Combining non perturbative models with the CSS formalism

J. Osvaldo Gonzalez-Hernandez
University of Turin & INFN Turin

Based on: J.O. Gonzalez-Hernandez, T.C. Rogers, N. Sato

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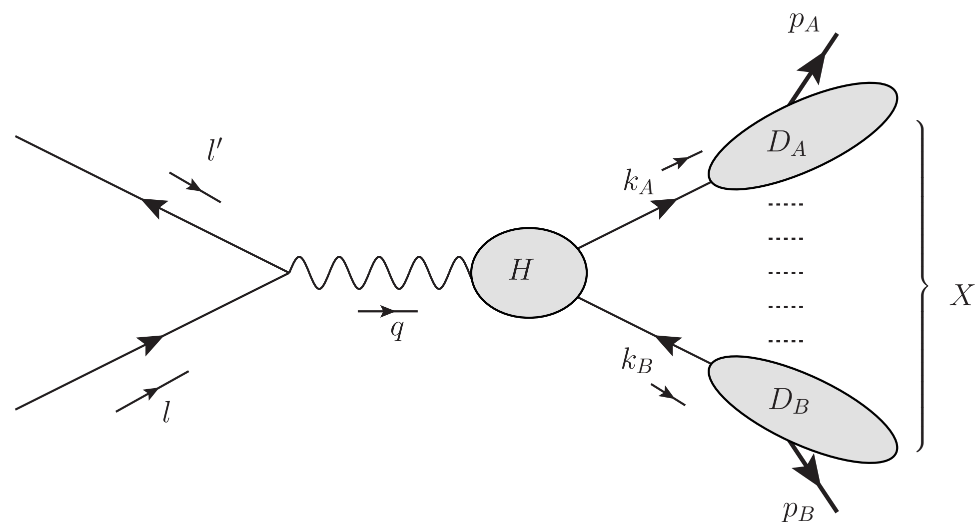
Outlook

- CSS formalism
- Potential issues in phenomenology
- “Bottom-up” approach to phenomenology.
- Final remarks.

CSS formalism

Consider



$$Q^2 \frac{\mathrm{d}\sigma^{A,B}}{\mathrm{d}z_A\,\mathrm{d}z_B\,\mathrm{d}q_\mathrm{T}^2}$$
$$= H_{j\bar{j}}(\mu_Q; C_2) \int \mathrm{d}^2\boldsymbol{k}_{A\mathrm{T}}\ \mathrm{d}^2\boldsymbol{k}_{B\mathrm{T}}\, D_{j/A}\left(z_A, z_A\boldsymbol{k}_{A\mathrm{T}}; \mu_Q, Q^2\right) D_{\bar{j}/B}\left(z_B, z_B\boldsymbol{k}_{B\mathrm{T}}; \mu_Q, Q^2\right) \delta^{(2)}\left(\boldsymbol{q}_\mathrm{T} - \boldsymbol{k}_{A\mathrm{T}} - \boldsymbol{k}_{B\mathrm{T}}\right) +$$
$$Y^{A,B}(q_\mathrm{T}, Q; \mu_Q) + O\left(m/Q\right)\ . \tag{9}$$

Large qT Corrections
important but focus on W for now.

W term

$$W(q_{\mathrm{T}}, Q) = H(\alpha_s(\mu_Q); C_2) \int \frac{\mathrm{d}^2 \boldsymbol{b}_{\mathrm{T}}}{(2\pi)^2} \; e^{-i \boldsymbol{q}_{\mathrm{T}} \cdot \boldsymbol{b}_{\mathrm{T}}} \; \tilde{D}_A(z_A, \boldsymbol{b}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2) \; \tilde{D}_B(z_B, \boldsymbol{b}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2)$$

$$\times \exp\left\{ \tilde{K}(b_{\mathrm{T}}; \mu_{Q_0}) \ln\left(\frac{Q^2}{Q_0^2}\right) + \int_{\mu_{Q_0}}^{\mu_Q} \frac{\mathrm{d}\mu'}{\mu'} \left[2\gamma(\alpha_s(\mu'); 1) - \ln \frac{Q^2}{\mu'^2} \gamma_K(\alpha_s(\mu')) \right] \right\} .$$

At Q0=Q resembles parton model picture

W term (with pQCD constraints from WOPE)

$$W(q_\mathrm{T},Q) = H(\alpha_s(\mu_Q);C_2)\int \frac{\mathrm{d}^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2}\, e^{-i\boldsymbol{q}_\mathrm{T}\cdot\boldsymbol{b}_\mathrm{T}}\ \tilde{D}_A(z_A,\boldsymbol{b}_\mathrm{T};\mu_{Q_0},Q_0^2)\ \tilde{D}_B(z_B,\boldsymbol{b}_\mathrm{T};\mu_{Q_0},Q_0^2)$$
$$\times \exp\left\{\tilde{K}(b_\mathrm{T};\mu_{Q_0})\ln\left(\frac{Q^2}{Q_0^2}\right) + \int_{\mu_{Q_0}}^{\mu_Q}\frac{\mathrm{d}\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu');1) - \ln\frac{Q^2}{\mu'^2}\gamma_K(\alpha_s(\mu'))\right]\right\}\,.$$

WOPE (pQCD)

$$W(q_\mathrm{T},Q) = H(\mu_Q;C_2)\int \frac{\mathrm{d}^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2}\, e^{-i\boldsymbol{q}_\mathrm{T}\cdot\boldsymbol{b}_\mathrm{T}}\,\tilde{D}_A(z_A,\boldsymbol{b}_*;\mu_{b_*},\mu_{b_*}^2)\tilde{D}_B(z_B,\boldsymbol{b}_*;\mu_{b_*},\mu_{b_*}^2)$$
$$\times \exp\left\{2\int_{\mu_{b_*}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu');1) - \ln\frac{Q}{\mu'}\gamma_K(\alpha_s(\mu'))\right] + \ln\frac{Q^2}{\mu_{b_*}^2}\tilde{K}(b_*;\mu_{b_*})\right\}$$
$$\times \exp\left\{-g_A(z_A,b_\mathrm{T}) - g_B(z_B,b_\mathrm{T}) - g_K(b_\mathrm{T})\ln\left(\frac{Q^2}{Q_0^2}\right)\right\}\,.$$

Further step: Constraints to small bT behaviour

$$W(q_{\rm T},Q) = H(\alpha_s(\mu_Q);C_2)\int \frac{{\rm d}^2\boldsymbol{b}_{\rm T}}{(2\pi)^2}\; e^{-i\boldsymbol{q}_{\rm T}\cdot\boldsymbol{b}_{\rm T}}\; \tilde{D}_A(z_A,\boldsymbol{b}_{\rm T};\mu_{Q_0},Q_0^2)\;\tilde{D}_B(z_B,\boldsymbol{b}_{\rm T};\mu_{Q_0},Q_0^2)$$
$$\times \exp\left\{\tilde{K}(b_{\rm T};\mu_{Q_0})\ln\left(\frac{Q^2}{Q_0^2}\right)+\int_{\mu_{Q_0}}^{\mu_Q}\frac{{\rm d}\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu');1)-\ln\frac{Q^2}{\mu'^2}\gamma_K(\alpha_s(\mu'))\right]\right\}\,.$$

$$W(q_{\rm T},Q) = H(\mu_Q;C_2)\int \frac{{\rm d}^2\boldsymbol{b}_{\rm T}}{(2\pi)^2}\; e^{-i\boldsymbol{q}_{\rm T}\cdot\boldsymbol{b}_{\rm T}}\,\tilde{D}_A(z_A,\boldsymbol{b}_*;\mu_{b_*},\mu_{b_*}^2)\tilde{D}_B(z_B,\boldsymbol{b}_*;\mu_{b_*},\mu_{b_*}^2)$$
$$\times \exp\left\{2\int_{\mu_{b_*}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu');1)-\ln\frac{Q}{\mu'}\gamma_K(\alpha_s(\mu'))\right]+\ln\frac{Q^2}{\mu_{b_*}^2}\tilde{K}(b_*;\mu_{b_*})\right\}$$
$$\times \exp\left\{-g_A(z_A,b_{\rm T})-g_B(z_B,b_{\rm T})-g_K(b_{\rm T})\ln\left(\frac{Q^2}{Q_0^2}\right)\right\}\,.$$

Models characterizing nonperturbative behaviour

Further step: Constraints to small bT behaviour

$$W(q_{\mathrm{T}},Q) = H(\alpha_s(\mu_Q);C_2)\int \frac{\mathrm{d}^2\boldsymbol{b}_{\mathrm{T}}}{(2\pi)^2}\; e^{-i\boldsymbol{q}_{\mathrm{T}}\cdot\boldsymbol{b}_{\mathrm{T}}}\;\tilde{D}_A(z_A,\boldsymbol{b}_{\mathrm{T}};\mu_{Q_0},Q_0^2)\;\tilde{D}_B(z_B,\boldsymbol{b}_{\mathrm{T}};\mu_{Q_0},Q_0^2)$$
$$\times\exp\left\{\tilde{K}(b_{\mathrm{T}};\mu_{Q_0})\ln\left(\frac{Q^2}{Q_0^2}\right)+\int_{\mu_{Q_0}}^{\mu_Q}\frac{\mathrm{d}\mu'}{\mu'}\left[2\gamma(\alpha_s(\mu');1)-\ln\frac{Q^2}{\mu'^2}\gamma_K(\alpha_s(\mu'))\right]\right\}.$$

$$W(q_{\mathrm{T}},Q) = H(\mu_Q;C_2)\int \frac{\mathrm{d}^2\boldsymbol{b}_{\mathrm{T}}}{(2\pi)^2}\; e^{-i\boldsymbol{q}_{\mathrm{T}}\cdot\boldsymbol{b}_{\mathrm{T}}}\tilde{D}_A(z_A,\boldsymbol{b}_*;\mu_{b_*},\mu_{b_*}^2)\tilde{D}_B(z_B,\boldsymbol{b}_*;\mu_{b_*},\mu_{b_*}^2)$$
$$\times\exp\left\{2\int_{\mu_{b_*}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[\gamma(\alpha_s(\mu');1)-\ln\frac{Q}{\mu'}\gamma_K(\alpha_s(\mu'))\right]+\ln\frac{Q^2}{\mu_{b_*}^2}\tilde{K}(b_*;\mu_{b_*})\right\}$$
$$\times\exp\left\{-g_A(z_A,b_{\mathrm{T}})-g_B(z_B,b_{\mathrm{T}})-g_K(b_{\mathrm{T}})\ln\left(\frac{Q^2}{Q_0^2}\right)\right\}.$$

Transition from small to large bT

$$\boldsymbol{b}_*(b_{\mathrm{T}}) = \frac{\boldsymbol{b}_{\mathrm{T}}}{\sqrt{1+b_{\mathrm{T}}^2/b_{\max}^2}}.$$

Scale setting in the OPE

$$\mu_{b_*}\equiv C_1/b_*.$$

Exact definition of W does not depend on the shape of b* nor on the value of bmax

$$g_K(b_{\rm T}) \equiv \tilde{K}(b_*, \mu) - \tilde{K}(b_{\rm T}, \mu)$$

$$-g_A(z, \boldsymbol{b}_{\rm T}) \equiv \ln \left(\frac{\tilde{D}_A(z, \boldsymbol{b}_{\rm T}; \mu_{Q_0}, Q_0^2)}{\tilde{D}_A(z, \boldsymbol{b}_*; \mu_{Q_0}, Q_0^2)} \right)$$

