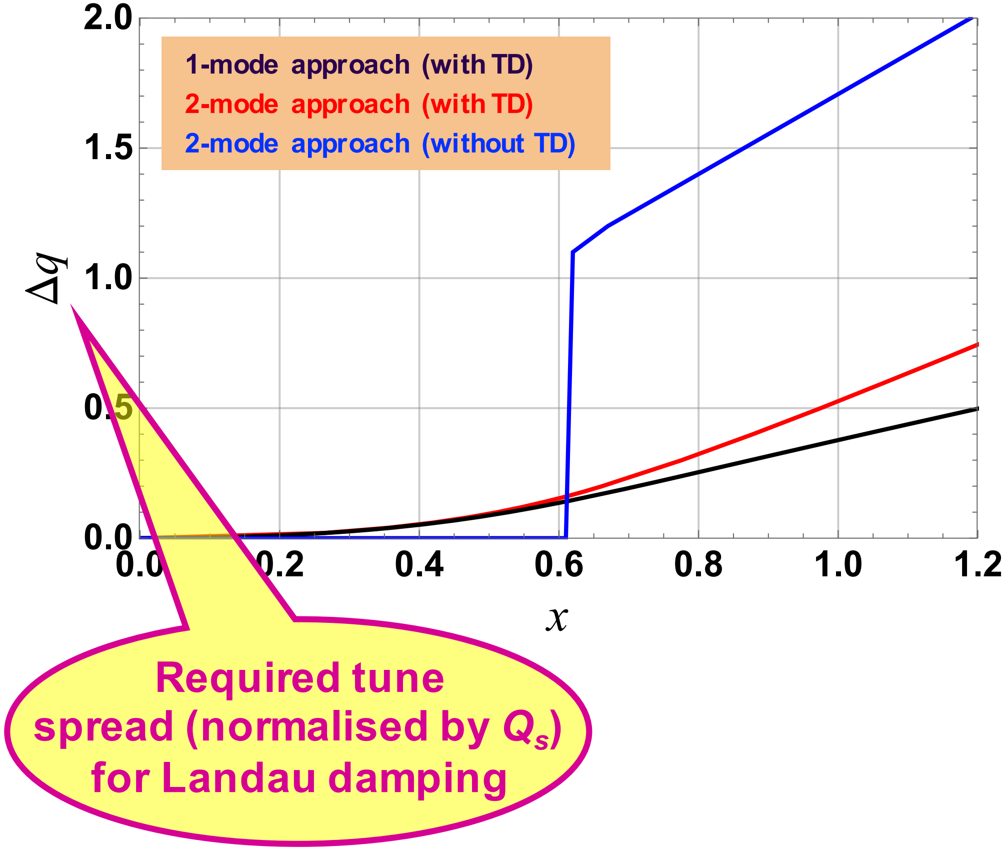
=> Intensity threshold with the new (Q20) optics: **~ 4.5 10^{11} p/b**

◆ Landau damping for TMCI: with vs. without Transverse Damper

See also https://cds.cern.ch/record/2674776/files/CERN-ACC-NOTE-2019-0018.pdf

(~ LHC case – "short-bunch regime" – zero chromaticity)





LONGITUDINAL: HIGH-INTENSITY

- Reminder: general equation of coherent motion considering the contributions from all the modes *m*

$$j\,\Delta\omega_{cm}\; j^{-m}\; g_m(\hat{\tau})\,\hat{\tau} = \frac{2\pi I_b\; m\,\omega_s}{\Omega_0^2 \hat{V}_T\; h\cos\phi_s}\,\frac{dg_0}{d\hat{\tau}}\sum_{p=-\infty}^{p=+\infty}\frac{Z_l(p)}{p}\,J_m(p\,\Omega_0\,\hat{\tau})\,\sigma(p)$$

- Multiplying both sides by $J_m(l\,\Omega_0\,\hat{\tau})$ and integrating over $\hat{\tau}$

$$\Rightarrow \qquad \Delta\omega_{cm}\,\sigma_m(l) = \sum_{p=-\infty}^{p=+\infty} K_{lp}^{m}\,\sigma(p)$$

- Dividing both sides by $\Delta\omega_{cm}^{x}$ and summing over *m*

$$\boxed{\sigma(l) = \varepsilon_{long}\sum_{p=-\infty}^{p=+\infty}\left[\,j\,\frac{Z_l(p)}{p}\right] M_{lp}\,\sigma(p)}$$

with

$$\boxed{M_{lp} = 2\,B \sum_m \frac{m}{\dfrac{\omega_c}{\omega_s} - m} \int_0^1 J_m\left(p\,\pi\,B\,u\right) J_m\left(l\,\pi\,B\,u\right) u\,du}$$

and

$$\boxed{\varepsilon_{long} = \frac{4\,I_b}{\pi^2\,B^3\,\hat{V}_{\mathrm{T}}\,h\cos\phi_s}}$$

