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In collaboration with
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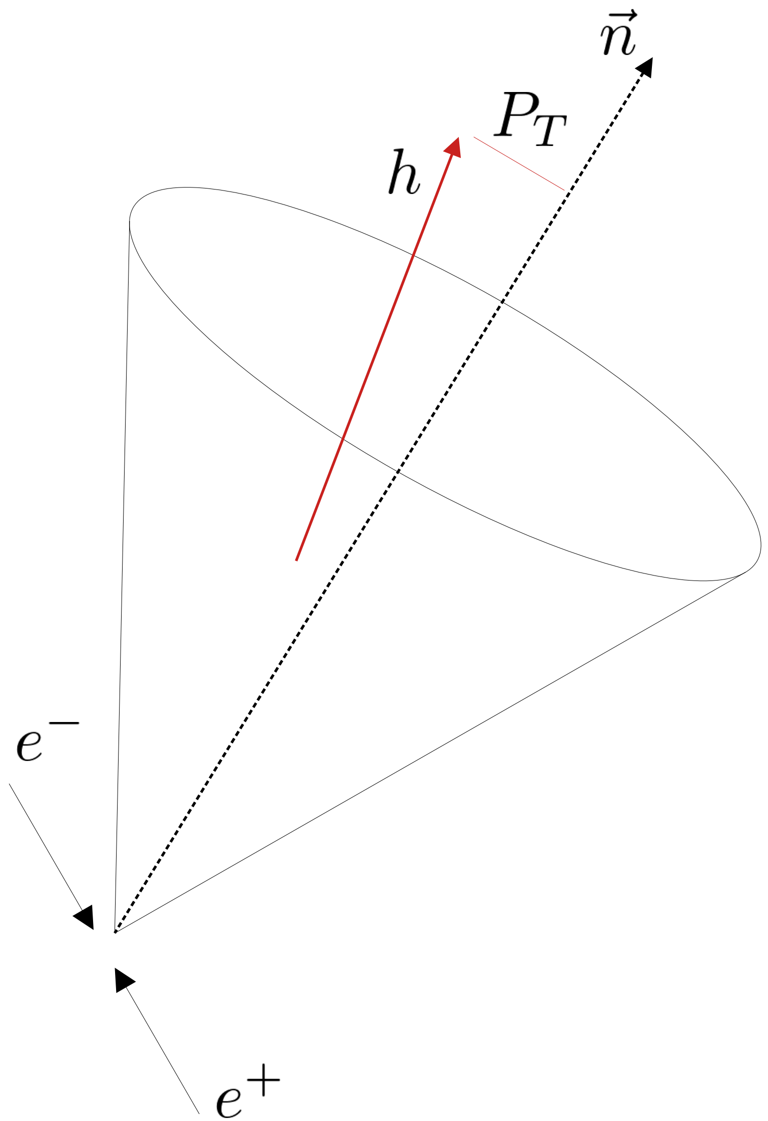
Istituto Nazionale di Fisica Nucleare
SEZIONE DI TORINO

Thrust resummation and TMD effects in e^+e^- annihilation

6th international workshop on transverse phenomena in hard processes

The process $e^+ e^- \to h\, X$ (thrust)

The cross section is differential in: $$z_h = \frac{E}{Q/2}, \quad T = \frac{\sum_i |\vec{P}_{(\mathrm{c.m.}),\,i} \cdot \widehat{n}|}{\sum_i |\vec{P}_{(\mathrm{c.m.}),\,i}|}, \quad P_T \text{ w.r.t } \vec{n}$$



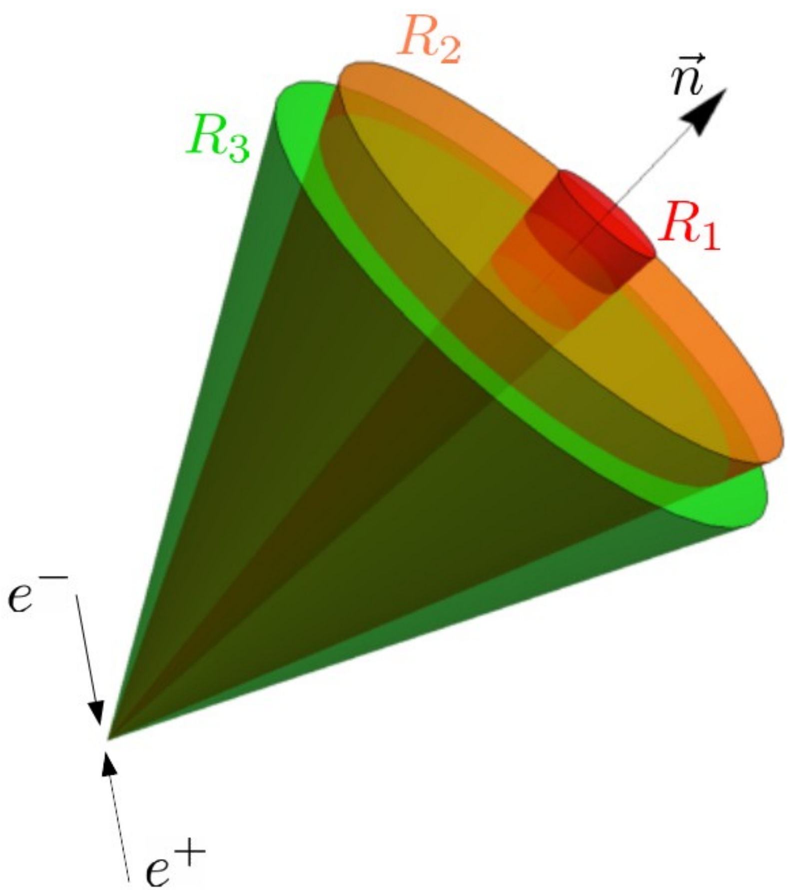
- **Recent data** (BELLE collab. 2019) that must be interpreted in the framework of factorization theorems.
- **Non-standard process**, as it is not covered by standard TMD factorization.
- The effects associated with the thrust intertwine those due to transverse momentum dependence. The final result should possess **features typical of both event-shape observables and TMD cross sections**.
- Naively, it is the **cleanest way to access a TMD** (FF). However, relating it with those encountered in standard TMD factorization is subtle and non-trivial.

THEORY

Kinematic regions of $e^+ e^- \to h\, X$ (thrust)

Depending on where the hadron is located within the jet the underlying kinematics can be remarkably different, resulting in different factorization theorems

Three Regions:



The hadron is detected very close to the axis of the jet:

- Extremely small P_T
- Soft radiation affects significantly the transverse deflection of the hadron from the thrust axis

TMD FF + non-pert. SOFT contribution

The hadron is detected in the central region of the jet:

- Most common scenario
- Majority of experimental data fall into this case

TMD FF

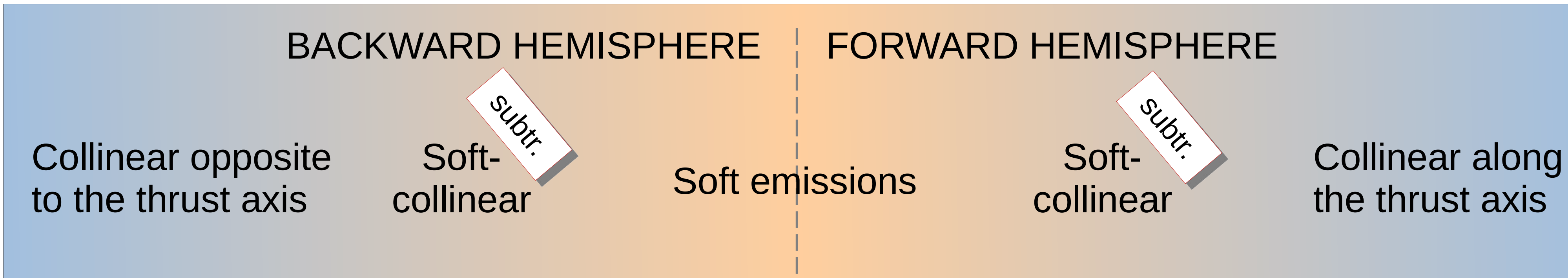
The hadron is detected near the boundary of the jet:

- Moderately small P_T
- The hadron P_T causes the spread of the jet affecting the topology of the final state (i.e. the value of thrust)

Generalized FJF

Each region is determined *uniquely* by which kind of radiation contributes to the TMD effects.