Exact definition of W does not depend on the shape of b* nor on the value of bmax

$$g_K(b_T) \equiv \tilde{K}(b_*, \mu) - \tilde{K}(b_T, \mu)$$

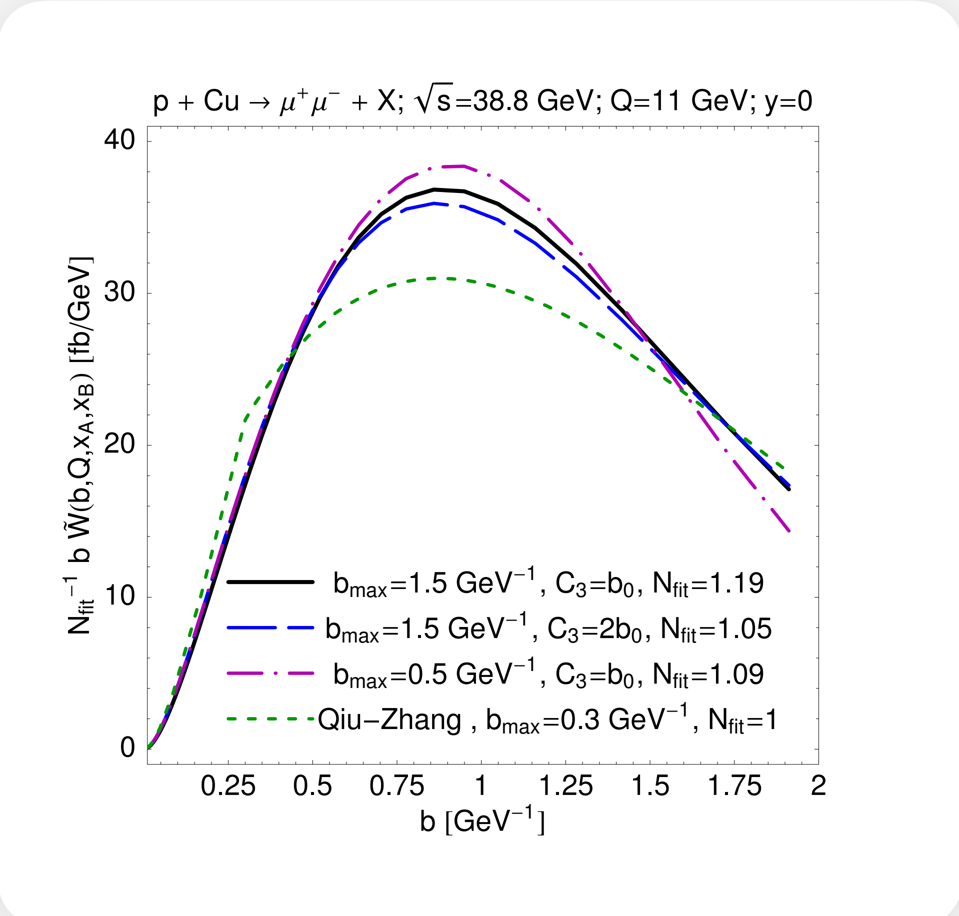
$$-g_A(z, \boldsymbol{b}_T) \equiv \ln\left(\frac{\tilde{D}_A(z, \boldsymbol{b}_T; \mu_{Q_0}, Q_0^2)}{\tilde{D}_A(z, \boldsymbol{b}_*; \mu_{Q_0}, Q_0^2)}\right)$$

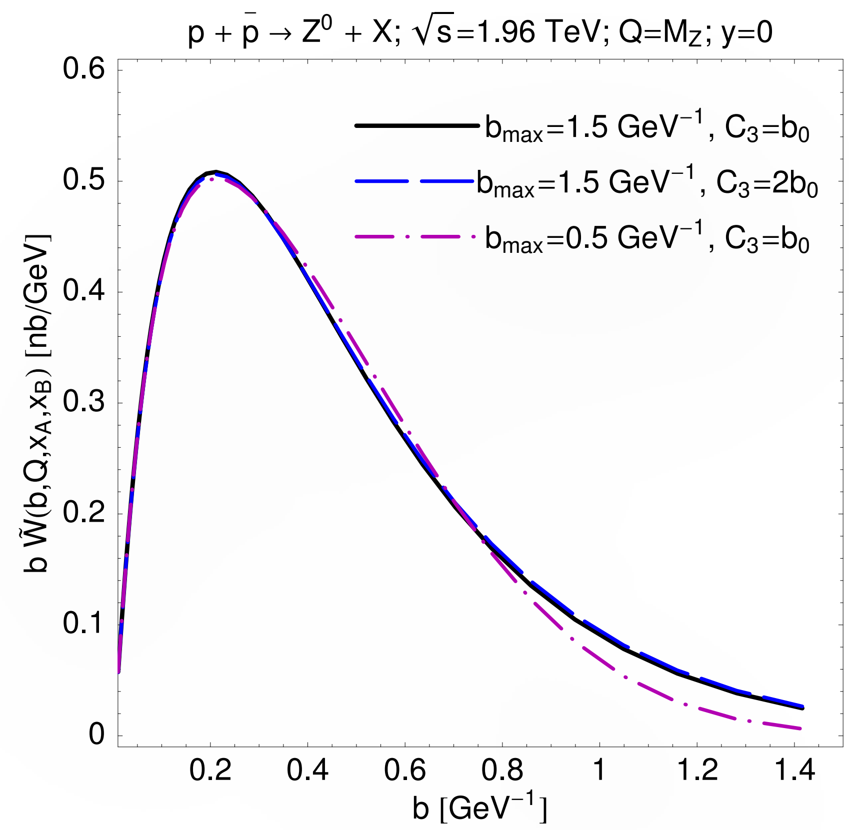
When modelling g-functions, should only allow for mild dependence on b* and bmax

Potential issues in phenomenology

At lower energies, more sensitivity to b*, bmax

Note that this dependence is due to a lack of constraints on g-functions

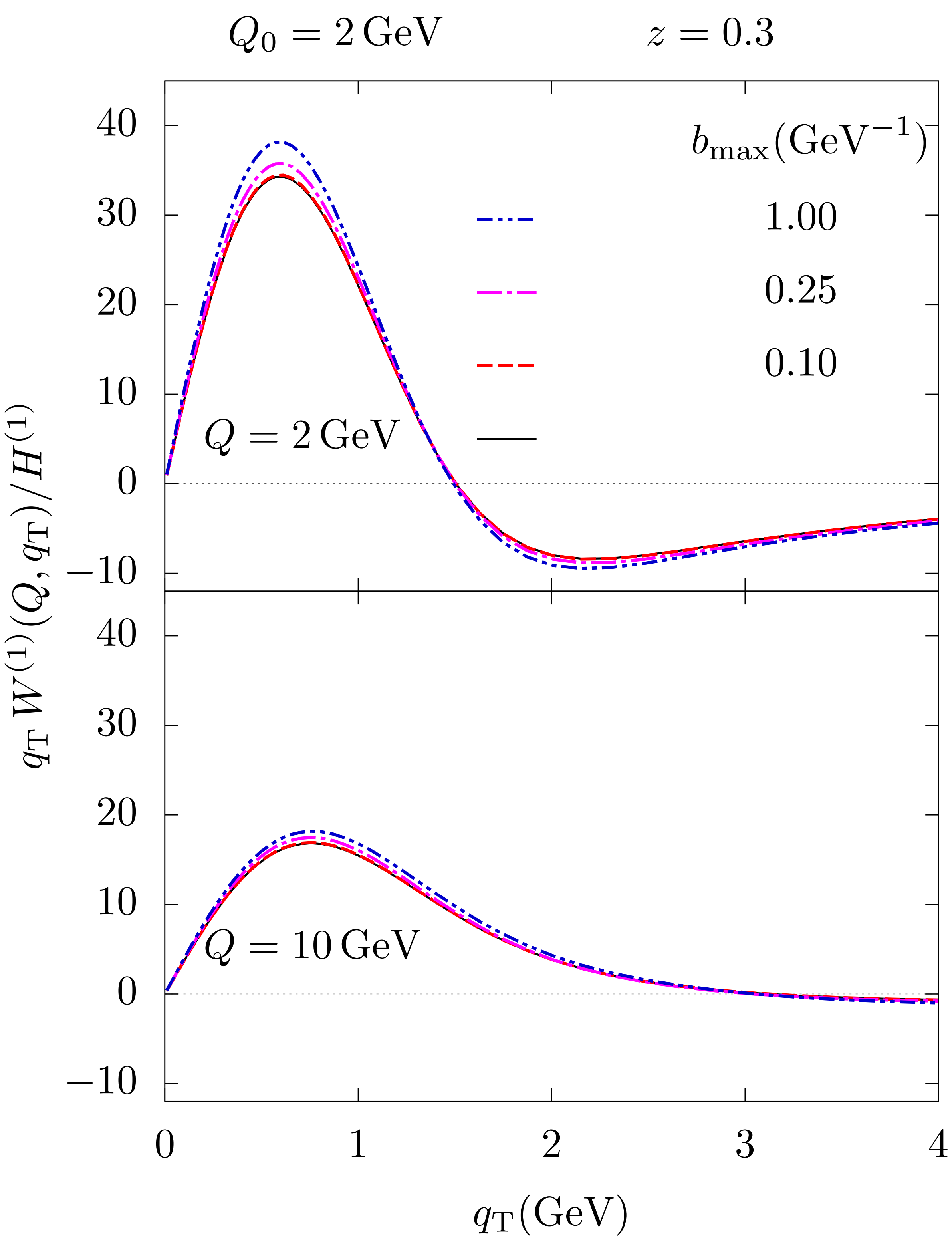




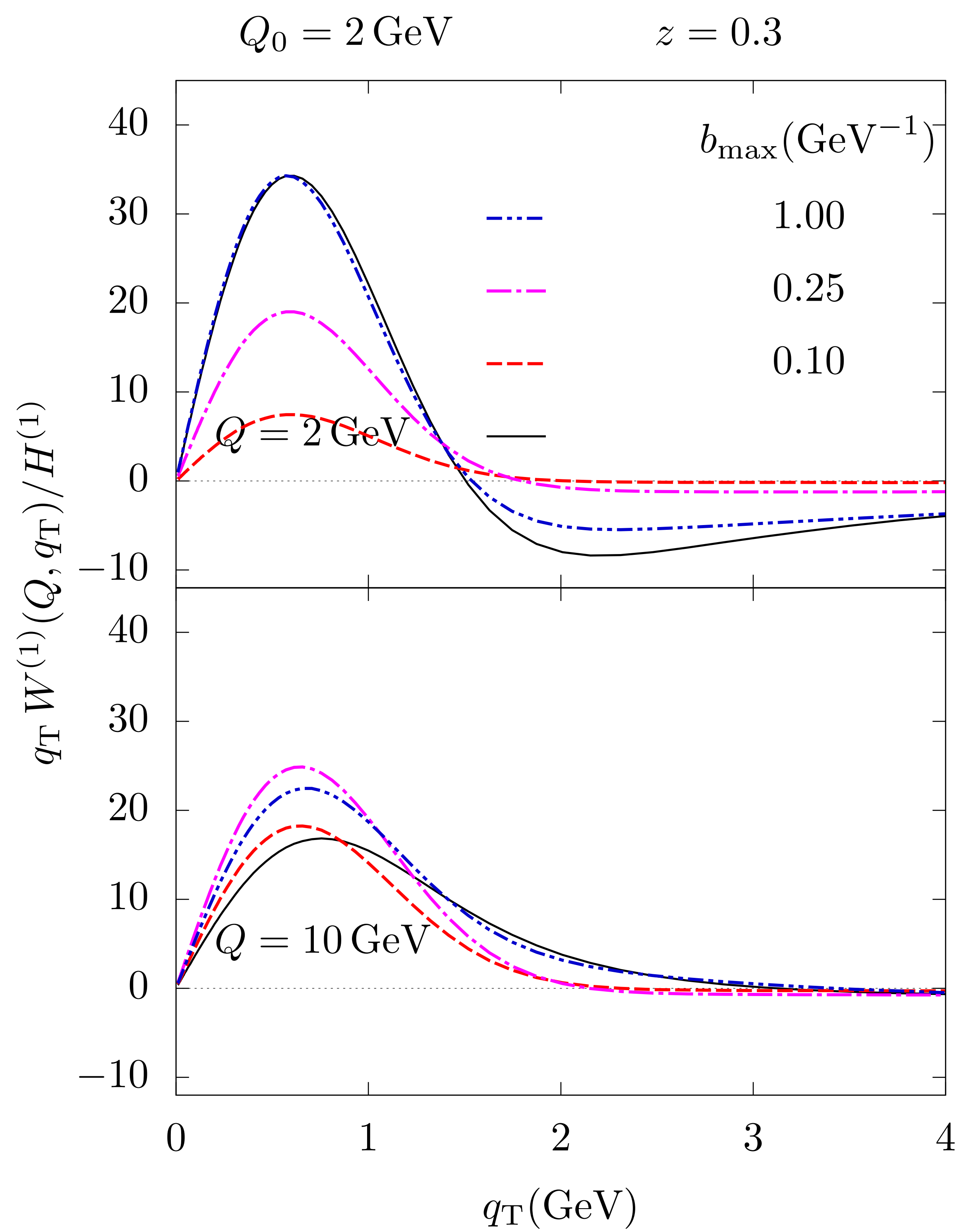
A. V. Konychev and P. M. Nadolsky, Phys. Lett. B633, 710 (2006), arXiv:hep-ph/0506225