- Or, decomposition on the low-intensity modes (following Garnier1987) => GALACLIC

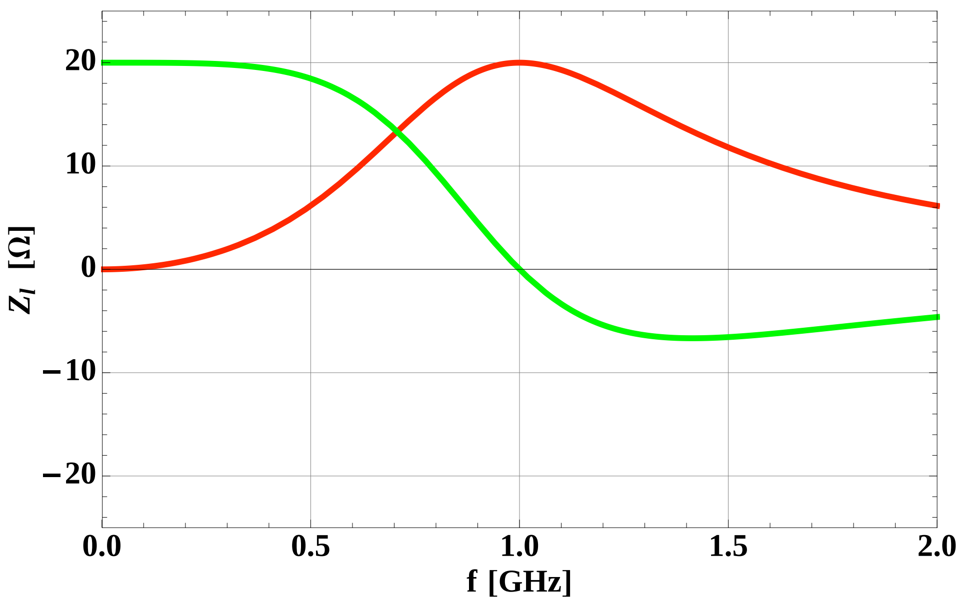
$$x = \frac{\mathrm{Im}\left[\dfrac{Z_l(p)}{p}\right]_{p=0} 4\,I_b}{\pi^2\,B^3\,\hat{V}_T\,h\cos\phi_s}$$

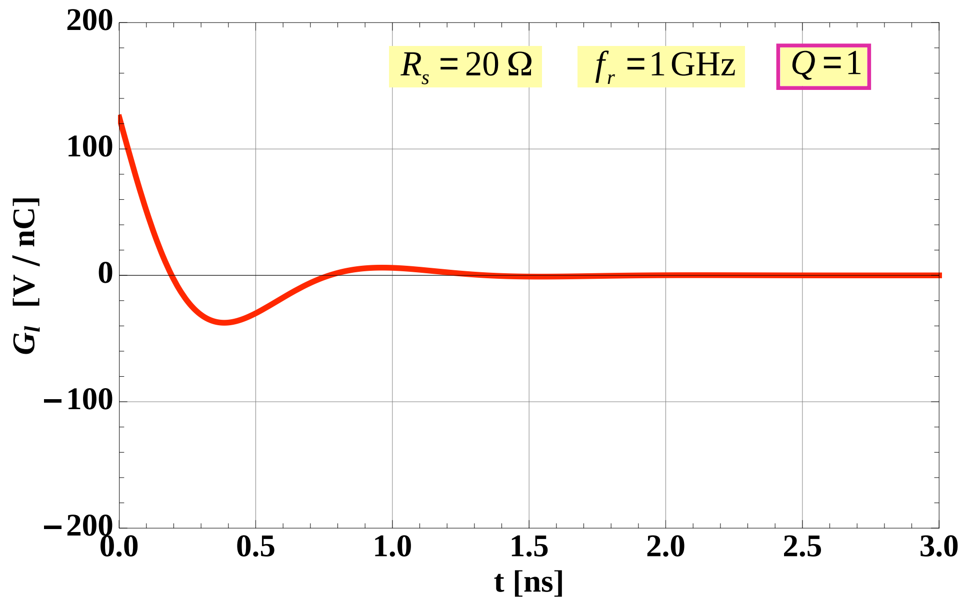
- Check between the 2 methods => **Broad-Band resonator impedance above transition**

$$\overline{\omega}_r = \omega_r \sqrt{1 - \frac{1}{4\,Q^2}} \qquad \alpha = \frac{\omega_r}{2\,Q}$$

$$\boxed{Z_l\,(\omega) = \frac{R_s}{1 + j\,Q\left(\dfrac{\omega}{\omega_r} - \dfrac{\omega_r}{\omega}\right)}}$$