- Virtual emissions, clearly, cannot produce any TMD effect
- Real emissions can be emitted in each of the two hemispheres associated with the thrust axis. Their direction crucially determines if they are able to produce TMD effects:



3 possible cases

Collinear opposite to the thrust axis	Soft-collinear	Soft emissions (backward)	Soft emissions (forward)	Soft-collinear	Collinear along the thrust axis	
✗ TMD	✗ TMD	✗ TMD	V TMD	V TMD	V TMD	**R1**
same	same	same	✗ TMD	V TMD	V TMD	**R2**
same	same	same	✗ TMD	✗ TMD	V TMD	**R3**

In this talk ⟶ Region 2

- Most interesting, as the majority of data is expected to fall into this case.
- Most debated, tension with SCET result (cf. *Makris et al.* [hep-ph:2009.11871])
- Most difficult, unexpected issues rising from factorization (absence of non-perturbative TMD soft contributions, unusual treatment of rapidity divergences).

Building a factorization theorem for Region 2:

$$d\sigma_{R_2} \sim H \cdot J \cdot \mathcal{S} \cdot \Gamma_{j \to h}^{p_T \text{ small}}$$

Generalized Fragmenting Jet function considered in its low-p_T limit. (coll. forward) — V TMD

Hard function (virtual em.) — X TMD

Jet Thrust function (coll. backward) — X TMD

Generalized Soft Thrust function (soft em.) — X TMD — NEW!

Same of the usual Soft Thrust function but defined with rapidity divergence regulators.
In Collins's formalism they are (two) rapidity cut-offs describing how much the Wilson lines are tilted off the light-cone.

$$d\sigma_{R_2} \sim H \cdot J \cdot \mathcal{S} \cdot \Gamma_{j\to h}^{p_T \text{ small}}$$

SUBTRACTIONS

$$d\sigma_{R_2} \sim \frac{H \cdot J \cdot \mathcal{S} \cdot \Gamma_{j\to h}^{p_T \text{ small}}}{\mathcal{Y}_L \cdot \Upsilon_R^{p_T \text{ small}}}$$

CRUCIAL!!
Considering this contribution NOT relevant for TMD effects leads to the SCET result

✗ TMD

Soft-Collinear Thrust function
(soft-coll. backward em.)

NEW!

No correspondence with any usual object in thrust observavles.
It is defined as the Generalized Soft Thrust function but the Wilson lines associated with forward propagation are on the light-cone.

V TMD

Soft-Collinear Thrust factor considered in its low-p_T limit.
(soft-coll. forward em.)

It coincides with the “thrust-TMD collinear-soft function” of *Makris et al.* [hep-ph:2009.11871]

TMD Fragmentation Function

A TMD FF appears *naturally* as an ingredient of the factorization theorem.

$$\widetilde{D}_{j\to h}(z,\, b_T;\, y_1) = \frac{\Gamma^{p_T \text{ small}}_{j\to h}(z,\, b_T,\, 1-T)}{\Upsilon^{p_T \text{ small}}_{R}(b_T,\, 1-T;\, y_1)}$$

However, this is NOT the same TMD FF appearing in usual TMD factorization (SIDIS, $e^+e^- \to h_1\, h_2$).

They differ in the non-perturbative regime (large $\mathrm{b_T}$): $\widetilde{D}^{\text{usual}} = \widetilde{D} \times \sqrt{M_S}$ ← Soft Model NEW!

Notice that the comparison of the two extraction would allow to access the long-distance behavior of TMD Soft Factor (trending topic!)

Finally:

$$d\sigma_{R_2} \sim H \cdot J \cdot \frac{\mathcal{S}}{\mathcal{Y}_L} \cdot \widetilde{D}_{j\to h}$$

Treatment of the rapidity cut-off

The left-out dependence on the rapidity cut-off is not interpreted as the signal of the breaking of factorization, but instead as the signal that **the two regulators, the thrust and the rapidity cut-off, must be related through some certain specific condition**.

$$T \longleftrightarrow y_1$$

Such condition is exposed by imposing the CS-invariance to the factorized cross-section:

$$\frac{d\sigma_{R_2}}{dy_1} = 0 \longrightarrow \quad \overline{y}_1 = -L_u \, \frac{1 - 2\,\lambda_u}{2\,\lambda_u} \left(1 - e^{\frac{2\beta_0}{\gamma_K^{[1]}} \widetilde{K}\big|_{\mu_S}(b_T)} \right) + \begin{smallmatrix}\text{power}\\ \text{suppr.}\end{smallmatrix}$$

with $L_u = \log u e^{\gamma_E}$, $\lambda_u = \dfrac{\alpha_S(Q)}{4\pi}\,\beta_0\,L_u$ and $\mu_S = \frac{Q}{u}\,e^{-\gamma_E}$.

The final factorization theorem must be considered fixing the rapidity cut-off to the function $\overline{y}_1$

$$\frac{d\sigma_{R_2}}{dz_h\,d^2\vec{P}_T\,dT} = \sigma_B\,N_C \int \frac{d^2\vec{b}_T}{(2\pi)^2}\, e^{i\,\frac{\vec{P}_T}{z_h}\cdot\vec{b}_T} \int \frac{du}{2\pi\,i}\, e^{u\,(1-T)} \times H\,\widehat{J}(u)\,\frac{\widehat{\mathcal{S}}(u,\overline{y}_1,y_2)}{\widehat{\mathcal{Y}}_L(u,y_2)} \sum_j e_j^2\,\widetilde{D}_{h/j}(z_h,\,b_T,\,\overline{y}_1)$$

N^2LL-accuracy in Thrust

$$
d\widetilde{\sigma}_{R_2} \sim -\widetilde{D}_{h/j}\left(z_h, a_S, L_b^\star;\, b_T\right)\,\frac{1}{1-T}\,\Big(1 + a_S\, C_1(g_K)\Big)\times
$$
$$
\exp\Big\{ -\log\left(1-T\right) g_1(\lambda,\, \lambda_b^\star,\, g_K) + g_2(\lambda,\, \lambda_b^\star,\, g_K) - \frac{1}{\log\left(1-T\right)}\, g_3(\lambda,\, \lambda_b^\star,\, g_K)\Big\}\times
$$
$$
\left(\gamma(\lambda,\, \lambda_b^\star,\, g_K) - \frac{1}{\log\left(1-T\right)}\,\rho(\lambda,\, \lambda_b^\star,\, g_K)\right)
$$

Where $\lambda = -a_S\,\beta_0\,\log\left(1-T\right)$, $\lambda_b^\star = 2\,a_S\,\beta_0\,\log\left(b_T^\star\,Q/c_1\right)$ and g_k encodes the non-perturbative content of the CS-kernel.
The TMD FF is expressed as usual (here at N^2LL in b_T):

$$
\widetilde{D}_{j\to h}(z_h,\, a_S,\, L_b^\star;\, b_T) = \sum_l \left(\widetilde{C}_{j\to l} \otimes d_{l\to h}(z_h)\right)^{\mathrm{N^2LO}}
$$
$$
\times\; e^{L_b^\star\, g_1^{\mathrm{TMD}}(\lambda_b^\star) + g_2^{\mathrm{TMD}}(\lambda_b^\star) + \frac{1}{L_b^\star}\, g_3^{\mathrm{TMD}}(\lambda_b^\star)}\; M_D(z_h,\, b_T,\, j\to h)\, e^{-\frac{1}{2}\, g_K(b_T)\,\log\frac{Q}{M_h}}
$$

Notice how g_k enters into the game in a very different way with respect to the usual TMD factorization.

For this reason, this process can be considered as a **new tool** for the investigation of the large-distance behavior of the CS-kernel (trending topic!)

PHENOMENOLOGY
(preliminary)

Sources of theoretical errors

Many sources of theoretical errors!