In both cases, using WOPE

Example: e+e- —> h h



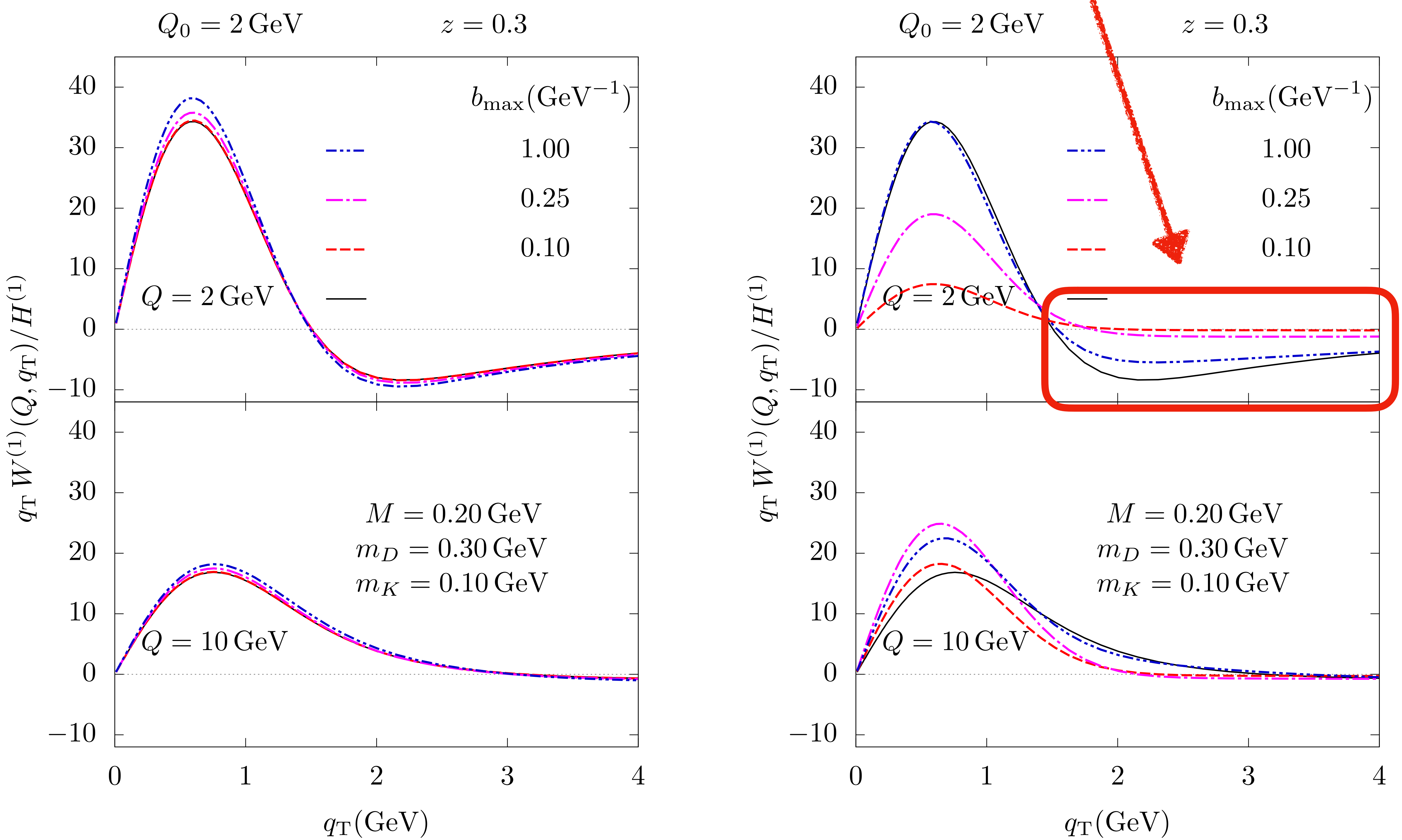
Explicitly constraining g-functions



Unconstrained g-functions

WOPE not enough to constraint large qT (small bT)