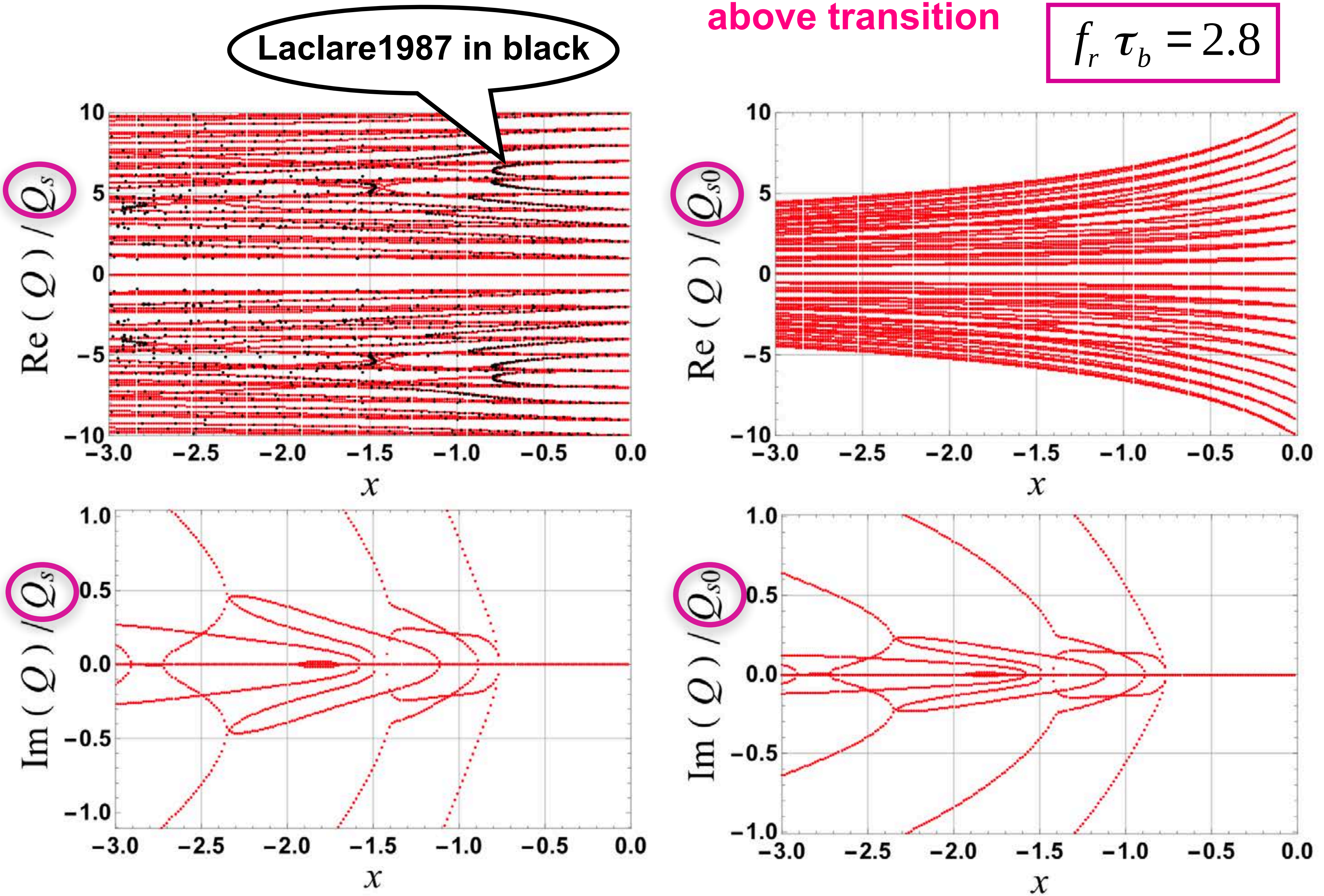
$$\boxed{G_l\,(t) = \frac{\omega_r\,R_s}{Q}\,e^{-\alpha t}\left[\cos\left(\overline{\omega}_r\,t\right) - \frac{\alpha}{\overline{\omega}_r}\sin\left(\overline{\omega}_r\,t\right)\right]}$$