- Factorization approximations, $P_T \ll P^+;\ M_h \ll Q$
- Crossing over into Region 1
- Crossing over into Region 3
- Contributions from higher order topologies
- Matching with Fixed Order (unapproximated, full QCD)
- Solution for the rapidity cut-off large and positive
- Collinear FFs uncertainties (LHAPDF)
- Arbitrariness in modelling the non-perturbative TMD functions
- Arbitrariness in modelling the non-perturbative function(s) associated with thrust

“Approximated model”

Boglione, Gonzalez, Simonelli, *Transverse Momentum Dependent Fragmentation Functions from recent BELLE data*, arXiv hep-ph:2205.xxyy

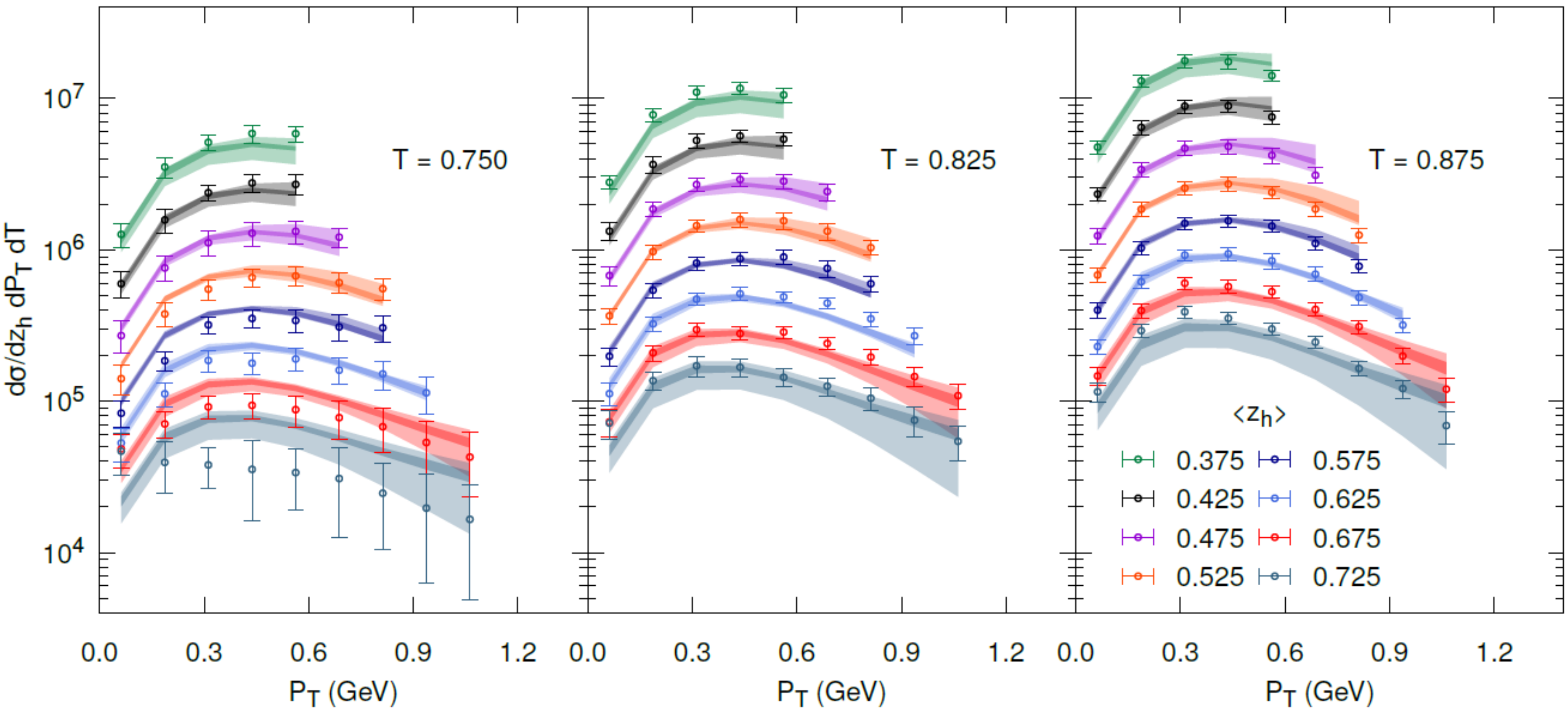
Strong approximations and assumptions:

- Rapidity cut-off associated directly (and naively) to thrust.
- Any b_T-dependence outside the TMD FF is integrated out.
- Formal resummation of thrust cannot be performed

Boglione, Simonelli, *Factorization of e+e-→HX cross section, differential in zh, PT and thrust, in the 2-jet limit*, JHEP 02 (2021) 076

$$\chi^2 = 1.25$$

with 3 free parameters



Preliminary FIT of BELLE data @ N²LL*

BELLE collab. Phys.Rev.D 99 (2019) 11, 112006

Many sources of theoretical errors!

- Factorization approximations, $P_T \ll P^+;\ M_h \ll Q$ → $P_T \lesssim 0.2\,P^+$, pions.
- Crossing over into Region 1
- Crossing over into Region 3

→ Data selection algorithm

M. Boglione and A. Simonelli, *Kinematic regions in the $e^+e^- \to hX$ factorized cross section in a 2-jet topology with thrust*, JHEP 02, 013 (2022)

- Contributions from higher order topologies → $T \gtrsim 0.8$
- Matching with Fixed Order (unapproximated, full QCD)
- Solution for the rapidity cut-off large and positive $e^{-\overline{y}_1} \ll 1$ →
- Collinear FFs uncertainties (LHAPDF)
- Arbitrariness in modelling the non-perturbative TMD functions
- Arbitrariness in modelling the non-perturbative function(s) associated with thrust

Errors affected by:
- log-accuracy (thrust)
- g_k function

The size can be determined only "a posteriori" (after fit)

*The TMD FF is still considered at NLL.

Khalek, Bertone, Khoudli, Nocera, *Pion and kaon fragmentation functions at next-to-next-to-leading order*, arXiv hep-ph:2204.10331

Two kind of Non-Perturbative effects

1) **TMD effects** are encoded into two functions:

- The TMD FF model (POWER LAW):

$$M_D(z_h,\, b_T) = \text{F.T.}\left\{ \frac{\Gamma(p)}{\pi\,\Gamma(p-1)} M^{2\,(p-1)} \left(M^2 + \frac{P_T^2}{z_h^2} \right)^{-p} \right\}$$

It depends on two free parameters z_0 and α, linked to M and p through:

$$\begin{cases} p = \frac{1}{2}\left(\frac{3}{1-R} - 1\right) \\ M = \frac{W}{z_h}\sqrt{\frac{3}{1-R}} \end{cases}$$

With:

$$R = 1 - (1-z)^{\alpha\, z_0/1-z_0}\; z^{\alpha},$$

$$W = \frac{M_h}{\sqrt{R}}$$

In total **4** free parameters: $z_0,\ \alpha,\ g_a,\ g_b$

- The g_K-function, depending on two free parameters g_a and g_b ruling its small-b_T and large asymptotic behavior.

$$g_K(b_T) = \frac{g_b x(b_T)\,\tanh\left(g_b x(b_T)\right)}{1 + x(b_T)^2}$$

With:

$$x(b_T) = \frac{b_T}{g_a\sqrt{1 + \left(\frac{b_T}{g_a\, g_b}\right)^2}}$$

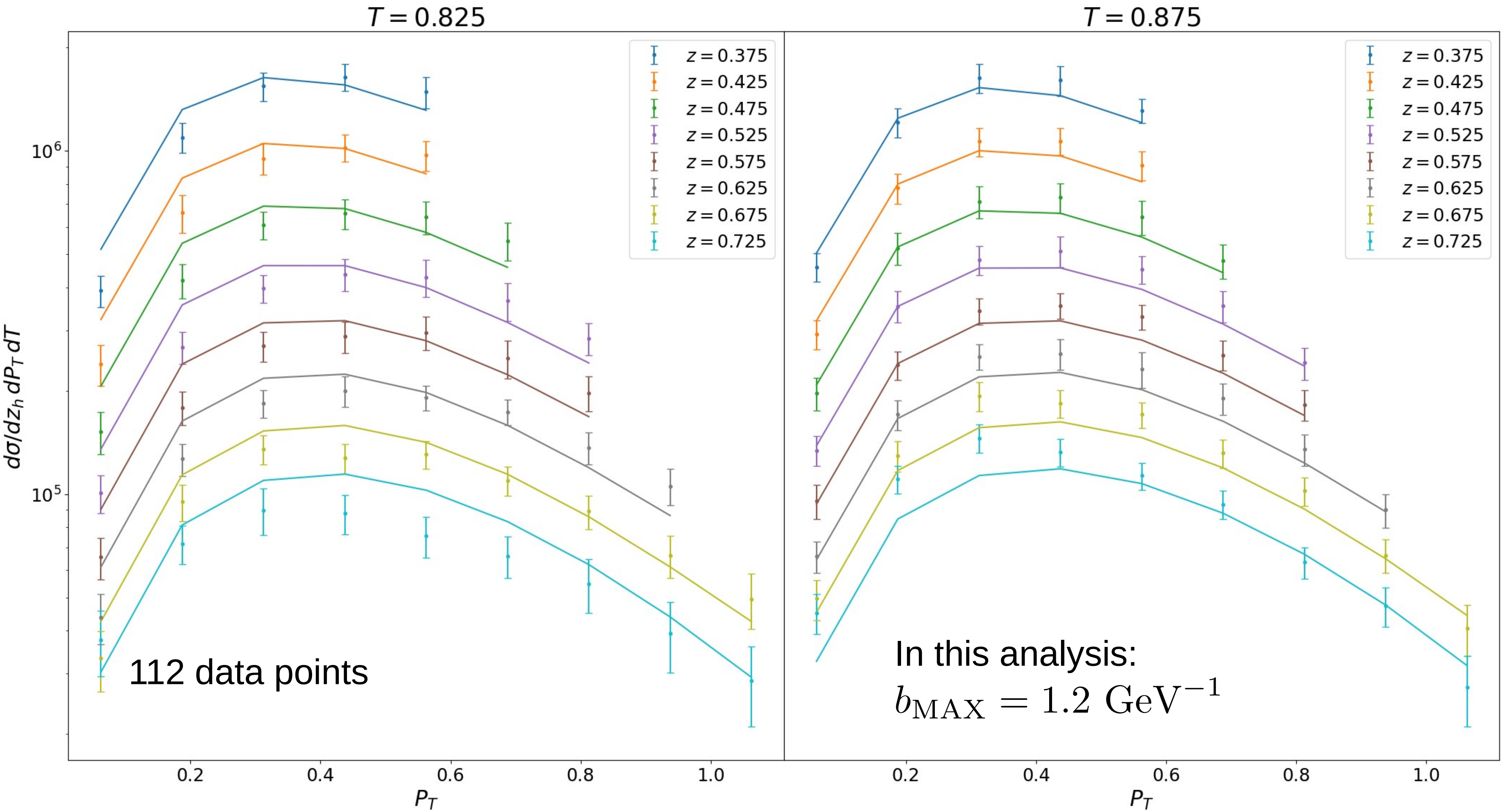
2) **Thrust effects** neglected as long as data are increasing in T. True if $T \lesssim 0.9$.