Example: e+e- —> h h



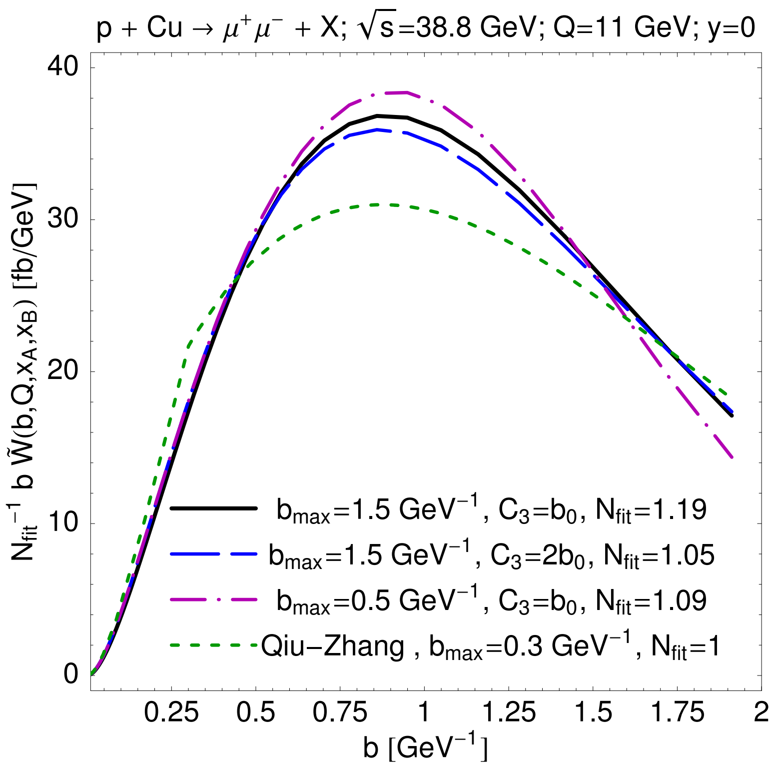
Explicitly constraining g-functions

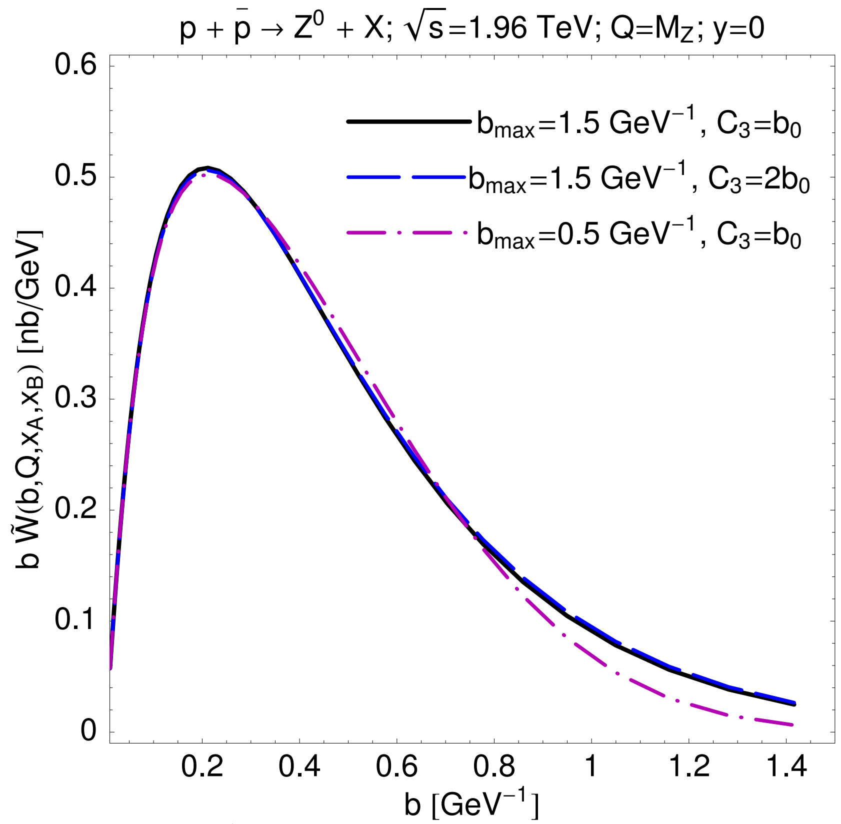
Unconstrained g-functions

Another important consideration

Consider sensitivity of bT at different energy scales

A. V. Konychev and P. M. Nadolsky, Phys. Lett. B633, 710 (2006), arXiv:hep-ph/0506225





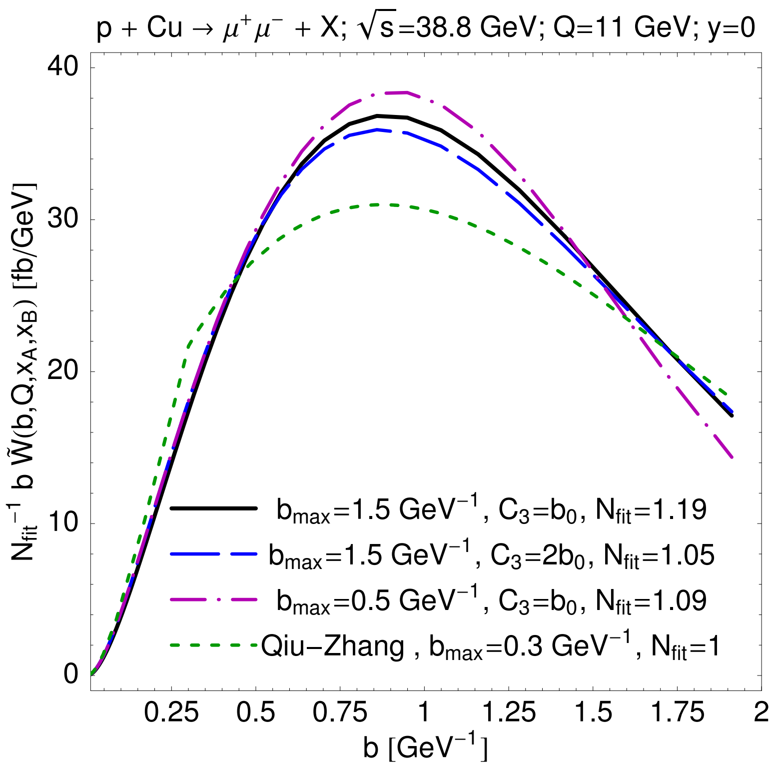
Access to larger bT

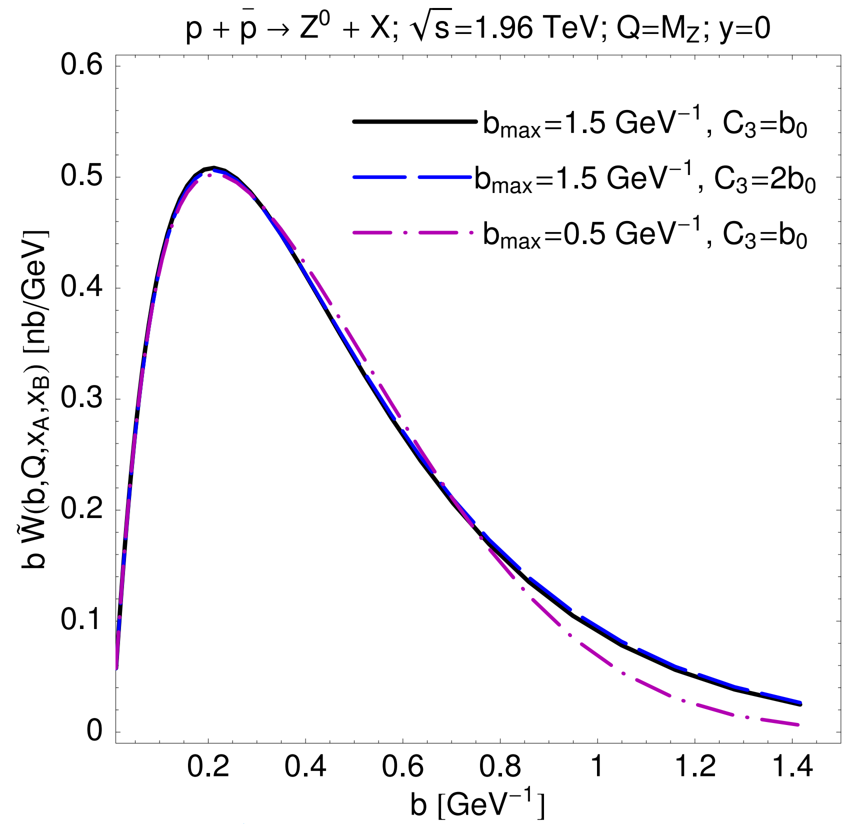
Small bT dominated

Backward evolution poorly Constrained

Recall TMD evolution involves a Fourier transform, thus, knowledge of full bT range needed

A. V. Konychev and P. M. Nadolsky, Phys. Lett. B633, 710 (2006), arXiv:hep-ph/0506225





Access to larger bT

Small bT dominated

Forward evolution
More stable

A good strategy for pheno is to look at smaller energy scale observables (more information on long distance behaviour)

$$W(q_{\mathrm{T}}, Q) = H(\mu_Q; C_2) \int \frac{\mathrm{d}^2\boldsymbol{b}_{\mathrm{T}}}{(2\pi)^2} \; e^{-i\boldsymbol{q}_{\mathrm{T}}\cdot\boldsymbol{b}_{\mathrm{T}}} \tilde{D}_A(z_A, \boldsymbol{b}_*; \mu_{b_*}, \mu_{b_*}^2) \tilde{D}_B(z_B, \boldsymbol{b}_*; \mu_{b_*}, \mu_{b_*}^2)$$
$$\times \exp\left\{ 2\int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[\gamma(\alpha_s(\mu'); 1) - \ln\frac{Q}{\mu'}\gamma_K(\alpha_s(\mu')) \right] + \ln\frac{Q^2}{\mu_{b_*}^2} \tilde{K}(b_*; \mu_{b_*}) \right\}$$
$$\times \exp\left\{ -g_A(z_A, b_{\mathrm{T}}) - g_B(z_B, b_{\mathrm{T}}) - g_K(b_{\mathrm{T}}) \ln\left(\frac{Q^2}{Q_0^2}\right) \right\} .$$

Transition from small to large bT

$$\boldsymbol{b}_*(b_{\mathrm{T}}) = \frac{\boldsymbol{b}_{\mathrm{T}}}{\sqrt{1 + b_{\mathrm{T}}^2/b_{\max}^2}} .$$

Scale setting in the OPE

$$\mu_{b_*} \equiv C_1/b_* .$$

A good strategy for pheno is to look at smaller energy scale observables (more information on long distance behaviour)

+

Must ensure models smoothly transition from small bT (predicted by pQCD) to large bT

$$W(q_{\mathrm{T}}, Q) = H(\mu_Q; C_2) \int \frac{\mathrm{d}^2 \boldsymbol{b}_{\mathrm{T}}}{(2\pi)^2} \, e^{-i \boldsymbol{q}_{\mathrm{T}} \cdot \boldsymbol{b}_{\mathrm{T}}} \tilde{D}_A(z_A, \boldsymbol{b}_*; \mu_{b_*}, \mu_{b_*}^2) \tilde{D}_B(z_B, \boldsymbol{b}_*; \mu_{b_*}, \mu_{b_*}^2)$$
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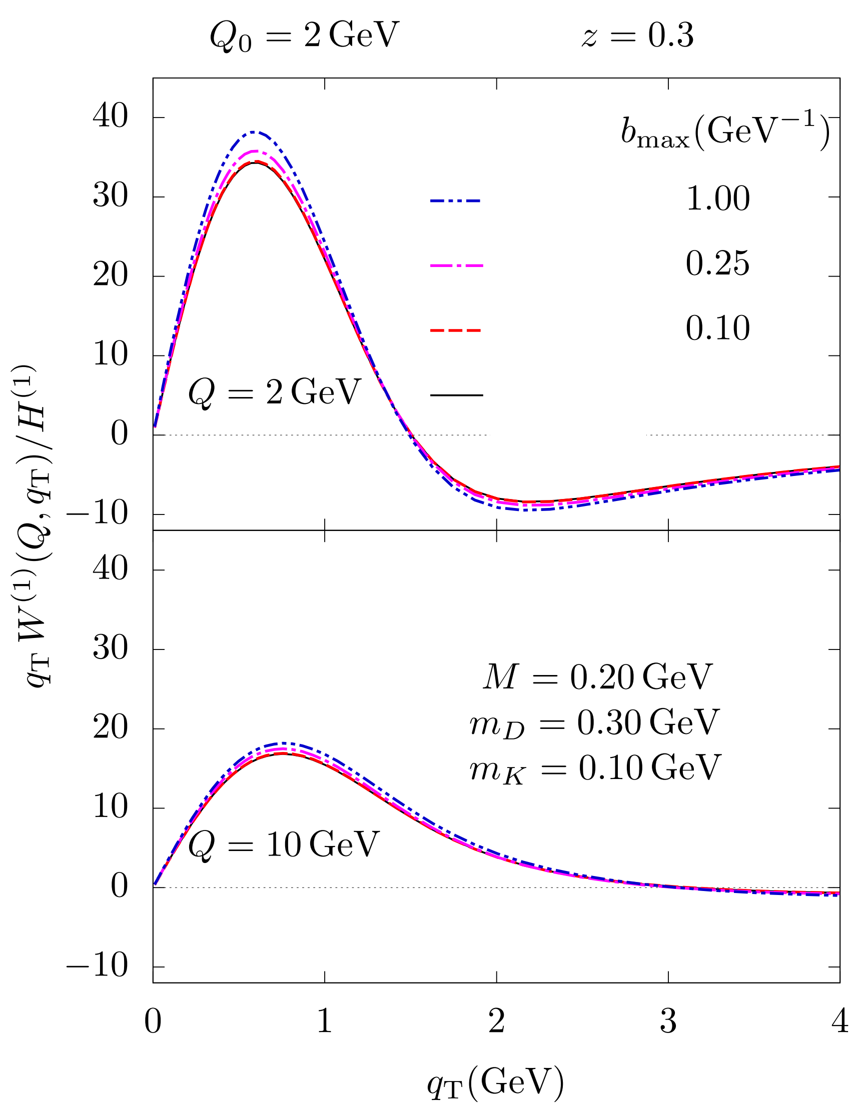


Integral relation very important:

$$d_c(z;\mu_Q) \equiv 2\pi z^2 \int_0^{\mu_Q} \mathrm{d}k_T\, k_T D(z, z\boldsymbol{k}_T; \mu_Q, Q^2)$$

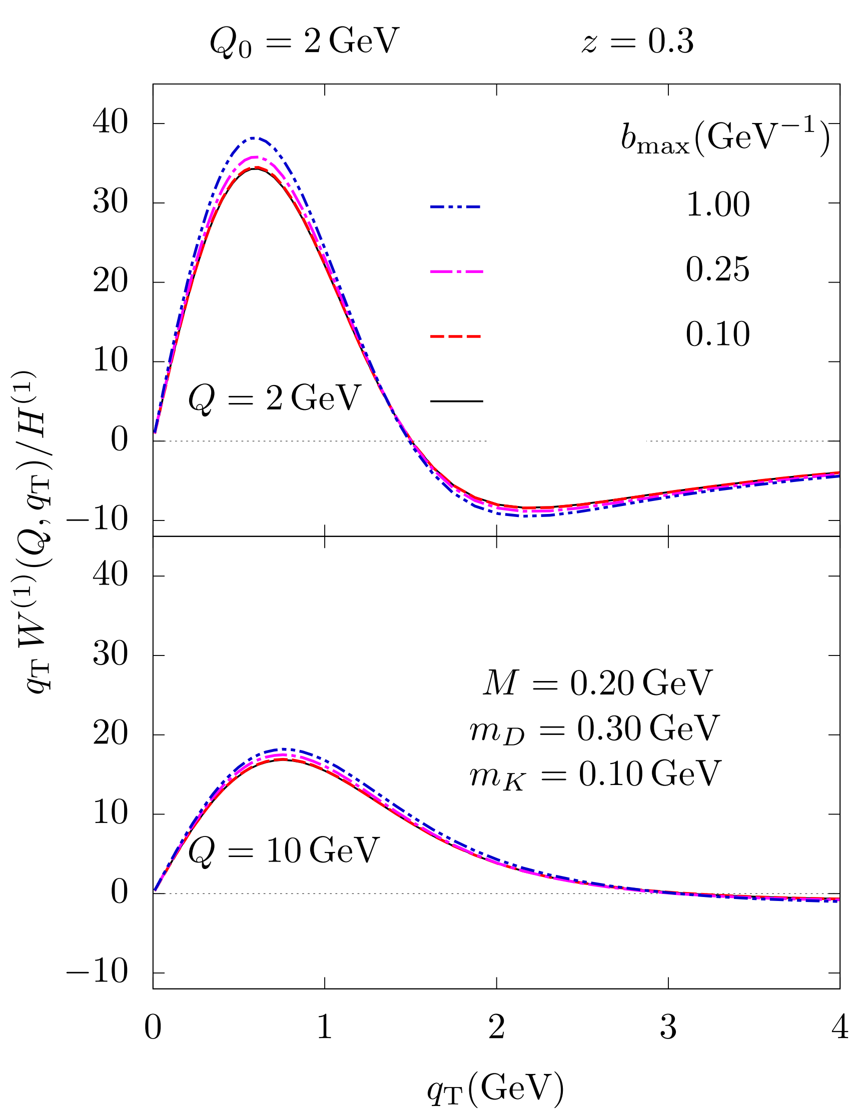
$$2\pi z^2 \int_0^{\mu_Q} \mathrm{d}k_T\, k_T D(z, z\boldsymbol{k}_T; \mu_Q, Q^2) = d_r(z;\mu_Q) + \Delta^{(n,d_r)}(\alpha_s(\mu_Q)) + O\left(\frac{m}{Q}, \alpha_s(\mu_Q)^{n+1}\right),$$

We used these in obtaining Fig. on the left

A good strategy for pheno is to look at smaller energy scale observables (more information on long distance behaviour)

+

Must ensure models smoothly transition from small bT (predicted by pQCD) to large bT



Integral relation very important:

$$d_c(z;\mu_Q) \equiv 2\pi z^2 \int_0^{\mu_Q} \mathrm{d}k_\mathrm{T}\, k_\mathrm{T} D(z, z\boldsymbol{k}_\mathrm{T};\mu_Q, Q^2)$$

$$2\pi z^2 \int_0^{\mu_Q} \mathrm{d}k_\mathrm{T}\, k_\mathrm{T} D(z, z\boldsymbol{k}_\mathrm{T};\mu_Q, Q^2) = d_r(z;\mu_Q) + \Delta^{(n,d_r)}(\alpha_s(\mu_Q)) + O\left(\frac{m}{Q}, \alpha_s(\mu_Q)^{n+1}\right),$$