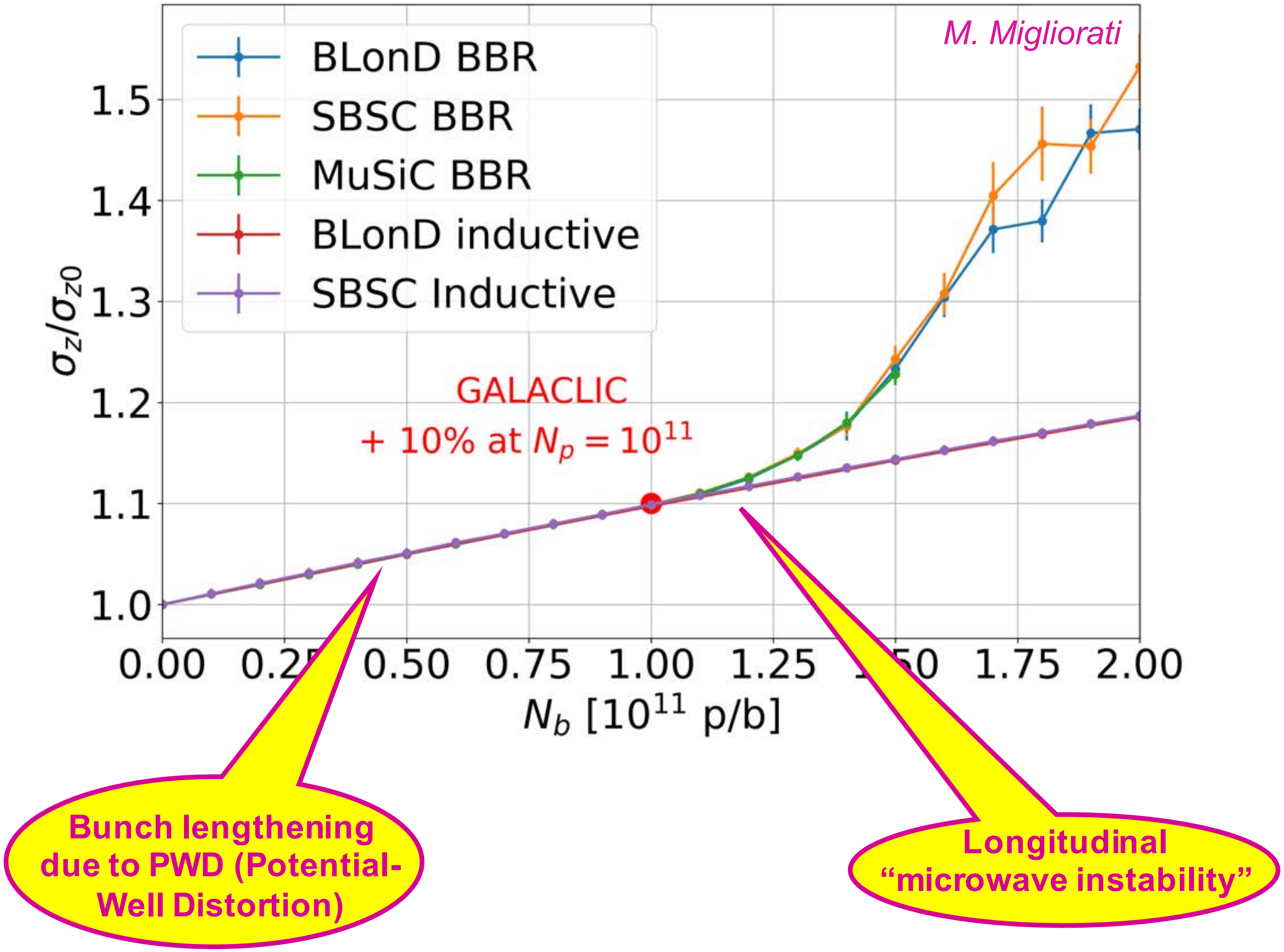


- Check between the 2 methods => **Broad-Band resonator impedance above transition**

