Recent progress on TMDs and PDFs of spin-1 hadrons

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6th international workshop on transverse phenomena in hard processes
and the transverse structure of the proton
(In-person/Online) Pavia, Italy, May 23-27, 2022
https://agenda.infn.it/event/19219/

May 27, 2022

Contents

1. Introduction

- Tensor-polarized structure functions, gluon transversity, TMDs

2. TMDs, PDFs, multiparton distribution functions of spin-1 hadrons

- TMDs and PDFs up to twist 4 [1]
- Twist-2 relation and sum rule for PDFs [2]
- Relations from equation of motion and a Lorentz-invariance relation [3]

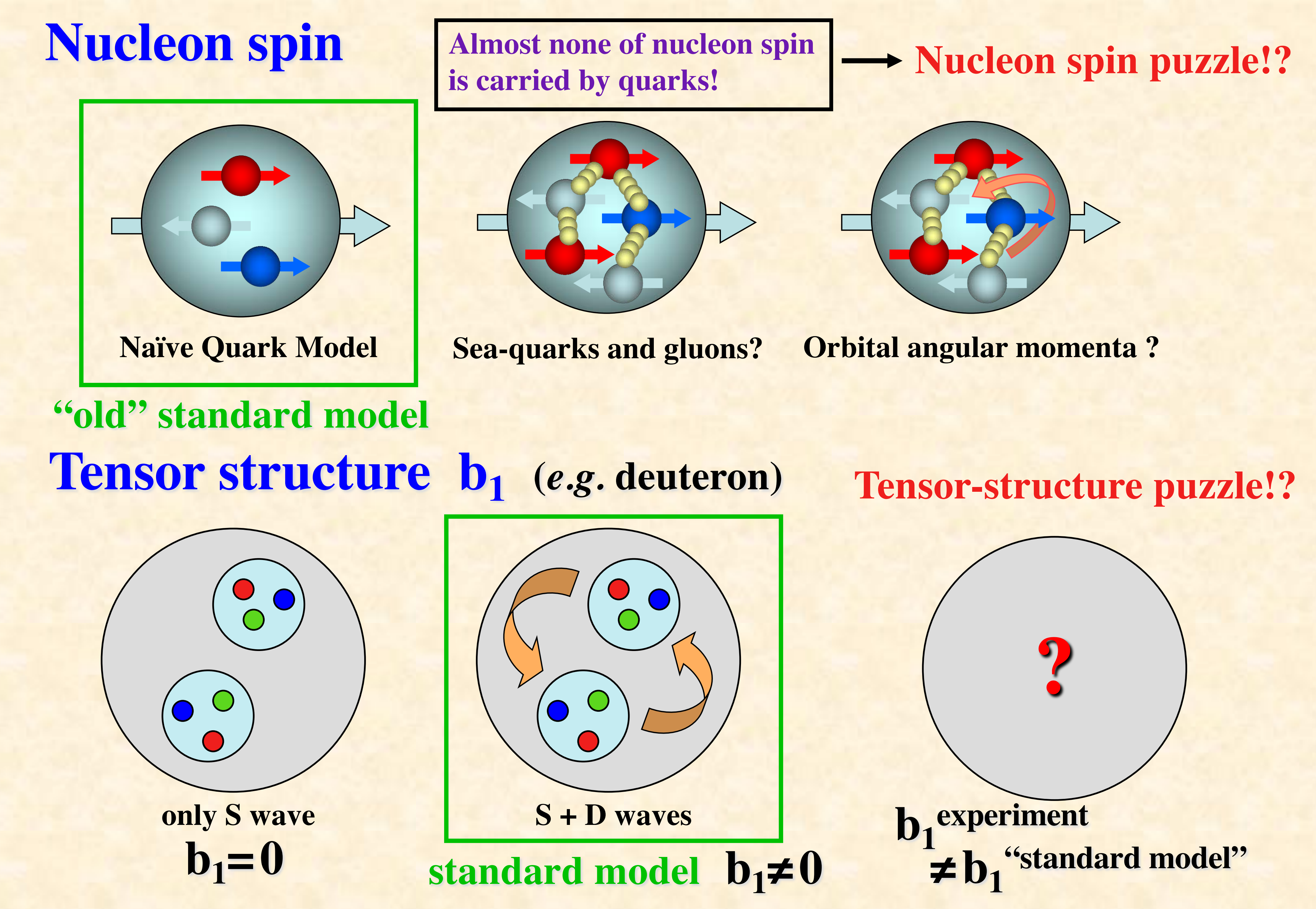
3. Future prospects and summary

References

[1] SK and Qin-Tao Song, PRD 103 (2021) 014025.