These corrections may be important at moderate energy scales

A good strategy for pheno is to look at
smaller energy scale observables
(more information on long distance behaviour)

+

Must ensure models smoothly transition from
small bT (predicted by pQCD) to large bT

This motivates our
“bottom-up” approach

bottom-up approach

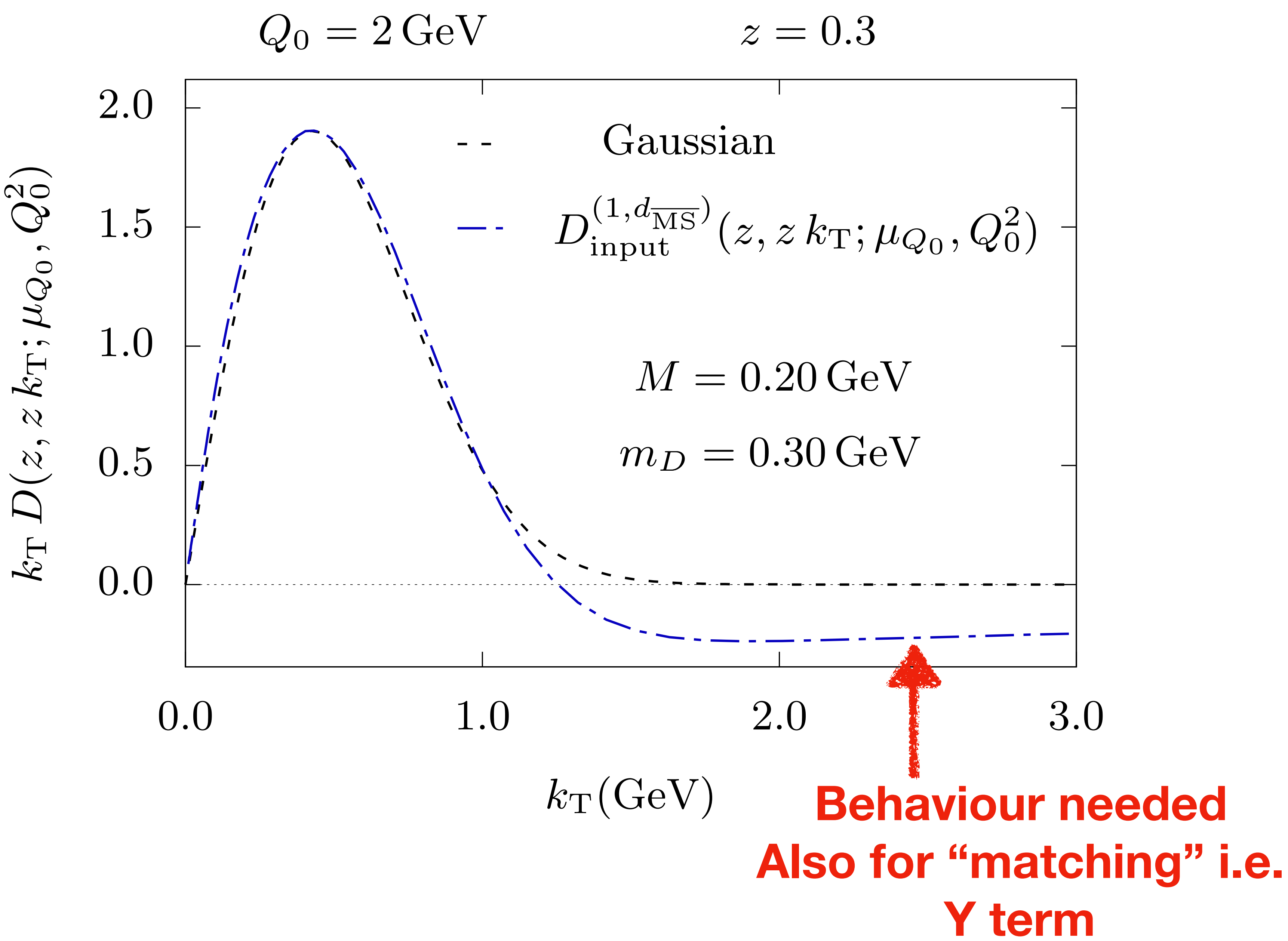
Work with this form of CSS

$$W(q_{\mathrm{T}}, Q) = H(\alpha_s(\mu_Q); C_2) \int \frac{\mathrm{d}^2 \boldsymbol{b}_{\mathrm{T}}}{(2\pi)^2} \, e^{-i \boldsymbol{q}_{\mathrm{T}} \cdot \boldsymbol{b}_{\mathrm{T}}} \, \tilde{D}_A(z_A, \boldsymbol{b}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2) \, \tilde{D}_B(z_B, \boldsymbol{b}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2)$$
$$\times \exp\left\{ \tilde{K}(b_{\mathrm{T}}; \mu_{Q_0}) \ln\left(\frac{Q^2}{Q_0^2}\right) + \int_{\mu_{Q_0}}^{\mu_Q} \frac{\mathrm{d}\mu'}{\mu'} \left[2\gamma(\alpha_s(\mu'); 1) - \ln \frac{Q^2}{\mu'^2} \gamma_K(\alpha_s(\mu')) \right] \right\} .$$

But work in momentum space when possible

model building

- **Choose models for smallest scale Q_0 at which factorization is trusted & constrain models using pQCD at kT~Q, Integral relation, etc.**



- **Choose models for smallest scale Q_0 at which factorization is trusted & constrain models using pQCD at kT~Q, Integral relation, etc.**

$$D_{\rm input}^{(1,d_r)}(z, z\boldsymbol{k}_{\rm T}; \mu_{Q_0}, Q_0^2) = \frac{1}{2\pi z^2}\frac{1}{k_{\rm T}^2 + m_D^2}\left[A^{(d_r)}(z;\mu_{Q_0}) + B^{(d_r)}(z;\mu_{Q_0})\ln\frac{Q_0^2}{k_{\rm T}^2 + m_D^2}\right] + \frac{C^{(d_r)}}{\pi M^2} e^{-z^2 k_{\rm T}^2/M^2}$$

$$K_{\rm input}^{(1)}(k_{\rm T};\mu_{Q_0}) = \frac{\alpha_s(\mu_{Q_0})C_F}{\pi^2}\frac{1}{k_{\rm T}^2 + m_K^2} + C_K\delta^{(2)}(\boldsymbol{k}_{\rm T})\,.$$

$$C_K = \frac{2\alpha_s(\mu_{Q_0})C_F}{\pi}\ln\left(\frac{m_K}{\mu_{Q_0}}\right)$$

$$
\begin{aligned}
&D_{\text{input}}^{(1,d_r)}(z, z\boldsymbol{k}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2) \\
&= \frac{1}{2\pi z^2} \frac{1}{k_{\mathrm{T}}^2 + m_D^2} \Bigg[A^{(d_r)}(z; \mu_{Q_0}) \\
&+ B^{(d_r)}(z; \mu_{Q_0}) \ln \frac{Q_0^2}{k_{\mathrm{T}}^2 + m_D^2} \Bigg] + \frac{C^{(d_r)}}{\pi M^2} e^{-z^2 k_{\mathrm{T}}^2 / M^2}
\end{aligned}
$$

↑

Pheno model: here a gaussian but any other model to be tested can go here

$$D_{\rm input}^{(1,d_r)}(z, z\boldsymbol{k}_{\rm T}; \mu_{Q_0}, Q_0^2)$$
$$= \frac{1}{2\pi z^2} \frac{1}{k_{\rm T}^2 + m_D^2} \left[A^{(d_r)}(z; \mu_{Q_0}) \right.$$
$$\left. + B^{(d_r)}(z; \mu_{Q_0}) \ln \frac{Q_0^2}{k_{\rm T}^2 + m_D^2} \right] + \frac{C^{(d_r)}}{\pi M^2} e^{-z^2 k_{\rm T}^2/M^2}$$

Constraints for kT ~Q0
Depend on collinear function

$$A^{(d_r)}(z; \mu) \equiv \frac{\alpha_s(\mu)}{\pi} \left\{ [(P_{qq} \otimes d_r)(z; \mu)] \right.$$
$$\left. - \frac{3C_F}{2} d_r(z; \mu) \right\} ,$$
$$B^{(d_r)}(z; \mu) \equiv \frac{\alpha_s(\mu) C_F}{\pi} d_r(z; \mu) .$$

Related to OPE in usual presentation Of CSS formula

$$
\begin{aligned}
&D^{(1,d_r)}_{\text{input}}(z, z\boldsymbol{k}_{\text{T}}; \mu_{Q_0}, Q_0^2) \\
&= \frac{1}{2\pi z^2}\frac{1}{k_{\text{T}}^2 + m_D^2}\Big[A^{(d_r)}(z;\mu_{Q_0}) \\
&+ B^{(d_r)}(z;\mu_{Q_0}) \ln\frac{Q_0^2}{k_{\text{T}}^2 + m_D^2}\Big] + \frac{C^{(d_r)}}{\pi M^2} e^{-z^2 k_{\text{T}}^2/M^2}
\end{aligned}
$$

Integral relation

$$
\begin{aligned}
&2\pi z^2 \int_0^{\mu_{Q_0}} \mathrm{d}k_{\text{T}}\, k_{\text{T}} D^{(n,d_r)}_{\text{input}}(z, z\boldsymbol{k}_{\text{T}}; \mu_{Q_0}, Q_0^2) \\
&\quad \equiv \underline{d}^{(n,d_r)}_c(z;\mu_{Q_0})\,.
\end{aligned}
$$

$$
\begin{aligned}
C^{(d_r)} = d^{(1,d_r)}_c(z;\mu_{Q_0}) - A^{(d_r)}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_D}\right) \\
- B^{(d_r)}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_D}\right) \ln\left(\frac{Q_0^2}{\mu_{Q_0} m_D}\right)
\end{aligned}
$$

Note C coefficient not independent from A,B. Integral relation reduces Number of parameters.

$$2\pi z^2 \int_0^{\mu_{Q_0}} \mathrm{d}k_\mathrm{T}\, k_\mathrm{T} D^{(n,d_r)}_{\mathrm{input}}(z, z\boldsymbol{k}_\mathrm{T}; \mu_{Q_0}, Q_0^2) \equiv \underline{d}_c^{(n,d_r)}(z;\mu_{Q_0})\,.$$

cannot be neglected.

+ kT~Q0 constraints guarantee:

$$D^{(n,d_r)}\left(z, z\boldsymbol{k}_\mathrm{T}; \mu_Q, Q^2\right) = \left[\mathcal{C}_D^{(n)}(zk_\mathrm{T}) \otimes d_r\right](z;\mu_Q)\,,$$

used in usual treatment, Not enough to guarantee Integral relation

Not the other way around

$$2\pi z^2 \int_0^{\mu_{Q_0}} \mathrm{d}k_\mathrm{T}\, k_\mathrm{T} D_{\mathrm{input}}^{(n,d_r)}(z, z\boldsymbol{k}_\mathrm{T}; \mu_{Q_0}, Q_0^2) \equiv \underline{d}_c^{(n,d_r)}(z; \mu_{Q_0}) \,.$$

This (and other) constraints Implied in usual CSS formula

$$W(q_\mathrm{T}, Q) = H(\mu_Q; C_2) \int \frac{\mathrm{d}^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2}\, e^{-i\boldsymbol{q}_\mathrm{T}\cdot\boldsymbol{b}_\mathrm{T}}\, \tilde{D}_A(z_A, \boldsymbol{b}_*; \mu_{b_*}, \mu_{b_*}^2) \tilde{D}_B(z_B, \boldsymbol{b}_*; \mu_{b_*}, \mu_{b_*}^2)$$
$$\times \exp\left\{ 2\int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[\gamma(\alpha_s(\mu'); 1) - \ln\frac{Q}{\mu'}\gamma_K(\alpha_s(\mu')) \right] + \ln\frac{Q^2}{\mu_{b_*}^2} \tilde{K}(b_*; \mu_{b_*}) \right\}$$
$$\times \exp\left\{ -g_A(z_A, b_\mathrm{T}) - g_B(z_B, b_\mathrm{T}) - g_K(b_\mathrm{T}) \ln\left(\frac{Q^2}{Q_0^2}\right) \right\} \,.$$

↑ ↑ ↑

Must not forget to include constraints explicitly.