$\boxed{f_r \, \tau_b = 2.8}$

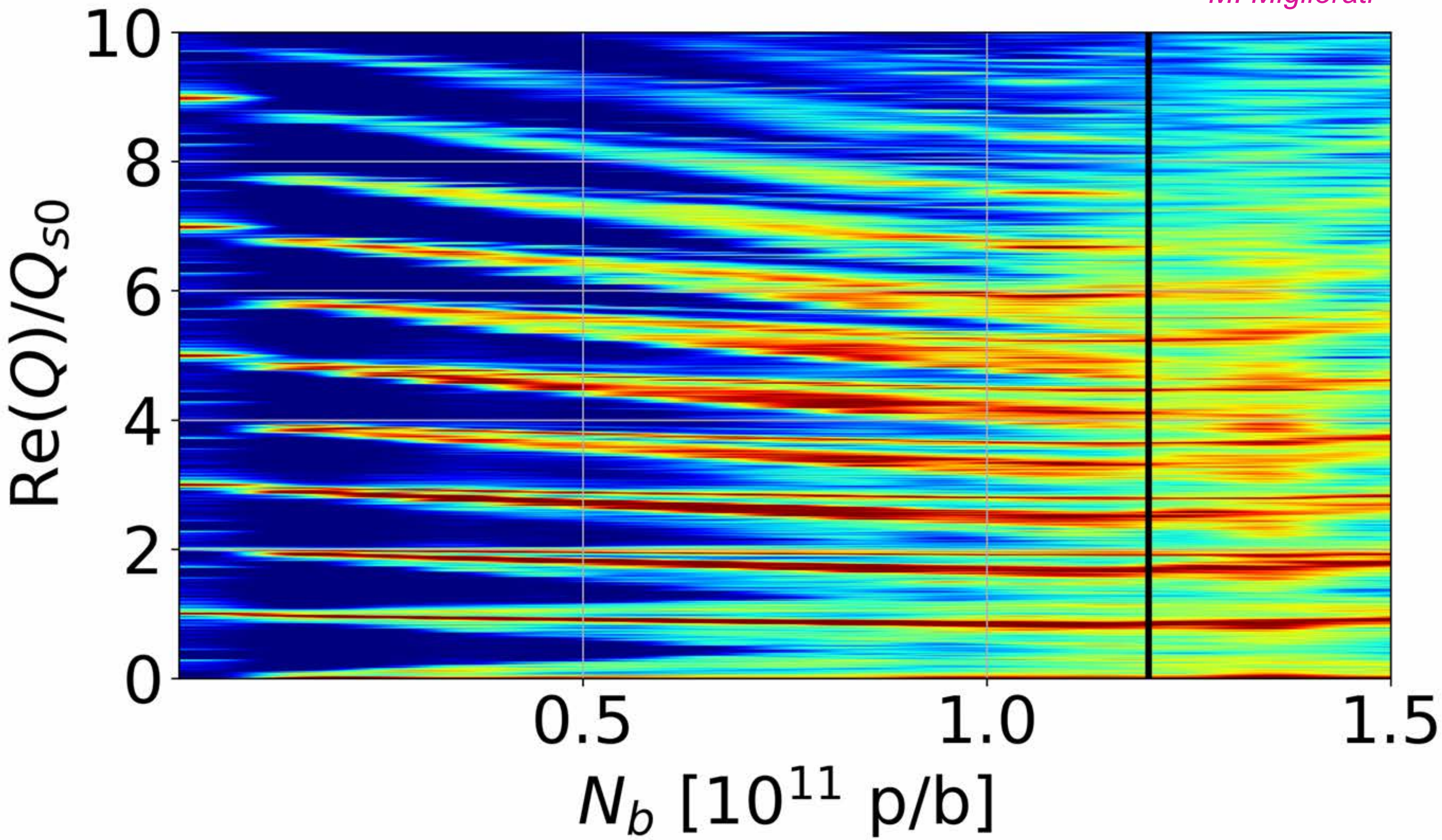


◆ Case of a **Broad-Band resonator impedance** $f_r \, \tau_b = 2.8$ (above transition) => Comparison between macroparticle tracking simulations…



See also IPAC19 paper (https://ipac2019.vrws.de/papers/mopgw089.pdf)

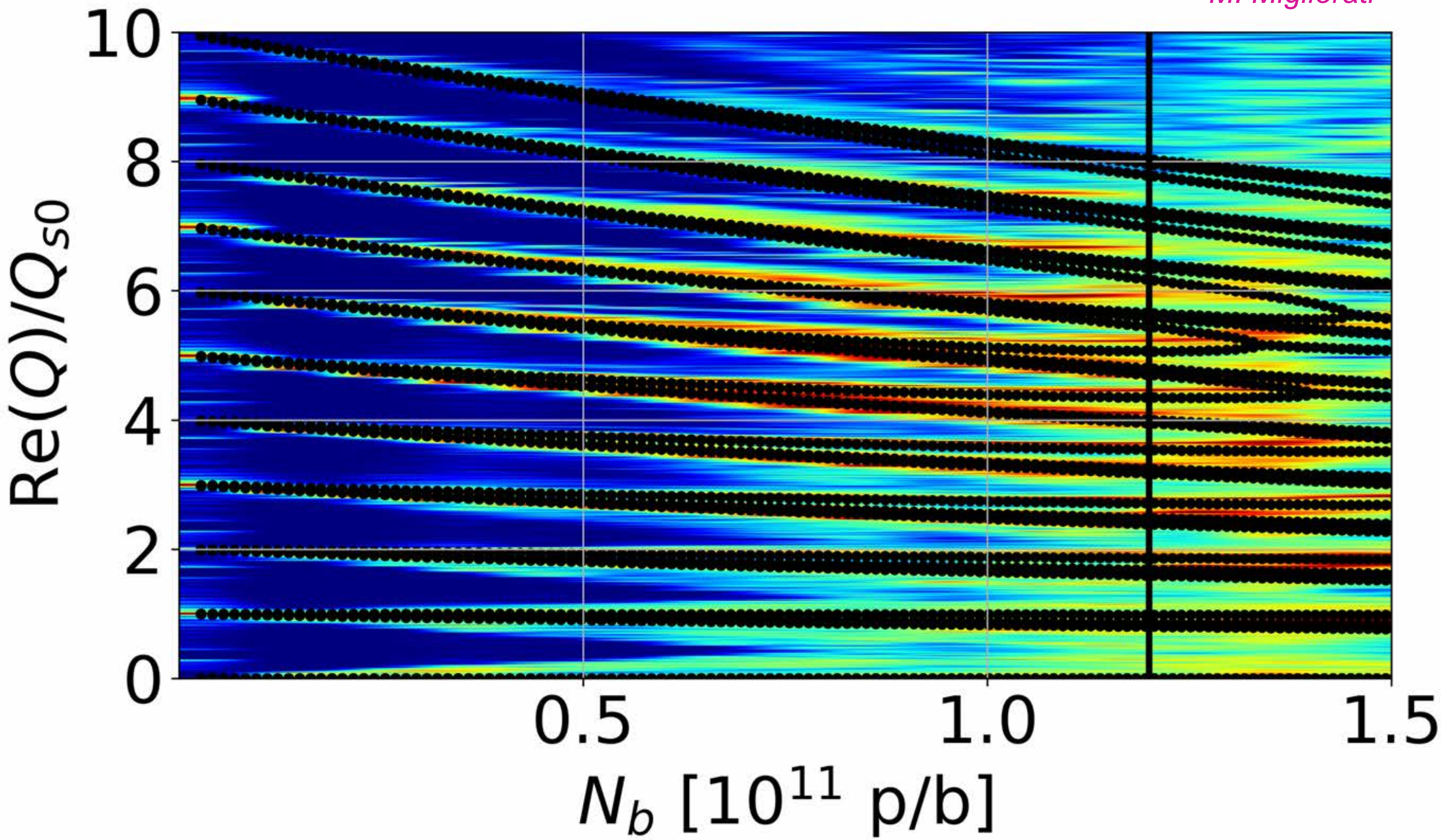
◆ Case of a **Broad-Band resonator impedance** $f_r \, \tau_b = 2.8$ (above transition) => Comparison between macroparticle tracking simulations…