Phenomenology: preliminary FIT of BELLE data @ N²LL*

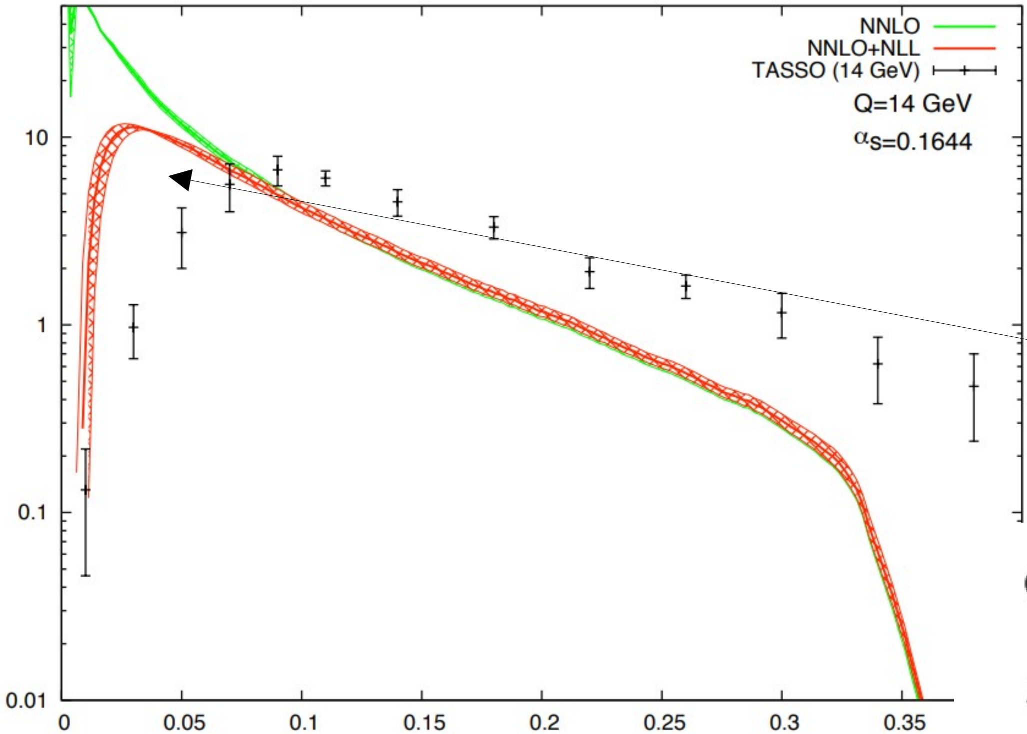
$\chi^2 = 1.25$

$\mathrm{M_D}: \quad z_0 = 0.729 \pm 0.011, \quad \alpha = 0.175 \pm 0.004$

$\mathrm{g_K}: \quad g_a = 0.445 \pm 0.026\ \mathrm{GeV}^{-1}, \quad g_b = 3.690 \pm 0.026$

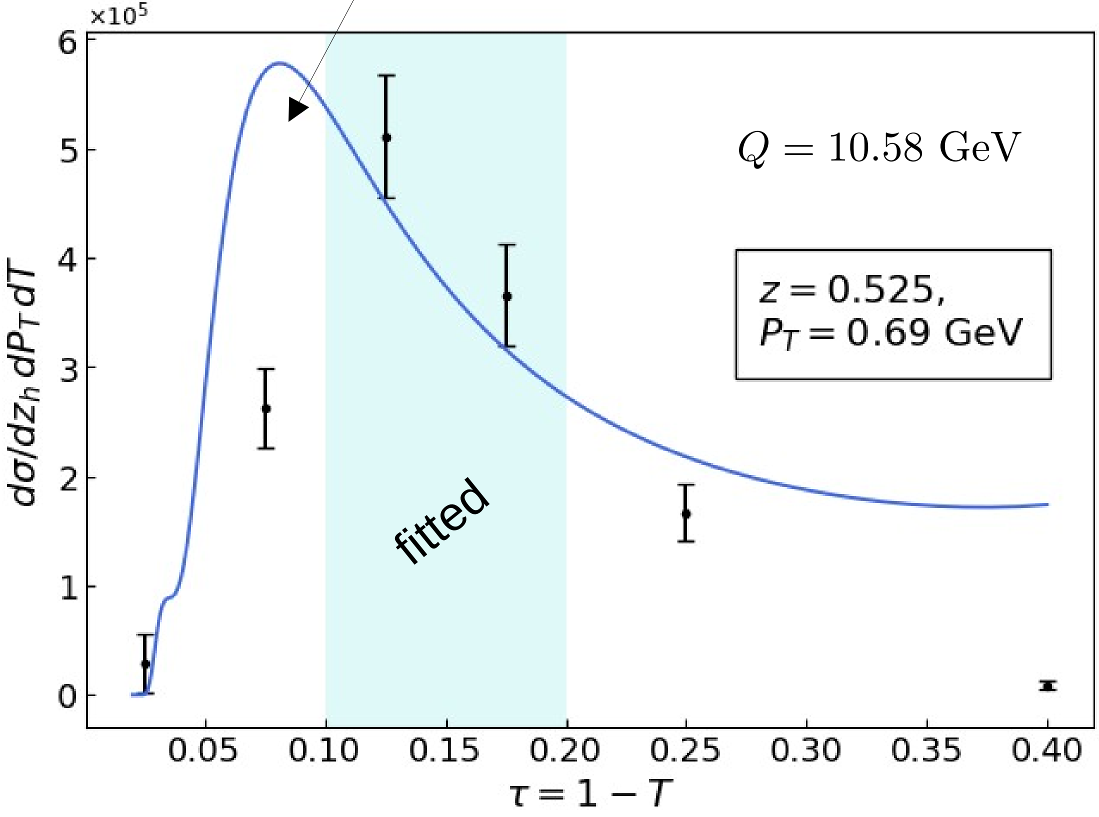


Peak shift and Thrust dependence



R.A. Davison and B.R. Webber, *Non-Perturbative Contribution to the Thrust Distribution in e^+e^- Annihilation*, Eur. Phys. J. C 59 (2009) 13 [0809.3326].

Same NON-PERTURBATIVE effect, *genuinely* due to thrust dependence.



Behaviour of g_K

- At small-b_T :

$$g_K = g_2\, b_T^2 + \ldots$$

with:

$$g2 = 68.879 \pm 8.016\ \text{GeV}^2$$

Two orders of magnitude larger than in previous extractions!