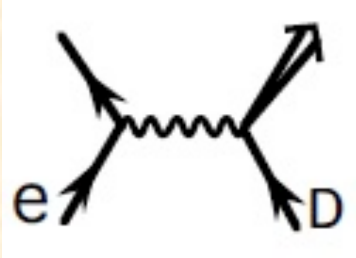
[2] JHEP 09 (2021) 141.

[3] PLB 826 (2022) 136908.

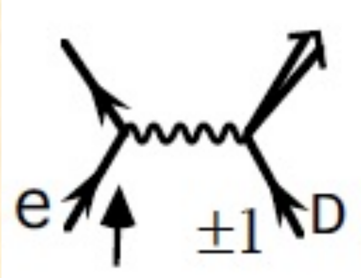

Nucleon spin
Almost none of nucleon spin is carried by quarks!
Nucleon spin puzzle!?
Naïve Quark Model
Sea-quarks and gluons?
Orbital angular momenta ?
“old” standard model
Tensor structure b_1 (*e.g.* deuteron)
Tensor-structure puzzle!?
?
only S wave
$b_1 = 0$
S + D waves
standard model $b_1 \neq 0$
$b_1^{\text{experiment}} \neq b_1^{\text{“standard model”}}$

Structure Functions

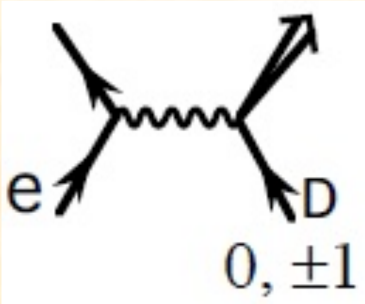
$$F_1 \propto \langle d\sigma \rangle$$



$$g_1 \propto d\sigma(\uparrow,+1) - d\sigma(\uparrow,-1)$$



$$b_1 \propto d\sigma(0) - \frac{d\sigma(+1) + d\sigma(-1)}{2}$$



note: $\sigma(0) - \frac{\sigma(+1)+\sigma(-1)}{2} = 3\langle\sigma\rangle - \frac{3}{2}[\sigma(+1)+\sigma(-1)]$

Parton Model

$$F_1 = \frac{1}{2}\sum_i e_i^2 \left(q_i + \bar{q}_i\right) \qquad q_i = \frac{1}{3}\left(q_i^{+1} + q_i^{0} + q_i^{-1}\right)$$