See also IPAC19 paper (https://ipac2019.vrws.de/papers/mopgw089.pdf)

- Case of a **Broad-Band resonator impedance** $f_r \, \tau_b = 2.8$ (above transition) => Comparison between macroparticle tracking simulations and GALACLIC in black

M. Migliorati



See also IPAC19 paper (https://ipac2019.vrws.de/papers/mopgw089.pdf)

- Case of a **Broad-Band resonator impedance** $f_r \, \tau_b = 2.8$

  - The threshold (mode-coupling) is reached when $\left| \varepsilon_{long}^{th} \right| \left| \frac{Z_l(p)}{p} \right|_{p=0} \approx 0.8$

    which can be re-written

$$\boxed{\left(\frac{\Delta p}{p_0} \right)^2_{\text{FWHH}} \geq \frac{10}{3\,\pi} \, \frac{I_{b,peak}}{\beta^2 \left(E_{total} / e \right) |\eta|} \left| \frac{Z_l(p)}{p} \right|_0}$$

using $\quad I_{b,peak} = \frac{3\, I_b}{2\, B} \qquad \left(\frac{\Delta p}{p_0} \right)^2_{\text{FWHH}} = \frac{\omega_s^2 \, \tau_b^2}{2\, \eta^2}$

  - This is the **Keil-Schnell-Boussard criterion** (i.e. the Keil-Schnell criterion for coasting beams applied with peak values for bunched beams as proposed by Boussard). **Note that PWD leads to different thresholds below and above transition**

- Case of a **Broad-Band resonator impedance** $f_r \, \tau_b = 2.8$

  - The threshold (mode-coupling) is reached when $\left| \varepsilon_{long}^{th} \right| \left| \dfrac{Z_l(p)}{p} \right|_{p=0} \approx 0.8$

    which can be re-written

$$\boxed{\left(\frac{\Delta p}{p_0} \right)^2_{\text{FWHH}} \geq \frac{10}{3\pi} \frac{I_{b,peak}}{\beta^2 \left(E_{total} / e \right) |\eta|} \left| \frac{Z_l(p)}{p} \right|_0}$$