- **use RG improvements in result for kT>Q0 region.**

Use scale transformation that satisfies

$$\overline{Q}_0(b_{\rm T}) = \begin{cases} C_1/b_{\rm T} & b_{\rm T} \ll C_1/Q_0\,, \\ Q_0 & \text{otherwise}\,, \end{cases}$$

↑ **Input scale**

Example:

Interpolates smoothly Between C1/bT and Q0

$$\overline{Q}_0(b_{\rm T}, a) = 2.0\ {\rm GeV}\left[1 - \left(1 - \frac{C_1}{(2.0\ {\rm GeV})b_{\rm T}}\right) e^{-b_{\rm T}^2 a^2}\right]$$

Input scale (Q0 = 2 GeV here)

Compare to

$$\boldsymbol{b}_*(b_{\rm T}) = \frac{\boldsymbol{b}_{\rm T}}{\sqrt{1 + b_{\rm T}^2/b_{\rm max}^2}}\,.$$

$$\mu_{b_*} \equiv C_1/b_*\,.$$

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$$\mu_{b_*} \equiv C_1/b_* .$$

RG improvement and interpolation
Between large and small bT
not disentangled

$$\overline{Q}_0(b_\mathrm{T}) = \begin{cases} C_1/b_\mathrm{T} & b_\mathrm{T} \ll C_1/Q_0\,, \\ Q_0 & \text{otherwise}\,, \end{cases}$$

At input scale Q0, either of “input” or “underlined” should work since W term is not relevant at kT>Q0

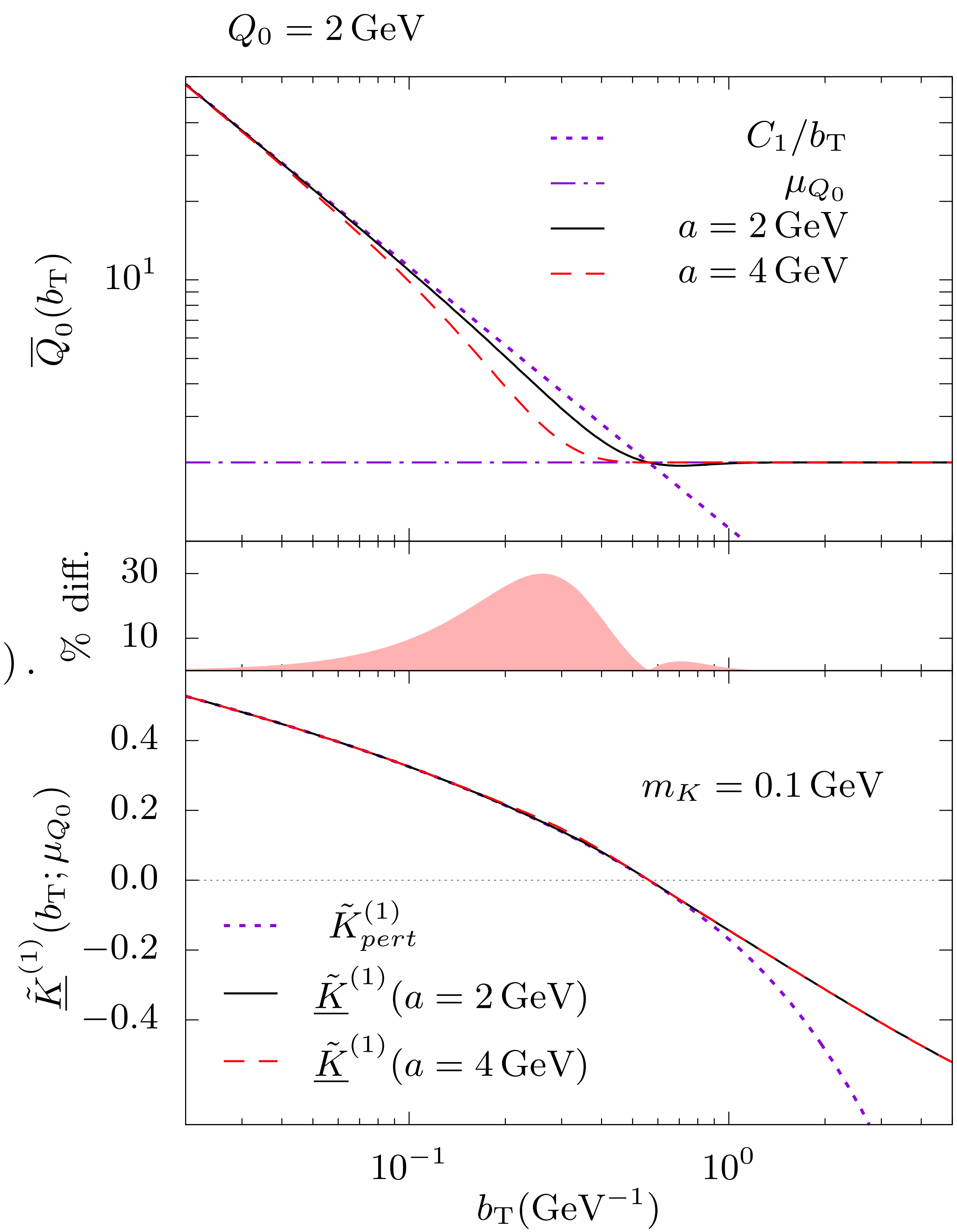
$$\underline{\tilde{K}}^{(n)}(b_\mathrm{T}; \mu_{Q_0}) \equiv \tilde{K}^{(n)}_{\text{input}}(b_\mathrm{T}; \mu_{\overline{Q}_0}) - \int_{\mu_{\overline{Q}_0}}^{\mu_{Q_0}} \frac{\mathrm{d}\mu'}{\mu'} \gamma_K^{(n)}(\alpha_s(\mu'))\,.$$

But to evolve to Q>>Q0, one needs to use RG improve version

$$\underline{\tilde{D}}^{(n,d_r)}(z, \boldsymbol{b}_\mathrm{T}; \mu_{Q_0}, Q_0^2) = \tilde{D}^{(n,d_r)}_{\text{input}}(z, \boldsymbol{b}_\mathrm{T}; \mu_{\overline{Q}_0}, \overline{Q}_0^2) \exp\left\{ \int_{\mu_{\overline{Q}_0}}^{\mu_{Q_0}} \frac{d\mu'}{\mu'} \left[\gamma^{(n)}(\alpha_s(\mu'); 1) - \ln\frac{Q_0}{\mu'} \gamma_K^{(n)}(\alpha_s(\mu')) \right] + \ln\frac{Q_0}{\overline{Q}_0} \tilde{K}^{(n)}_{\text{input}}(b_\mathrm{T}; \mu_{\overline{Q}_0}) \right\}$$

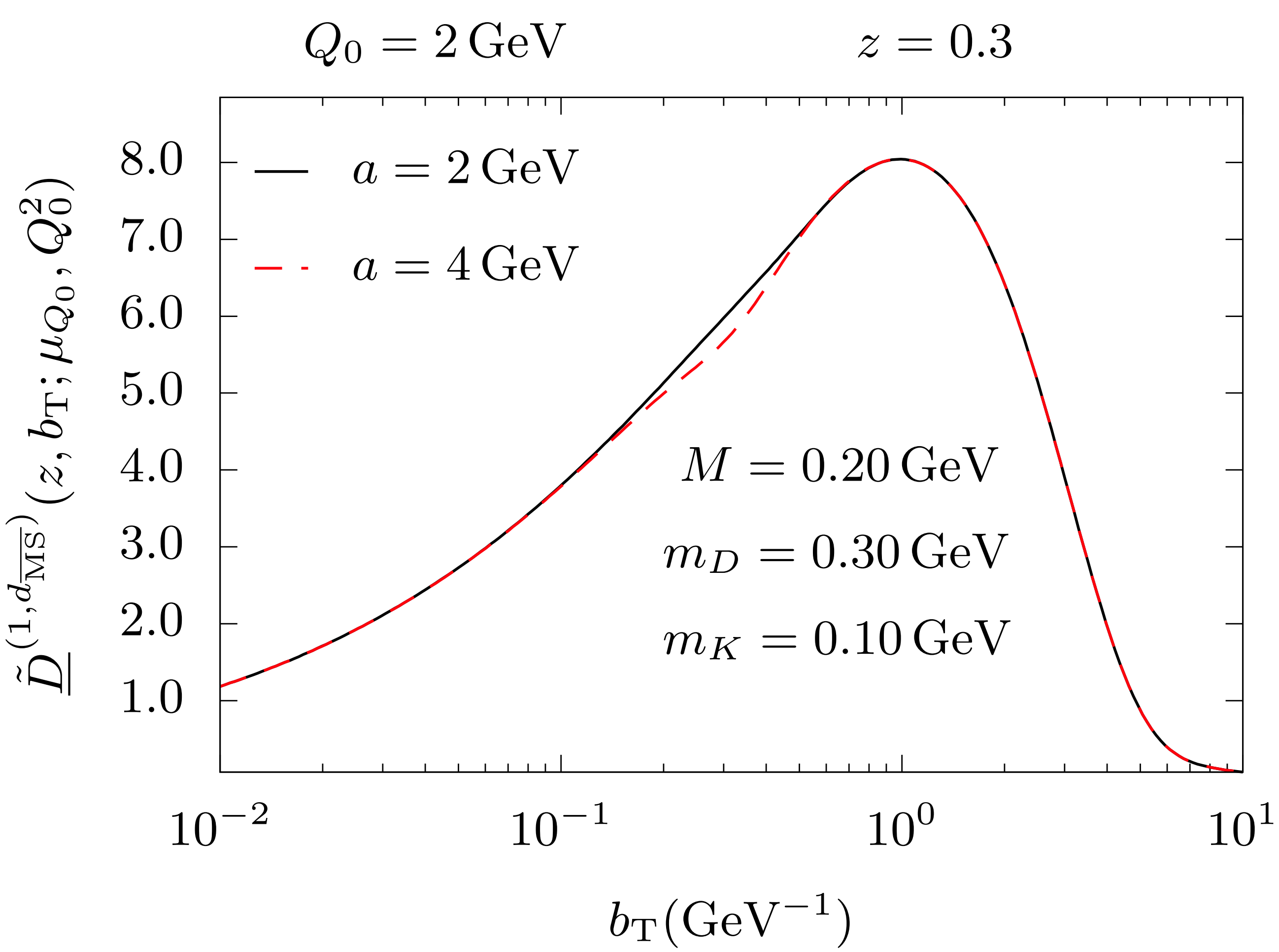
$$\overline{Q}_0(b_T) = \begin{cases} C_1/b_T & b_T \ll C_1/Q_0\,, \\ Q_0 & \text{otherwise}\,, \end{cases}$$

$$\underline{\tilde{K}}^{(n)}(b_T;\mu_{Q_0}) \equiv \tilde{K}^{(n)}_{\rm input}(b_T;\mu_{\overline{Q}_0}) - \int_{\mu_{\overline{Q}_0}}^{\mu_{Q_0}} \frac{d\mu'}{\mu'}\gamma_K^{(n)}(\alpha_s(\mu'))\,.$$