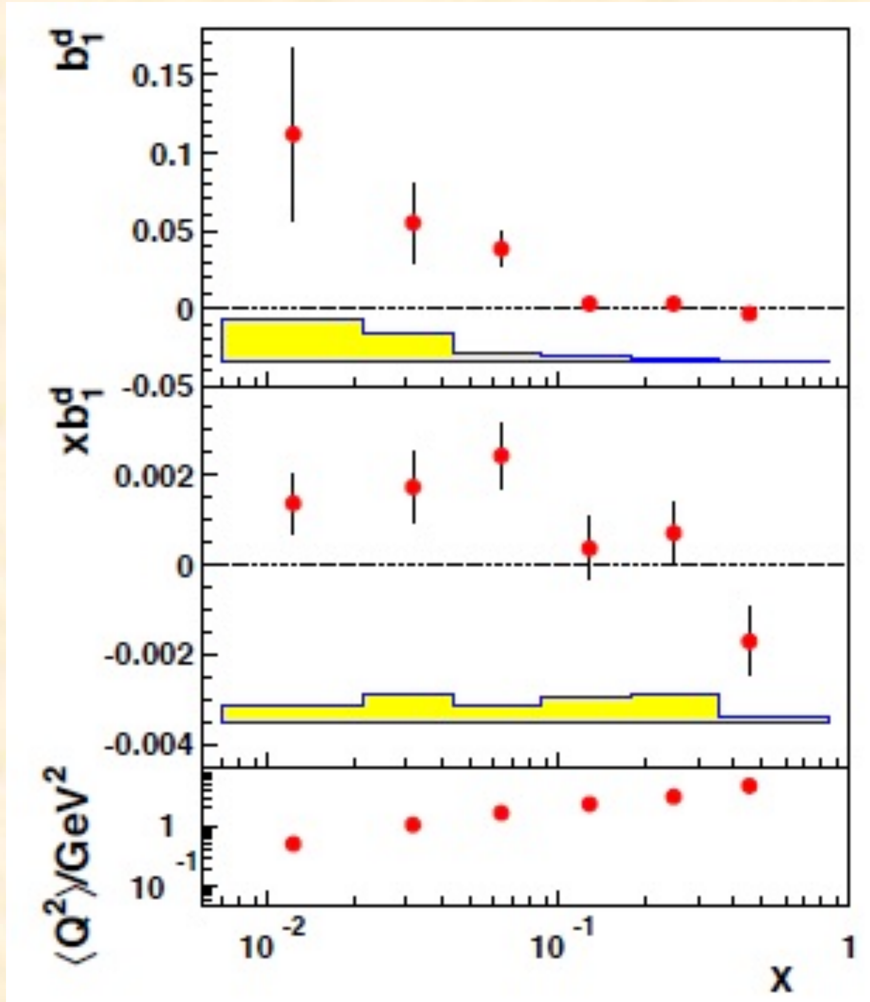
$$g_1 = \frac{1}{2}\sum_i e_i^2 \left(\Delta q_i + \Delta \bar{q}_i\right) \qquad \Delta q_i = q_{i\uparrow}^{+1} - q_{i\downarrow}^{+1}$$

$$\left[q_{\uparrow}^{H}\left(x,Q^2\right)\right]$$

$$b_1 = \frac{1}{2}\sum_i e_i^2 \left(\delta_T q_i + \delta_T \bar{q}_i\right) \qquad \delta_T q_i = q_i^0 - \frac{q_i^{+1} + q_i^{-1}}{2}$$

HERMES results on b_1

A. Airapetian *et al.* (HERMES), PRL 95 (2005) 242001.



27.6 GeV/c $\rightleftarrows$, 0

positron → ● **deuteron**

b_1 **measurement in the kinematical region**

$$0.01 < x < 0.45, \;\; 0.5 \text{ GeV}^2 < Q^2 < 5 \text{ GeV}^2$$

b_1 **sum in the restricted** Q^2 **range** $Q^2 > 1 \text{ GeV}^2$

$$\int_{0.02}^{0.85} dx\, b_1(x) = [0.35 \pm 0.10(\text{stat}) \pm 0.18(\text{sys})] \times 10^{-2}$$

at $Q^2 = 5 \text{ GeV}^2$

$$\int dx\, b_1^D(x) = \lim_{t\to 0} -\frac{5}{12}\frac{t}{M^2} F_Q(t) + \sum_i e_i^2 \int dx\, \delta_T \bar{q}_i(x) = 0 \; ?$$

b_1 sum rule: F. E. Close and SK, PRD 42 (1990) 2377.

$$\int \frac{dx}{x}\left[F_2^p(x) - F_2^n(x)\right] = \frac{1}{3}\int dx\left[u_v - d_v\right] + \frac{2}{3}\int dx\left[\bar{u} - \bar{d}\right] \neq 1/3$$

Drell-Yan experiments probe these antiquark distributions.

Standard model prediction for b_1 of deuteron

$$b_1(x) = \int \frac{dy}{y} \delta_T f(y) F_1^N(x/y, Q^2), \quad y = \frac{M p\cdot q}{M_N P\cdot q} \simeq \frac{2p^-}{P^-}$$