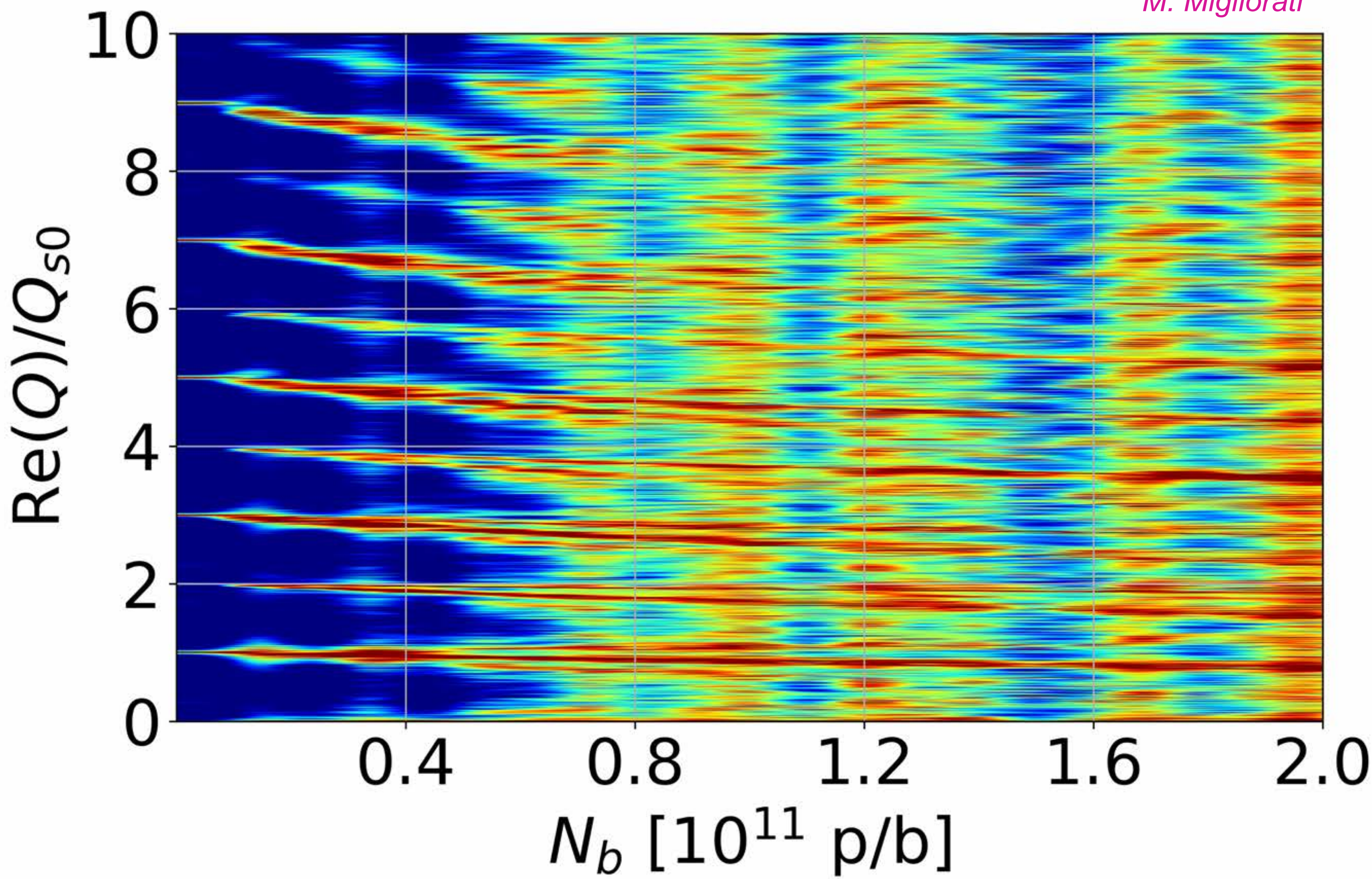
using $$I_{b,peak} = \frac{3 \, I_b}{2 \, B} \qquad \left(\frac{\Delta p}{p_0} \right)^2_{\text{FWHH}} = \frac{\omega_s^2 \, \tau_b^2}{2 \, \eta^2}$$

  - This is the **Keil-Schnell-Boussard criterion** (i.e. the Keil-Schnell criterion for coasting beams applied with peak values for bunched beams as proposed by Boussard). **Note that PWD leads to different thresholds below and above transition**

** No dependence on Q_s!*
*** The same formula can also be obtained by considering only the mode-coupling between the 2 adjacent modes overlapping the maximum of the resonator impedance*

- Case of a **constant inductive impedance** => Comparison between macroparticle tracking simulations…