Dependence on scale transformation is a higher Order correction

$$\overline{Q}_0(b_{\mathrm{T}}) = \begin{cases} C_1/b_{\mathrm{T}} & b_{\mathrm{T}} \ll C_1/Q_0\,, \\ Q_0 & \text{otherwise}\,, \end{cases}$$



$$\underline{\tilde{D}}^{(n,d_r)}(z, \boldsymbol{b}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2)$$
$$= \tilde{D}^{(n,d_r)}_{\text{input}}(z, \boldsymbol{b}_{\mathrm{T}}; \mu_{\overline{Q}_0}, \overline{Q}_0^2) \exp\left\{ \int_{\mu_{\overline{Q}_0}}^{\mu_{Q_0}} \frac{d\mu'}{\mu'} \left[\gamma^{(n)}(\alpha_s(\mu'); 1) - \ln \frac{Q_0}{\mu'} \gamma_K^{(n)}(\alpha_s(\mu')) \right] + \ln \frac{Q_0}{\overline{Q}_0} \tilde{K}^{(n)}_{\text{input}}(b_{\mathrm{T}}; \mu_{\overline{Q}_0}) \right\}$$

Dependence on scale transformation is a higher Order correction

Phone at Q~Q0

$$\overline{Q}_0(b_\mathrm{T}) = \begin{cases} C_1/b_\mathrm{T} & b_\mathrm{T} \ll C_1/Q_0\,, \\ Q_0 & \text{otherwise}\,, \end{cases}$$

$$\underline{\tilde{K}}^{(n)}(b_\mathrm{T};\mu_{Q_0}) \equiv \tilde{K}^{(n)}_{\text{input}}(b_\mathrm{T};\mu_{\overline{Q}_0}) - \int_{\mu_{\overline{Q}_0}}^{\mu_{Q_0}} \frac{\mathrm{d}\mu'}{\mu'}\gamma_K^{(n)}(\alpha_s(\mu'))\,.$$

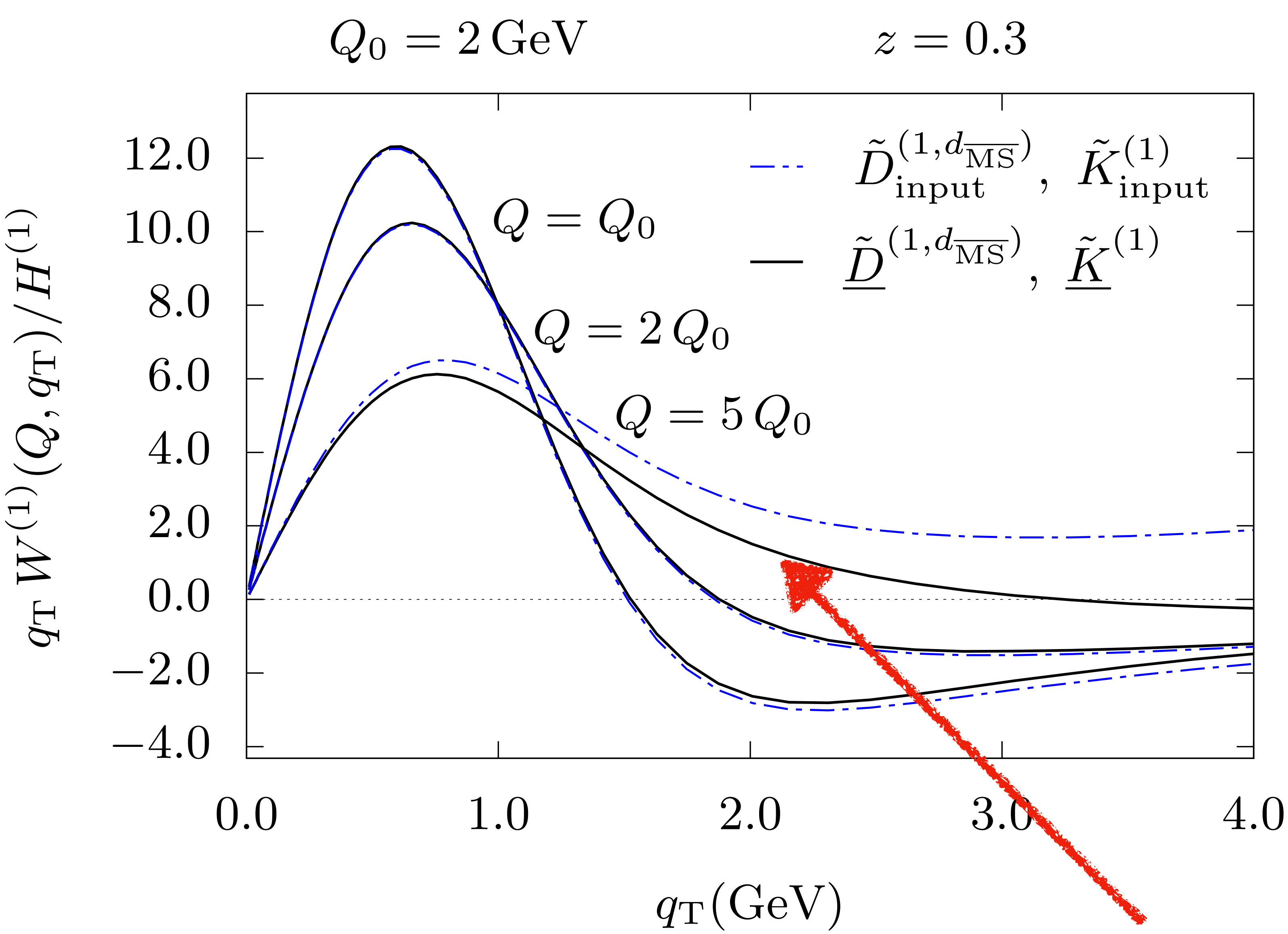
$$\underline{\tilde{D}}^{(n,d_r)}(z,\boldsymbol{b}_\mathrm{T};\mu_{Q_0},Q_0^2) = \tilde{D}^{(n,d_r)}_{\text{input}}(z,\boldsymbol{b}_\mathrm{T};\mu_{\overline{Q}_0},\overline{Q}_0^2)\exp\left\{\int_{\mu_{\overline{Q}_0}}^{\mu_{Q_0}}\frac{d\mu'}{\mu'}\left[\gamma^{(n)}(\alpha_s(\mu');1) - \ln\frac{Q_0}{\mu'}\gamma_K^{(n)}(\alpha_s(\mu'))\right] + \ln\frac{Q_0}{\overline{Q}_0}\tilde{K}^{(n)}_{\text{input}}(b_\mathrm{T};\mu_{\overline{Q}_0})\right\}$$

At input scale Q0, either of “input” or “underlined” should work since W term is not relevant at kT>Q0