$$\delta_T f(y) = f^0(y) - \frac{f^+(y) + f^-(y)}{2}$$

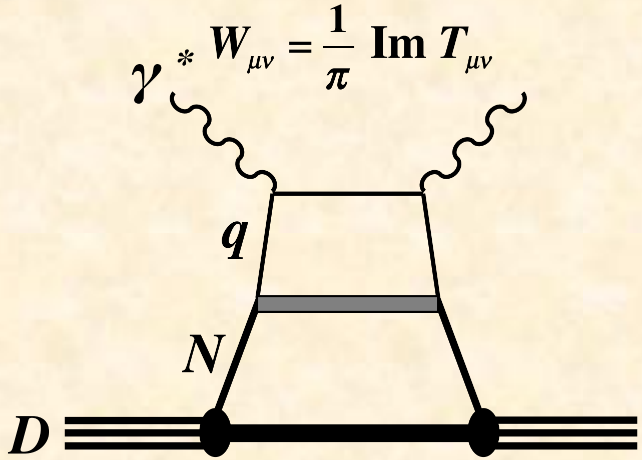
$$= \int d^3p\, y \left[-\frac{3}{4\sqrt{2}\pi} \phi_0(p)\phi_2(p) + \frac{3}{16\pi} |\phi_2(p)|^2 \right] (3\cos^2\theta - 1)\,\delta\left(y - \frac{p\cdot q}{M_N \nu} \right)$$

S-D term D-D term

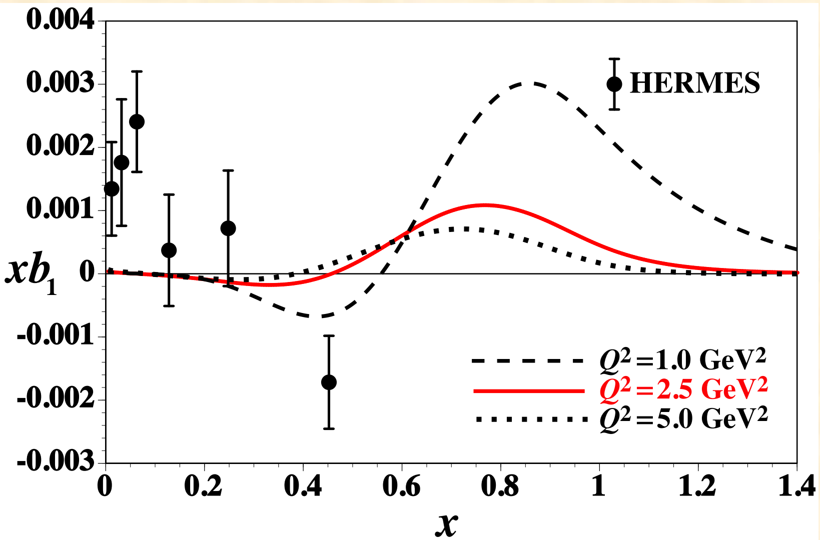
Nucleon momentum distribution:

$$f^H(y) \equiv f_\uparrow^H(y) + f_\downarrow^H(y) = \int d^3p\, y\, |\phi^H(\vec{p})|^2 \delta\left(y - \frac{E - p_z}{M_N} \right)$$

D-state admixture: $\phi^H(\vec{p}) = \phi^H_{\ell=0}(\vec{p}) + \phi^H_{\ell=2}(\vec{p})$



Standard model of the deuteron



W. Cosyn, Yu-Bing Dong, SK, M. Sargsian, Phys. Rev. D 95 (2017) 074036.

$|b_1(\text{theory})| \ll |b_1(\text{HERMES})|$ at $x < 0.5$

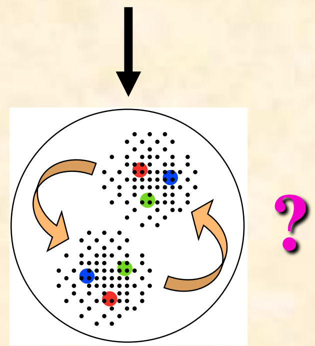
Standard convolution model does not work for the deuteron tensor structure!?