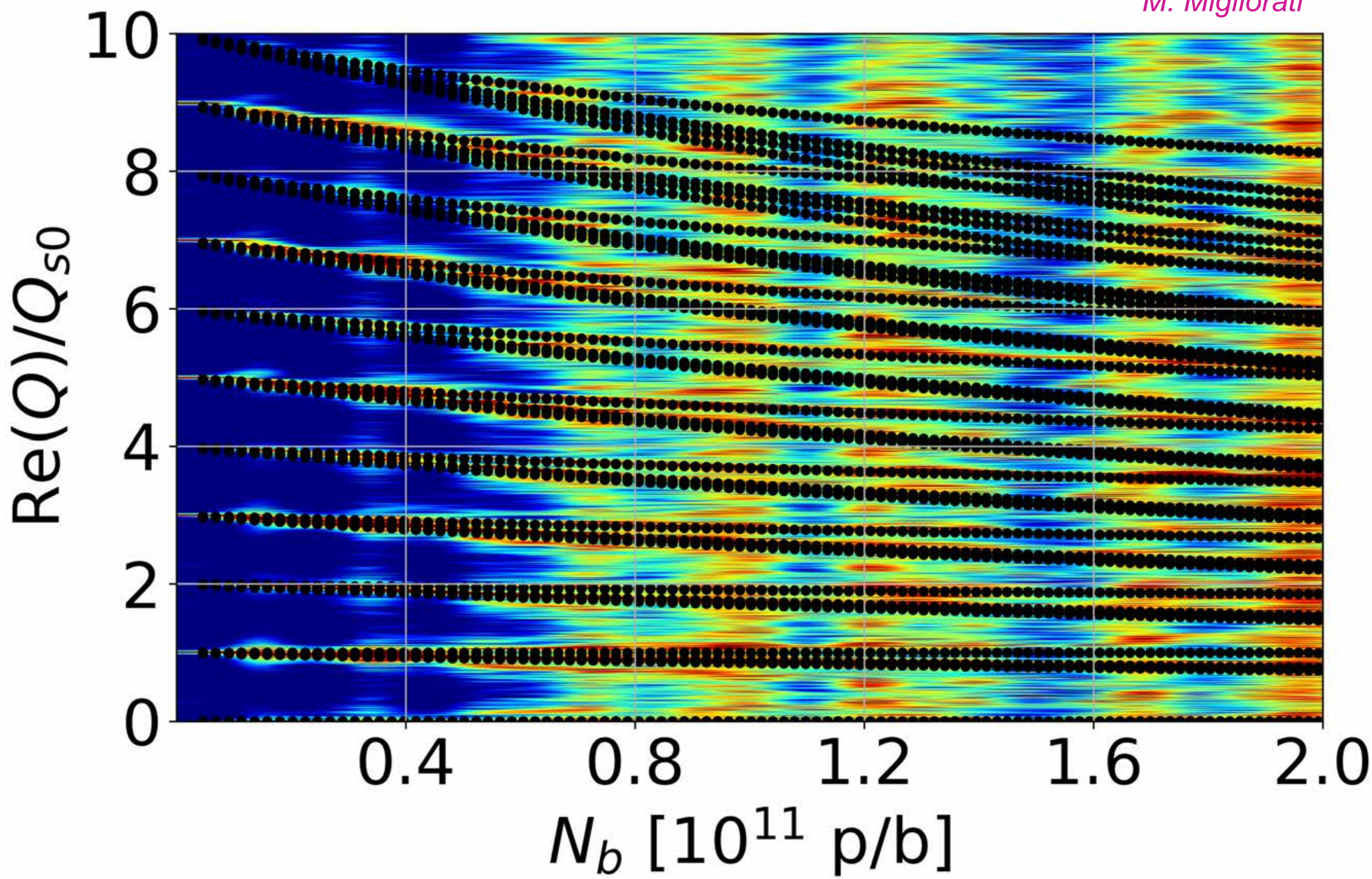
M. Migliorati



See also IPAC19 paper (https://ipac2019.vrws.de/papers/mopgw089.pdf)

◆ Case of a **constant inductive impedance** => Comparison between macroparticle tracking simulations and GALACLIC in black (no instability)

M. Migliorati



See also IPAC19 paper (https://ipac2019.vrws.de/papers/mopgw089.pdf)

CONCLUSION

- **Low-intensity** (for both Longitudinal and Transverse)
  - **Each mode can be treated individually**
  - Eigen-value system to be solved in general
  - Solution can be approximated by Sacherer formula
  - Landau damping used to stabilize these instabilities
    - From the non-linearity of the RF bucket in L
    - From external (controlled) nonlinearities in T: (Landau) octupoles

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- **High-intensity** (for both Longitudinal and Transverse)
  - **The modes cannot be treated independently** => Mode influencing and mode-coupling
  - 2 “new” Vlasov solvers
    - GALAC-TIC in transverse
    - GALAC-LIC in longitudinal
  - In the case of a Broad-Band resonator impedance and in the long-bunch regime, the same formulae as for coasting beams are recovered (using the peak values) in both L and T

- **Good understanding of impedance-induced beam instabilities** **BUT** what is usually missing is a precise model of the machine impedance (e.g.: **huge effort in the CERN SPS machine => Now the transverse impedance model can reproduce very well all the observables!**)

- **Good understanding of impedance-induced beam instabilities BUT** what is usually missing is a precise model of the machine impedance (e.g.: **huge effort in the CERN SPS machine => Now the transverse impedance model can reproduce very well all the observables!**)

- **FURTHERMORE**, in a machine like the LHC, not only all the mechanisms have to be understood separately, but **(ALL) the possible interplays between the different phenomena need to be analyzed in detail as they can play important roles in the beam stability**

  - Linear (Q') and nonlinear (Q") chromaticity
  - Landau octupoles (and other nonlinearities) or RFQs (under study)
  - Transverse damper (using realistic models)
  - Space charge
  - Beam-beam: head-on and long-range
  - Electron cloud
  - Linear coupling strength
  - Tune separation between the transverse planes (bunch by bunch)
  - Tune split between the two beams (bunch by bunch)
  - Transverse beam separation between the two beams
  - Noise, etc.