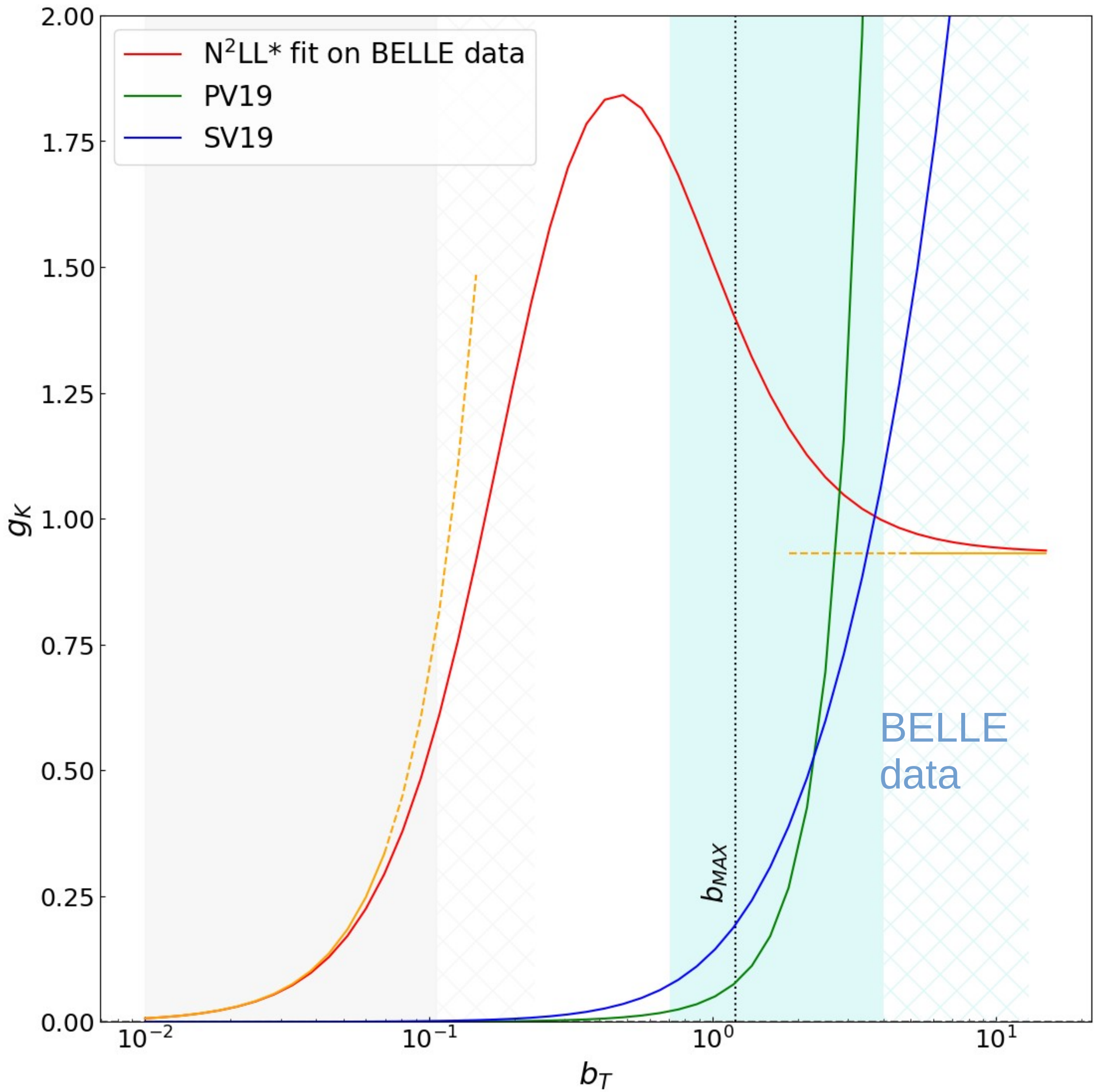
- At large-b_T :

$$g_K \sim g_0$$

with:

$$g0 = 0.932 \pm 0.001$$

Asymptotic constant behaviour



John Collins, Ted Rogers, Understanding the large-distance behavior of transverse-momentum-dependent parton densities and the Collins-Soper evolution kernel, Phys.Rev.D 91 (2015) 7, 074020

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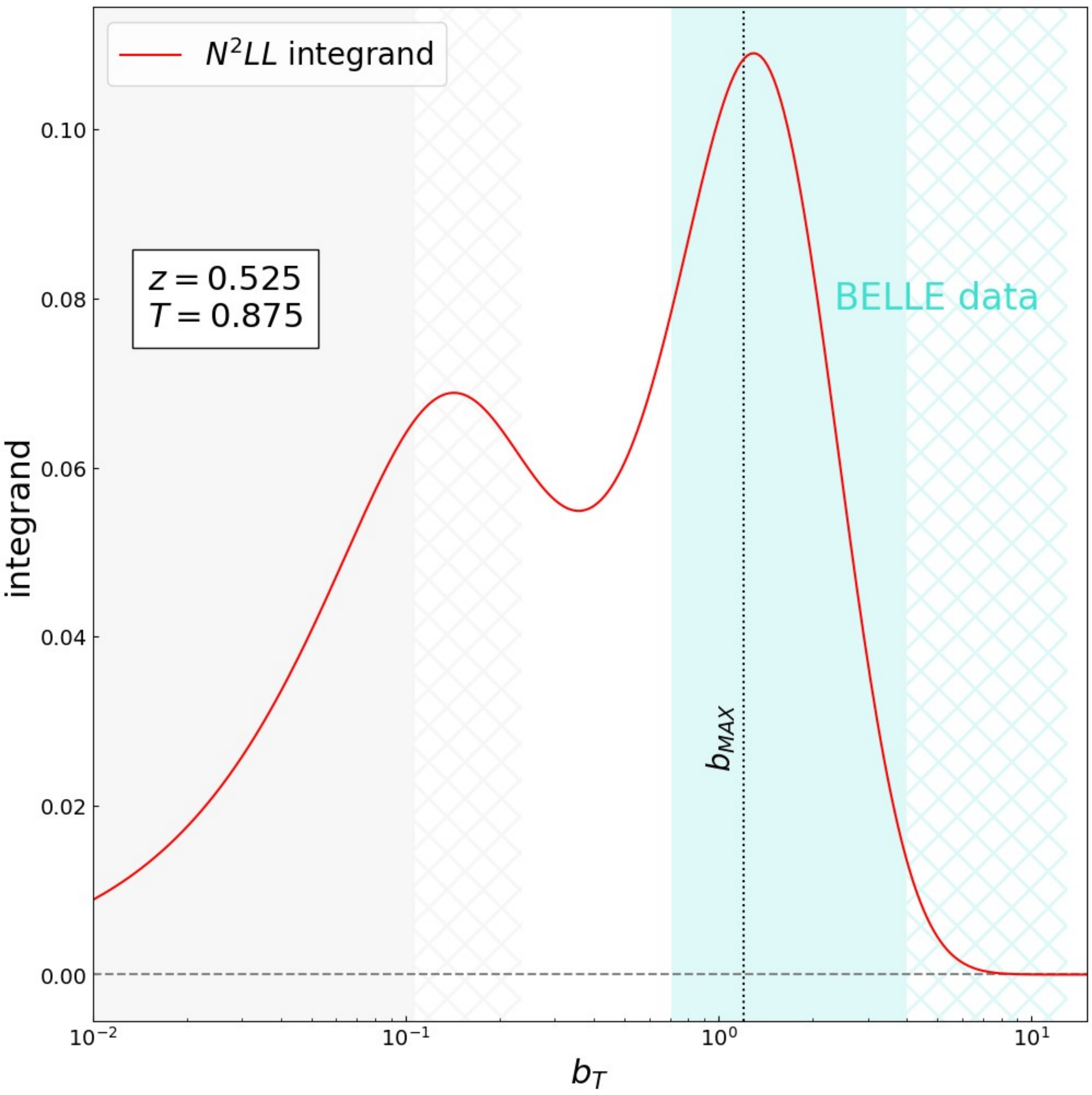


Conclusions and Future perspectives

Summarizing…

- We have set the theoretical framework to treat the process $e^+e^- \to h\, X$, where the transverse momentum of the detected hadron is measured w.r.t. the thrust axis.

- the TMD FF of Region 2 differs from that appearing (e.g.) in SIDIS at large distances. This is due to the absence of TMD non-perturbative contributions associated with soft radiation.

- In Region 2, the rapidity regulator used in TMD factorization correlates the thrust and the transverse momentum. This is a completely **new feature** for a factorization theorem.

- The thrust bins of BELLE data for $0.8 < \mathrm{T} < 0.9$, associated with Region 2, can be successfully fitted at **N^{2}LL-accuracy** in thrust, by using 4 free parameters for the (unpolarized, pion) TMD FF. This is the first time that the thrust behavior of such data is described within a consistent theoretical formalism.

  - Going to higher values of thrust requires to include genuinely non-perturbative effects associated to thrust dependence.

  - Going to lower values of thrust requires the matching with fixed order.

Conclusions and Future perspectives

In the future…

- Refining of the fit, taking into account all the error sources.
- Inclusion of the non-perturbative effects genuinely due to the thrust dependence in order to describe properly the **peak shift** observed for $T > 0.9$.
- Comparison of the result obtained from Region 2 with the extraction of the TMD FF obtained from the standard TMD processes (or from Region 1). This would allow to access the **soft model**, shedding light on its characteristic large-distance behavior.
- Extension of this theoretical framework to other non-standard TMD observables.

THANK YOU FOR YOUR ATTENTION!