But to evolve to Q>>Q0, one needs to use RG improve version

Verify these claims

Examples



This allows for pheno Close to input scale Q0 With "input" functions

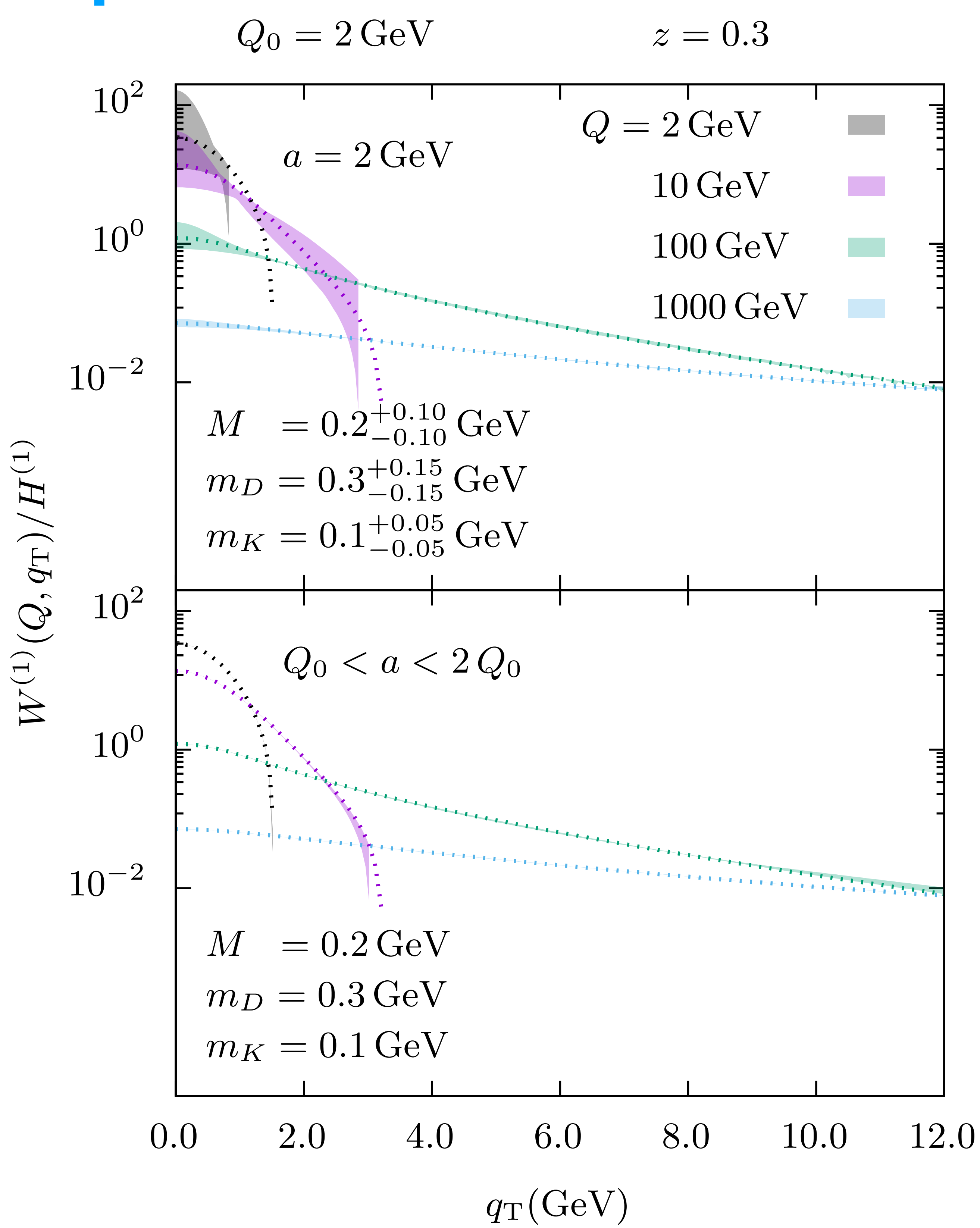
OR

Use existing results from pheno fits

Need RG improvement for larger Q

Examples

Making a case for "bottom-up" approach

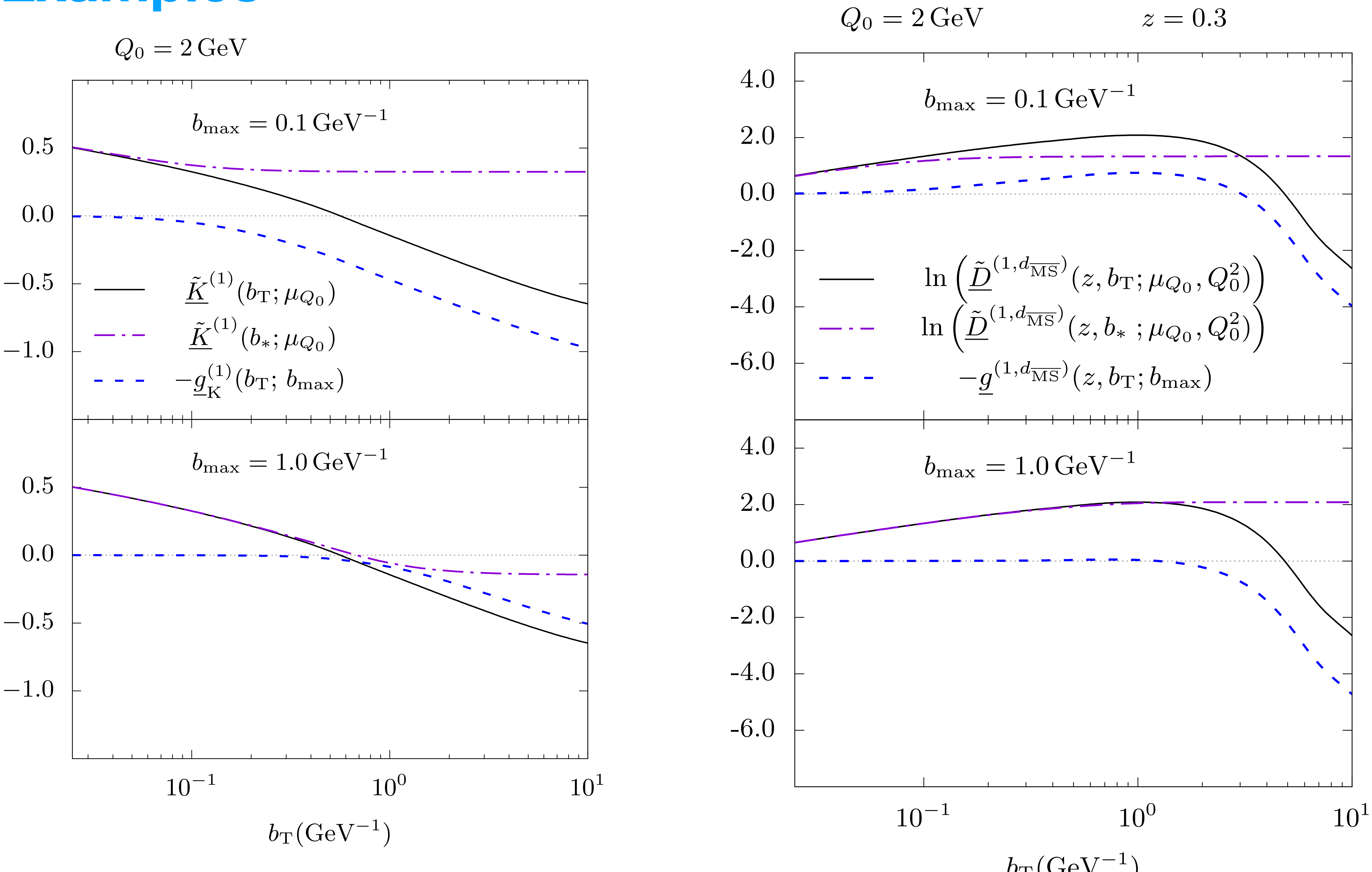


Larger sensitivity on non-perturbative parameters close to Q0 means it is harder to predict data At low energies from large Energies

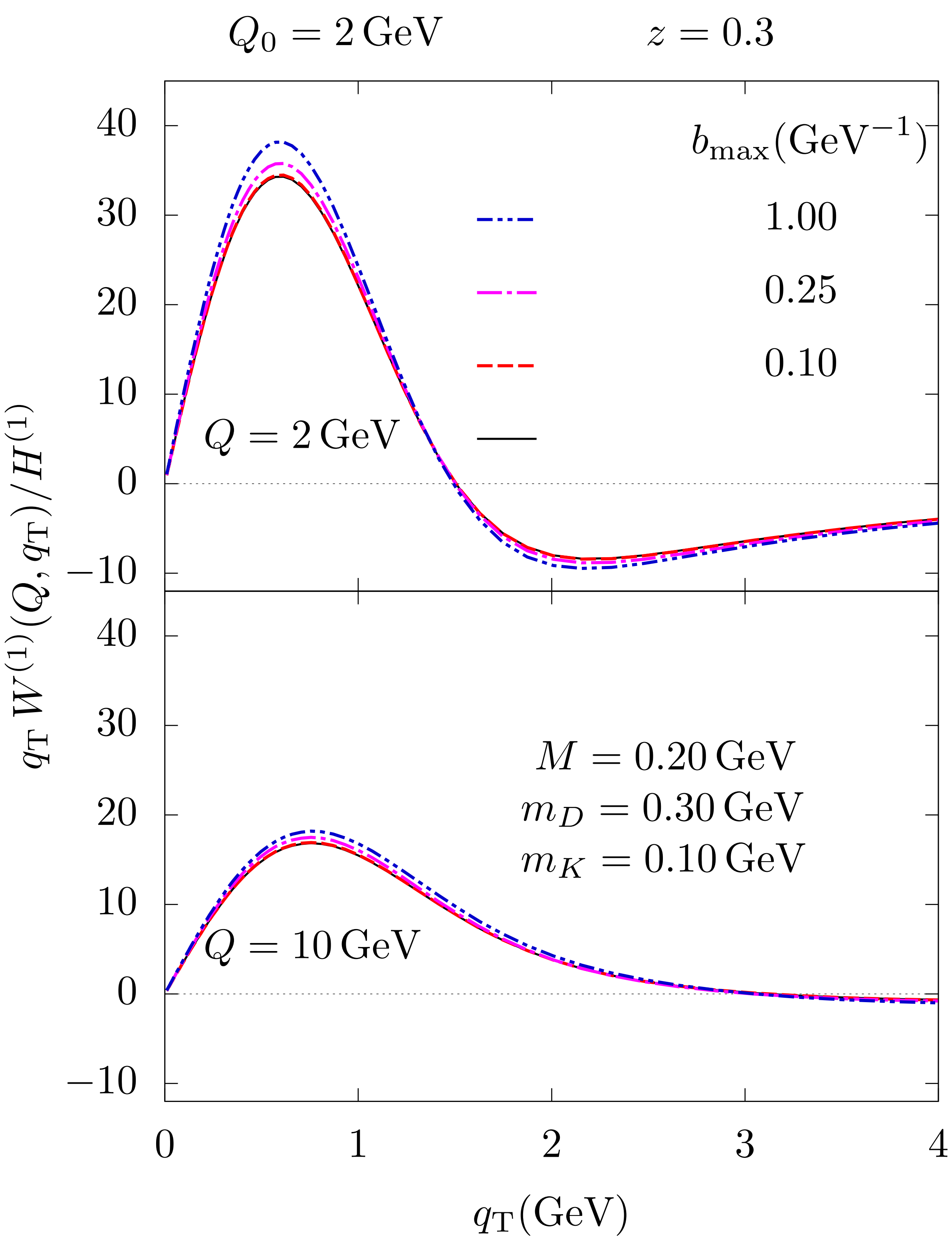
Note scale sensitivity Is much smaller

Examples

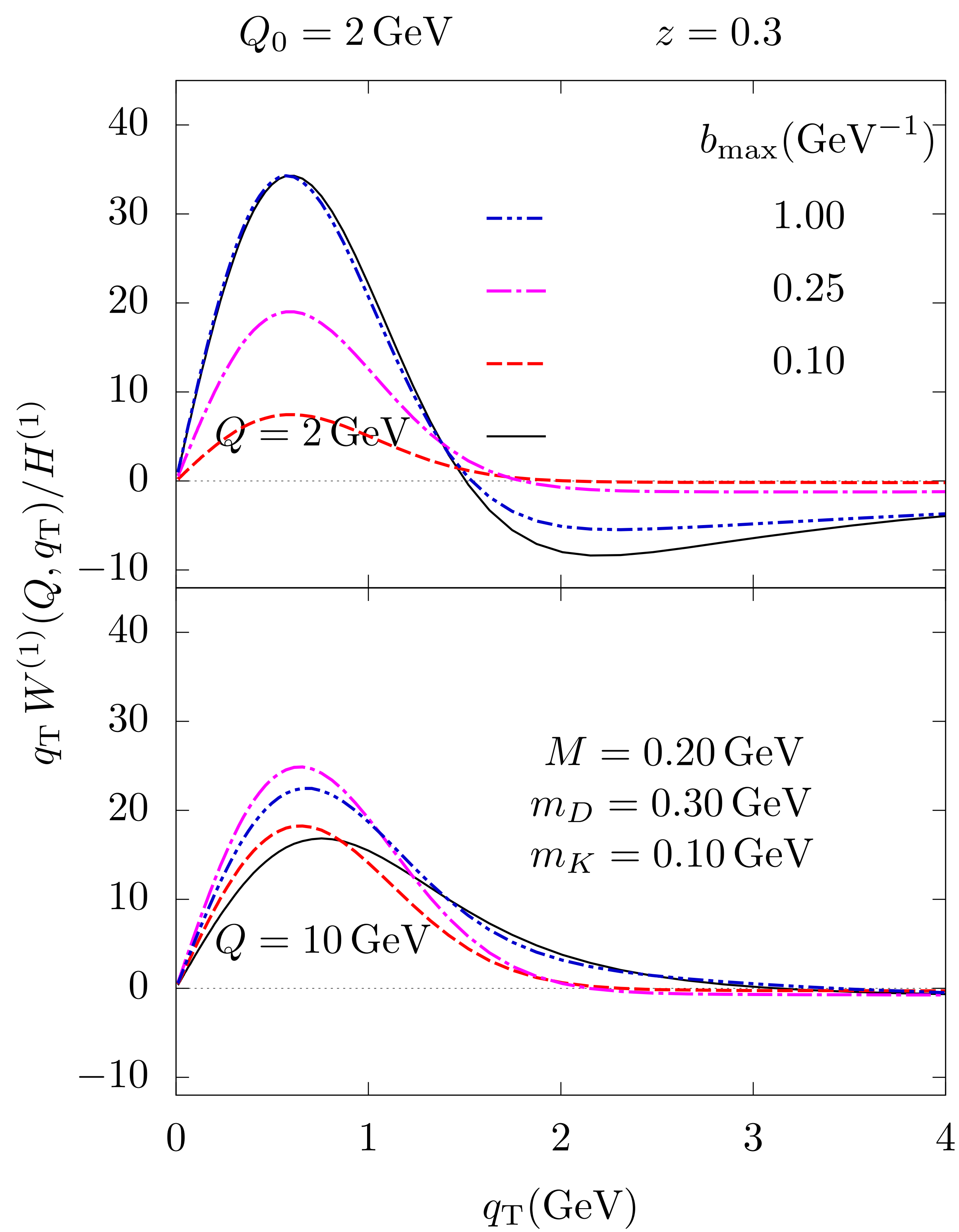
Comparing to usual formulation



In this example by construction W is independent of bmax.
This mimics what the original exact formula Implies



Using g's from our Underline functions



Unconstrained g's

Final Remarks

Bottom up approach advantages:

- **Allows to use existing pheno models/results**
- **Easy to constrain nonperturbative models (in relevant region)**
- **Defined "underlined" functions obey exact evolution equations**

$$\frac{\mathrm{d}\underline{\tilde{K}}^{(n)}(b_{\mathrm{T}};\mu)}{\mathrm{d}\ln\mu} = -\gamma_K^{(n)}(\alpha_s(\mu)) \,,$$

$$\frac{\partial \ln \underline{\tilde{D}}^{(n,d_r)}(z,\boldsymbol{b}_{\mathrm{T}};\mu_{Q_0},Q_0^2)}{\partial \ln Q_0} = \underline{\tilde{K}}^{(n)}(b_{\mathrm{T}};\mu_{Q_0}) \,,$$

$$\frac{\mathrm{d} \ln \underline{\tilde{D}}^{(n,d_r)}(z,\boldsymbol{b}_{\mathrm{T}};\mu_{Q_0},Q_0^2)}{\mathrm{d} \ln \mu_{Q_0}} = \gamma^{(n)}(\alpha_s(\mu_{Q_0});1) - \gamma_K^{(n)}(\alpha_s(\mu_{Q_0})) \ln\left(\frac{Q_0}{\mu_{Q_0}}\right) \,.$$

- **Can compare models agains each other:**
  - **a) do pheno at input scale Q0**
  - **b) evolve to larger scales to decide which model is better**

This is related to predictive power, the more you can predict, the better
The formulation + models + approximations work

Thanks