G. A. Miller, PRC 89 (2014) 045203,

Interesting suggestions:

hidden-color, 6-quark, · · ·

$$|6q\rangle = |NN\rangle + |\Delta\Delta\rangle + |CC\rangle + \cdots$$



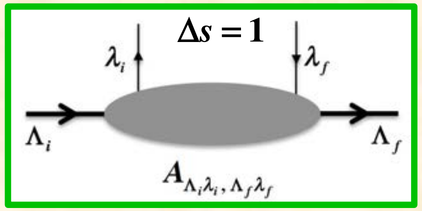
Gluon transversity $\Delta_T g$

Note on our notations:
Tensor-polarized gluon distribution: $\delta_T g$
Gluon transversity: $\Delta_T g$

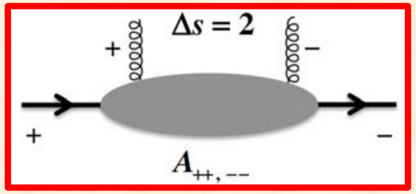
Helicity amplitude $A(\Lambda_i, \lambda_i, \Lambda_f, \lambda_f)$, conservation $\Lambda_i - \lambda_i = \Lambda_f - \lambda_f$

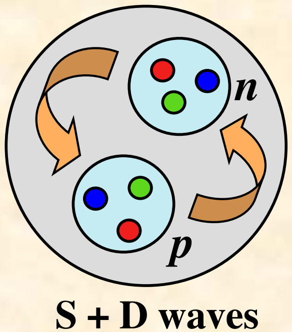
Longitudinally-polarized quark in nucleon: $\Delta q(x) \sim A\left(+\frac{1}{2}+\frac{1}{2}, +\frac{1}{2}+\frac{1}{2}\right) - A\left(+\frac{1}{2}-\frac{1}{2}, +\frac{1}{2}-\frac{1}{2}\right)$

Quark transversity in nucleon: $\Delta_T q(x) \sim A\left(+\frac{1}{2}+\frac{1}{2}, -\frac{1}{2}-\frac{1}{2}\right)$, $\lambda_i = +\frac{1}{2} \to \lambda_f = -\frac{1}{2}$ quark spin flip ($\Delta s = 1$)



Gluon transversity in deuteron: $\Delta_T g(x) \sim A(+1+1, -1-1)$, ~~$A\left(+\frac{1}{2}+1, -\frac{1}{2}-1\right)$~~ not possible for nucleon





Note: Gluon transversity does not exist for spin-1/2 nucleons.

$b_1\ (\delta_T q,\ \delta_T g) \neq 0 \iff$ still $\Delta_T g = 0$

What would be the mechanism(s) for creating $\Delta_T g \neq 0$?

Spin-1 deuteron experiments from the middle of 2020's

JLab

The Deuteron Tensor Structure Function b_1

A Proposal to Jefferson Lab PAC-38.
(Update to LOI-11-003)

J.-P. Chen (co-spokesperson), P. Solvignon (co-spokesperson), K. Allada, A. Camsonne, A. Deur, D. Gaskell, C. Keith, S. Wood, J. Zhang
Thomas Jefferson National Accelerator Facility, Newport News, VA 23606

N. Kalantarians (co-spokesperson), O. Rondon (co-spokesperson) Donal B. Day, Hovhannes Baghdasaryan, Charles Hanretty Richard Lindgren, Blaine Norum, Zhihong Ye
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K. Slifer† (co-spokesperson), A. Atkins, T. Badman, J. Calarco, J. Maxwell, S. Phillips, R. Zielinski
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Hebrew University of Jerusalem, Jerusalem

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Seonho Choi, Hoyoung Kang, Hyekoo Kang, Yoomin Oh
Seoul National University, Seoul 151-747 Korea

Proposal (approved),
Experiment: middle of 2020's

A Letter of Intent to Jefferson Lab PAC 44, June 6, 2016
Search for Exotic Gluonic States in the Nucleus

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Thomas Jefferson National Accelerator Facility, Newport News, VA 23606
W. Detmold, R. Jaffe, R. Milner, P. Shanahan
Laboratory for Nuclear Science, MIT, Cambridge, MA 02139
D. Crabb, D. Day, D. Keller, O. A. Rondon
University of Virginia, Charlottesville, VA 22904
J. Pierce
Oak Ridge National Laboratory, Oak Ridge, TN 37831

Fermilab

The Transverse Structure of the Deuteron with Drell-Yan

D. Keller[1]
[1] *University of Virginia, Charlottesville, VA 22904*

Proposal,
Fermilab-PAC: 2022
Experiment: 2020's

NICA

Progress in Particle and Nuclear Physics 119 (2021) 103858
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Progress in Particle and Nuclear Physics
ELSEVIER
journal homepage: www.elsevier.com/locate/ppnp