BACK-UP SLIDES

SCET interpretation:

$$d\sigma_{R_2} \sim \frac{H \cdot J \cdot \mathcal{S} \cdot \Gamma_{j\to h}^{p_T \text{ small}}}{\mathcal{Y}_L \cdot \mathcal{Y}_R}$$

✗ TMD

Soft-Collinear Thrust function (soft-coll. backward em.)

✗ TMD

Soft-Collinear Thrust function (soft-coll. forward em.)

The usual Soft Thrust function appears from:

$$S(1-T) = \frac{\mathcal{S}(1-T;\, y_1,\, y_2)}{\mathcal{Y}_L(1-T;\, y_2) \cdot \mathcal{Y}_R(1-T;\, y_1)}$$

Finally:

Multiply&Divide by $\Upsilon_R^{p_T \text{ small}}$

$$d\sigma_{R_2} \sim H \cdot J \cdot S \cdot \Gamma_{j\to h}^{p_T \text{ small}} = \underbrace{H \cdot J \cdot S \cdot \Upsilon_R^{p_T \text{ small}} \cdot \widetilde{D}_{j\to h}}$$

cf. Eq. (2.21) in *Makris et al.* [hep-ph:2009.11871]

Our result differs from that obtained in the framework of SCET…
…but now at least we know *where!*

Our interpretation (CSS):

$$d\sigma_{R_2} \sim H \cdot J \cdot \frac{\mathcal{S}}{\mathcal{Y}_L} \cdot \widetilde{D}_{j\to h}$$

- A TMD FF appears *naturally* as an ingredient of the factorization theorem. However, this is NOT the same TMD FF appearing in usual TMD factorization (SIDIS, $e^+e^- \to h_1\, h_2$).

  The difference is in the non-perturbative regime (large b_T):

$$\widetilde{D}^{\text{usual}} = \widetilde{D} \times \sqrt{M_S}$$

← Soft Model (NEW!)

- Rapidity divergences are regulated in two different *and overlapping* ways:

  - The Thrust (T)
  - The rapidity cut-off (y_1)

  This redundancy results in a (fictitious) left-out dependence of the rapidity cut-off.

SCET interpretation:

$$d\sigma_{R_2} \sim H \cdot J \cdot S \cdot \Gamma^{p_T \text{ small}}_{j\to h}$$

- The factorization theorem does not include a TMD FF, but instead a Generalized FJF at low-p_T.
  This is a consequence of having considered the same decomposition of radiation of Region 3 and, in fact, the two factorization theorems are almost identical.

- Rapidity divergences are regulated solely by the thrust, and hence there are no issues regarding the rapidity cut-off.

Thrust resummation starts by using Evolution Equations. The final result can be recasted as:

$$H\left(a_S(\mu), \log \mu/Q\right)\; \widehat{J}\left(a_S(\mu), L_J\right)\; \frac{\widehat{S}\left(a_S(\mu), L_S, \overline{y}_1, y_2; u\right)}{\widehat{Y}_L\left(a_S(\mu), L_S + y_2\right)}\; \widetilde{D}_{h/j}\left(z_h, a_S(\mu), L_b, \overline{y}_1;\, b_T\right) =$$

$$= H\left(a_S(Q), 0\right)\; \widehat{J}\left(a_S(\mu_J), 0\right)\; \frac{\widehat{S}\left(a_S(\mu_S), 0, 0; u\right)}{\widehat{Y}_L\left(a_S(\mu_S), 0\right)}$$

] u-dep. at ref. scale

$$\times\; \exp\left\{ \int_{\mu_J}^{Q} \frac{d\mu'}{\mu'} \gamma_J\left(a_S(\mu'), L_J\right) + \frac{1}{2} \int_{\mu_S}^{Q} \frac{d\mu'}{\mu'} \gamma_S\left(a_S(\mu'), L_S\right) \right\}$$

] u-dep. evolution

$$\times\; \widetilde{D}_{h/j}\left(z_h, a_S(Q), L_b, ;\, b_T\right)$$

] TMD FF at $\mu = Q;\ y_1 = 0$

$$\times\; \exp\left\{ -\frac{1}{2} \left(\int_{\mu_S}^{\overline{\mu_R}} \frac{d\mu'}{\mu'} \left[\widehat{g}\left(a_S(\mu')\right) + \gamma_K\left(a_S(\mu')\right) \log \frac{\mu'}{\overline{\mu_R}} \right] + \right.\right.$$
$$\left.\left. + \log \frac{\overline{\mu_R}}{\mu_S}\; \widetilde{K}\left(a_S(\mu_S), \log\left(\frac{b_T\, Q}{2u}\right);\, b_T \right) \right) \right\}$$

] u-b_T correlation

This result can be now computed at the desired log-accuracy, both in thrust (u) and in transverse momentum (b_T).

The rapidity cut-off induces a correlation between thrust and transverse momentum

Treatment of the rapidity cut-off

The left-out dependence on the rapidity cut-off is not interpreted as the signal of the breaking of factorization, but instead as the signal that **the two regulators, the thrust and the rapidity cut-off, must be related through some certain specific condition**.

$$T \longleftrightarrow y_1$$

Such condition is exposed by imposing the CS-invariance to the factorized cross-section:

$$\frac{d\sigma_{R_2}}{dy_1} = 0 \longrightarrow \boxed{\widehat{G}_R(u,\, y_1) + \widetilde{K}(b_T) = 0}$$

← RG-invariant combination!

NEW!

(right) **G-kernel**.
It rules the evolution of the Generalized Soft Thrust function together with its left counterpart:

$$\widehat{G}_R(u,\, y_1) = -2 \lim_{y_2 \to -\infty} \frac{\partial \log \widehat{\mathcal{S}}(u;\, y_1,\, y_2)}{\partial y_1}$$

$$\widehat{G}_L(u,\, y_2) = 2 \lim_{y_1 \to +\infty} \frac{\partial \log \widehat{\mathcal{S}}(u;\, y_1,\, y_2)}{\partial y_2}$$

Collins-Soper kernel.
It rules the CS-evolution of TMDs and the evolution of the 2-h Soft Factor:

$$\widetilde{K}(b_T) = \pm 2 \lim_{y_{2,1} \to \mp\infty} \frac{\partial \log \widehat{\mathbb{S}}_{\text{2-h}}(b_T;\, y_1,\, y_2)}{\partial y_{1,2}}$$

$$\frac{\partial \log \widetilde{D}(z,\, b_T;\, y_1)}{\partial y_1} = -\frac{1}{2}\widetilde{K}(b_T)$$

Solving the condition gives y_1 as a function of thrust and transverse momentum (implicitly encoded into u and b_T, respectively):

$$\overline{y}_1 = -L_u \, \frac{1 - 2\,\lambda_u}{2\,\lambda_u} \left(1 - e^{\frac{2\beta_0}{\gamma_K^{[1]}} \widetilde{K}\big|_{\mu_S}} \right) + \begin{matrix}\text{power}\\ \text{suppr.}\end{matrix} \quad \text{with} \quad \begin{aligned} L_u &= \log u e^{\gamma_E}, \\ \lambda_u &= \frac{\alpha_S(Q)}{4\pi} \beta_0 \, L_u, \\ \mu_S &= \frac{Q}{u} \, e^{-\gamma_E}. \end{aligned}$$

The final factorization theorem must be considered fixing the rapidity cut-off to the function $\overline{y}_1$

$$\frac{d\sigma_{R_2}}{dz_h \, d^2\vec{P}_T \, dT} = \sigma_B \, N_C \int \frac{d^2\vec{b}_T}{(2\pi)^2} \, e^{i \, \frac{\vec{P}_T}{z_h} \cdot \vec{b}_T} \int \frac{du}{2\pi \, i} \, e^{u\,(1-T)} \times H \, \widehat{J}(u) \, \frac{\widehat{\mathcal{S}}(u, \overline{y}_1, y_2)}{\widehat{\mathcal{Y}}_L(u, y_2)} \sum_j e_j^2 \widetilde{D}_{h/j}(z_h, \, b_T, \, \overline{y}_1)$$

Notice that the factorization theorem can only be trusted where $\overline{y}_1$ is large and positive, i.e. at large u (two-jet limit) and small-moderate bT (according to Region decomposition).

Phenomenology: preliminary FIT of BELLE data @ N^{2}LL*

With this simplifications, all the non-perturbative information is encoded into **two functions**:

- The TMD FF model (POWER LAW):

$$M_D(z_h,\, b_T) = \frac{2^{2-p}\,(M\, b_T)^{p-1}}{\Gamma(p-1)}\, K_{p-1}(M\, b_T) \overset{\text{F.T.}}{\Leftarrow} \frac{\Gamma(p)}{\pi\,\Gamma(p-1)} M^{2\,(p-1)} \left(M^2 + \frac{P_T^2}{z_h^2}\right)^{-p}$$

It depends on a single free parameter z_0, through:

$$\begin{cases} p = \dfrac{1}{2}\left(\dfrac{3}{1-R} - 1\right) \\ M = \dfrac{W}{z_h}\sqrt{\dfrac{3}{1-R}} \end{cases}$$

With:

$$\begin{cases} R = 1 - (1-z)^{z_0/1-z_0}, \\ W = \dfrac{M_h}{R^2} \end{cases}$$

In total **3** free parameters: $z_0,\ g_R,\ p_1$

- The g_K-function, depending on two free parameters g_r and p_1:

$$g_K(b_T) = a(b_T)\left(\frac{b_T}{b_{\text{MAX}}}\right)^{c(b_T)}$$

With:

$$a(b_T) = g_r + \tanh p_1 - \tanh\left[p_1\left(\frac{b_T^2}{b_{\text{MAX}}^2} - 1\right)\right]$$

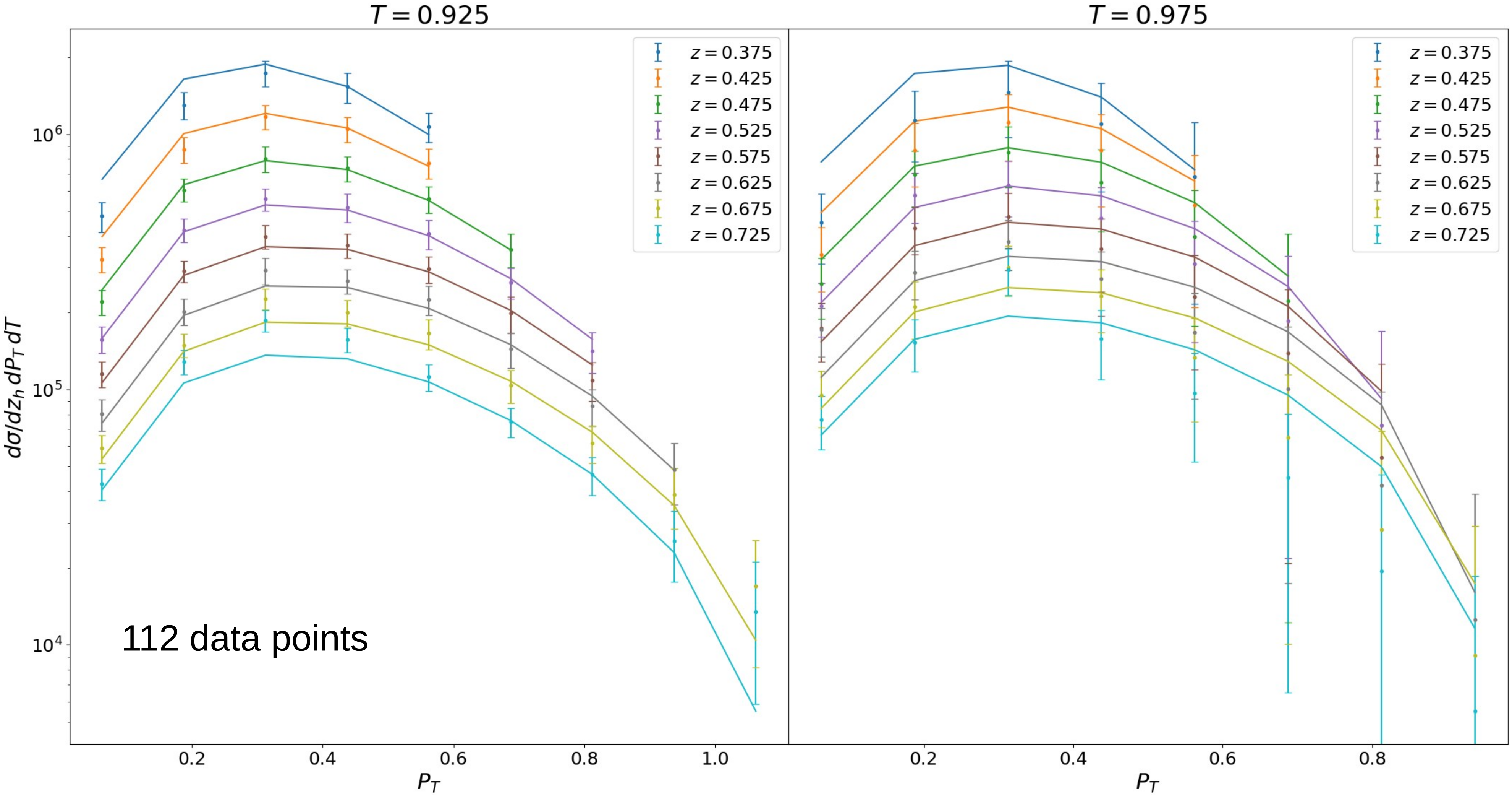
$$c(b_T) = 2\left(1 - \tanh\left(\frac{b_T^2}{b_{\text{MAX}}^2}\right)\right)$$

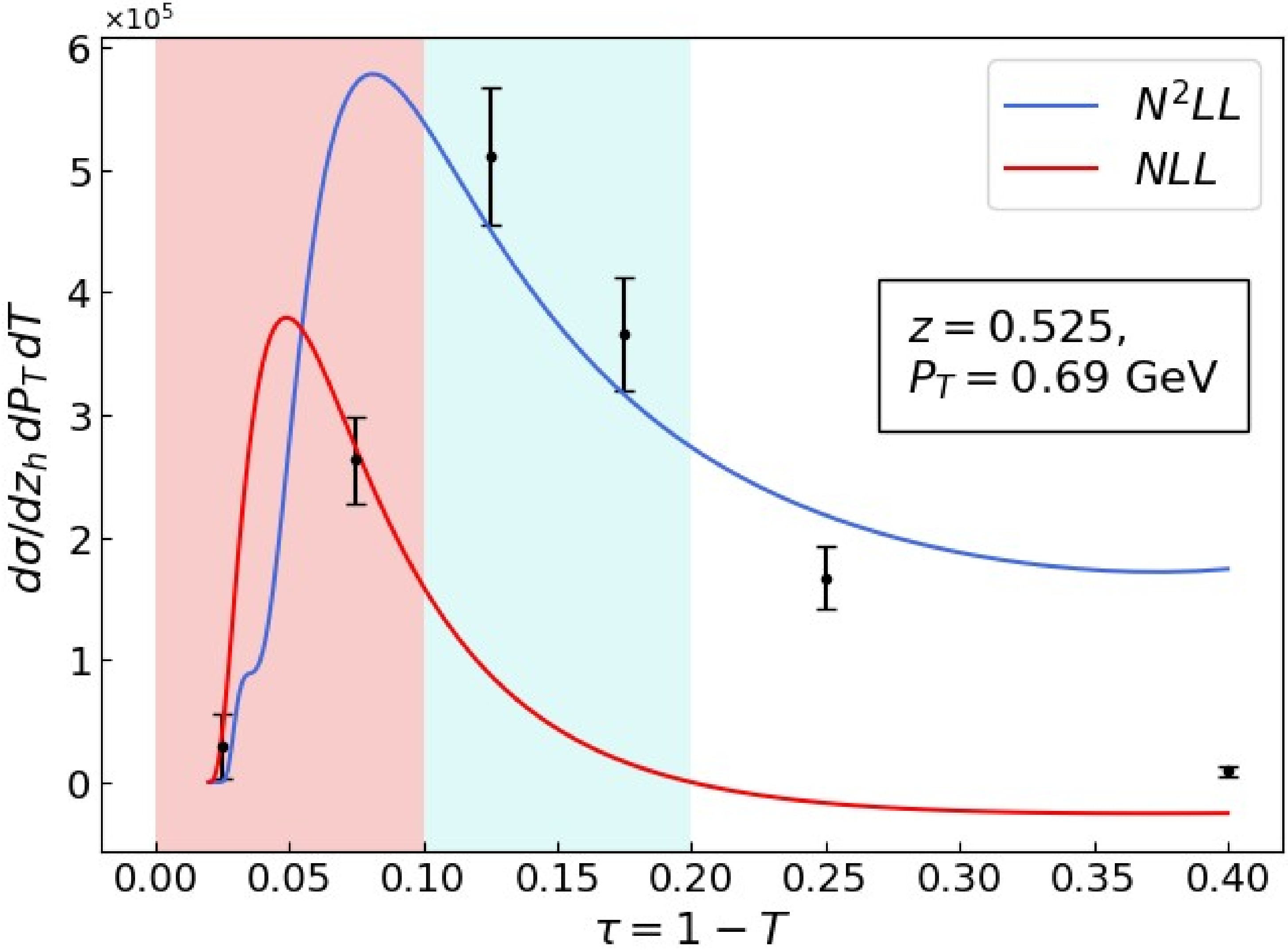
Phenomenology: preliminary FIT of BELLE data @ NLL