Review

On the physics potential to study the gluon content of proton and deuteron at NICA SPD

A. Arbuzov[a], A. Bacchetta[b,c], M. Butenschoen[d], F.G. Celiberto[b,c,e,f], U. D'Alesio[g,h], M. Deka[a], I. Denisenko[a], M.G. Echevarria[i], A. Efremov[a], N.Ya. Ivanov[a,j], A. Guskov[a,k,*], A. Karpishkov[l,a], Ya. Klopot[a,m], B.A. Kniehl[d], A. Kotzinian[j,o], S. Kumano[p], J.P. Lansberg[q], Keh-Fei Liu[r], F. Murgia[h], M. Nefedov[l], B. Parsamyan[a,n,o], C. Pisano[g,h], M. Radici[c], A. Rymbekova[a], V. Saleev[l,a], A. Shipilova[l,a], Qin-Tao Song[s], O. Teryaev[a]

Prog. Nucl. Part. Phys.
119 (2021) 103858,
Experiment: middle of 2020's

LHCspin

CERN-ESPP-Note-2018-111

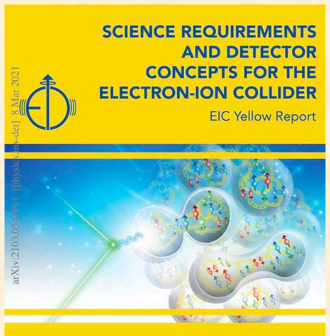
The LHCSpin Project

C. A. Aidala[1], A. Bacchetta[2,3], M. Boglione[4,5], G. Bozzi[2,3], V. Carassiti[6,7], M. Chiosso[4,5], R. Cimino[8], G. Ciullo[6,7], M. Contalbrigo[6,7], U. D'Alesio[9,10], P. Di Nezza[8], R. Engels[11], K. Grigoryev[11], D. Keller[12], P. Lenisa[6,7], S. Liuti[12], A. Metz[13], P.J. Mulders[14,15], F. Murgia[10], A. Nass[11], D. Panzieri[5,16], L. L. Pappalardo[6,7], B. Pasquini[2,3], C. Pisano[9,10], M. Radici[3], F. Rathmann[11], D. Reggiani[17], M. Schlegel[18], S. Scopetta[19,20], E. Steffens[21], A. Vasilyev[22]

arXiv:1901.08002,
Experiment: ~2028

see Appendix V
for some history

2030's EIC/EicC



R. Abdul Khalek *et al*.
arXiv:2103.05419.

D. P. Anderle *et al*.,
Front. Phys. 16 (2021) 64701.

Frontiers of Physics
https://doi.org/10.1007/s11467-021-1062-0
Front. Phys. 16(6), 64701 (2021)