$\chi^2 = 0.971$

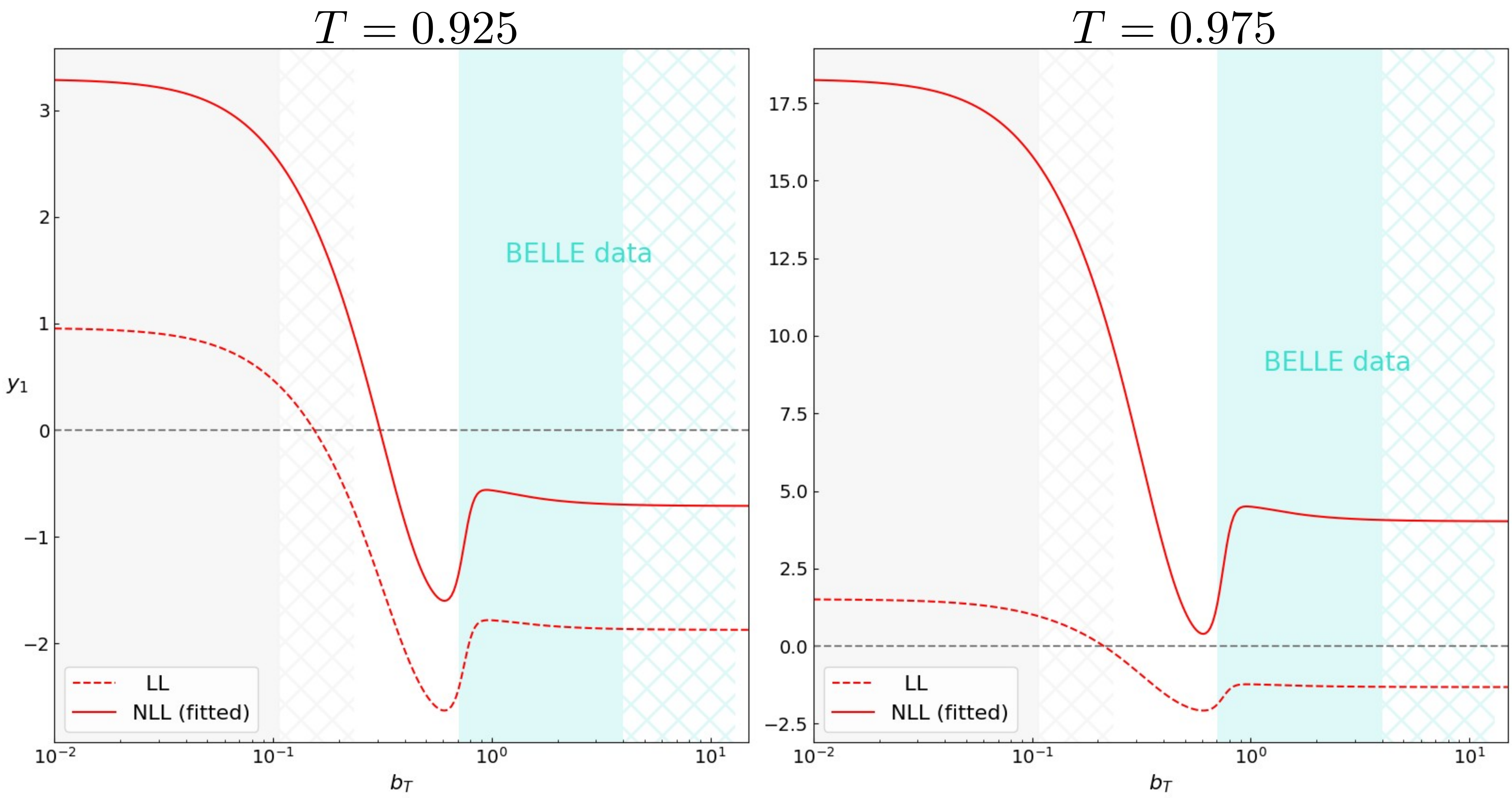
$$z_0 = 0.631 \pm 0.005$$
$$p_1 = 3.614 \pm 0.773$$
$$g_r = 0.704 \pm 0.010$$

In this analysis:
$b_{\text{MAX}} = 0.7\ \text{GeV}^{-1}$



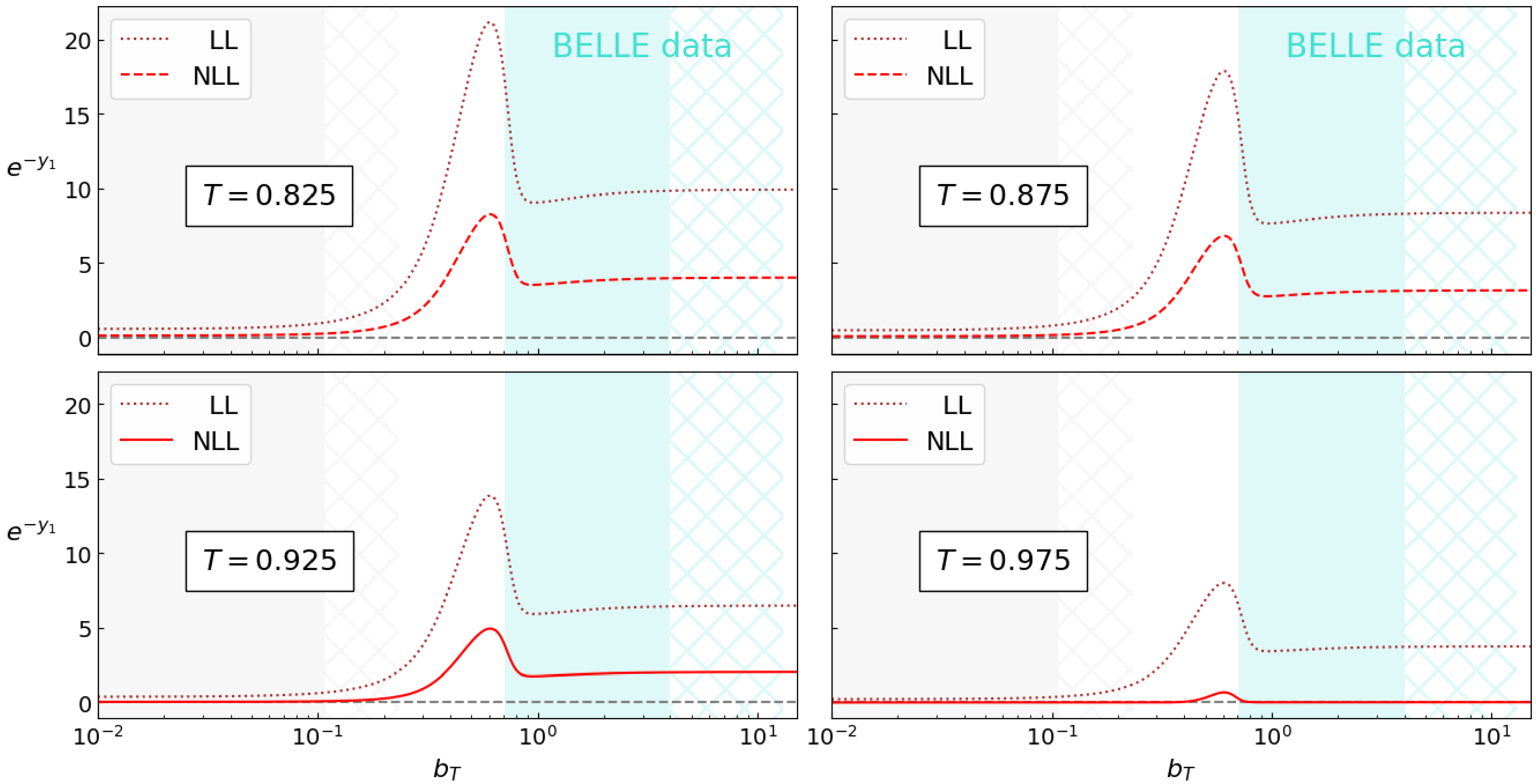

×10⁵
6
5
4
3
2
1
0
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
$d\sigma/dz_h\,dP_T\,dT$
$\tau = 1 - T$
N^2LL
NLL
$z = 0.525,$
$P_T = 0.69$ GeV

Is the rapidity cut-off large and positive?



- LL is not enough to have the errors associated with the largeness of y_1 under control.
- NLL seems enough for $T = 0.975$, but not satisfactory for $T = 0.925$. Here y_1 becomes negative around $b_T \sim 0.3\ \mathrm{GeV}^{-1}$.

Is the rapidity cut-off large and positive?



Is the rapidity cut-off large and positive?