REVIEW ARTICLE

Electron-ion collider in China

Daniele P. Anderle[1], Valerio Bertone[2], Xu Cao[3,4], Lei Chang[5], Ningbo Chang[6], Gu Chen[7], Xurong Chen[3,4], Zhuojun Chen[8], Zhufang Cui[9], Lingyun Dai[8], Weitian Deng[10], Minghui Ding[11], Xu Feng[12], Chang Gong[12], Longcheng Gui[13], Feng-Kun Guo[4,14], Chengdong Han[3,4], Jun He[15], Tie-Jiun Hou[16], Hongxia Huang[15], Yin Huang[17], Krešimir Kumerički[18], L. P. Kaptari[3,19], Demin Li[20], Hengne Li[1], Minxiang Li[3,21], Xueqian Li[5], Yutie Liang[3,4], Zuotang Liang[22], Chen Liu[22], Chuan Liu[12], Guoming Liu[1], Jie Liu[3,4], Liuming Liu[3,4], Xiang Liu[21], Tianbo Liu[22], Xiaofeng Luo[23], Zhun Lyu[24], Boqiang Ma[12], Fu Ma[3,4], Jianping Ma[4,14], Yugang Ma[4,25,26], Lijun Mao[3,4], Cédric Mezrag[2], Hervé Moutarde[2], Jialun Ping[15], Sixue Qin[27], Hang Ren[3,4], Craig D. Roberts[9], Juan Rojo[28,29], Guodong Shen[3,4], Chao Shi[30], Qintao Song[20], Hao Sun[31], Paweł Sznajder[32], Enke Wang[1], Fan Wang[9], Qian Wang[1], Rong Wang[3,4], Ruiru Wang[3,4], Taofeng Wang[33], Wei Wang[34], Xiaoyu Wang[20], Xiaoyun Wang[35], Jiajun Wu[4], Xinggang Wu[27], Lei Xia[36], Bowen Xiao[23,37], Guoqing Xiao[3,4], Ju-Jun Xie[3,4], Yaping Xie[3,4], Hongxi Xing[1], Hushan Xu[3,4], Nu Xu[3,4,23], Shusheng Xu[38], Mengshi Yan[12], Wenbiao Yan[36], Wencheng Yan[20], Xinhu Yan[39], Jiancheng Yang[3,4], Yi-Bo Yang[4,14], Zhi Yang[40], Deliang Yao[8], Zhihong Ye[41], Peilin Yin[38], C.-P. Yuan[42], Wenlong Zhan[3,4], Jianhui Zhang[43], Jinlong Zhang[22], Pengming Zhang[44], Yifei Zhang[36], Chao-Hsi Chang[4,14], Zhenyu Zhang[45], Hongwei Zhao[3,4], Kuang-Ta Chao[12], Qiang Zhao[4,46], Yuxiang Zhao[3,4], Zhengguo Zhao[36], Liang Zheng[47], Jian Zhou[22], Xiang Zhou[45], Xiaorong Zhou[36], Bingsong Zou[4,14], Liping Zou[3,4]

Twist-2 TMDs for spin-1/2 nucleons and spin-1 hadrons

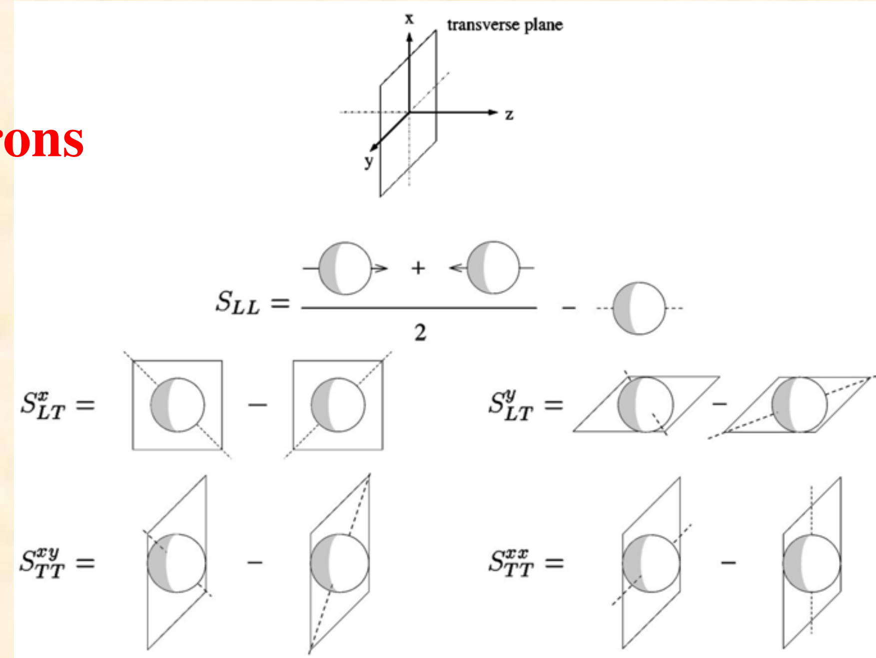
Twist-2 TMDs

Bacchetta-Mulders, PRD 62 (2000) 114004.

Hadron \ Quark	U (γ^+) T-even	U (γ^+) T-odd	L ($\gamma^+\gamma_5$) T-even	L ($\gamma^+\gamma_5$) T-odd	T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$) T-even	T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$) T-odd
U	f_1					$[h_1^\perp]$
L			g_{1L}		$[h_{1L}^\perp]$	
T		$f_{1T}^\perp$	g_{1T}		$[h_1], [h_{1T}^\perp]$	
LL	f_{1LL}					$[h_{1LL}^\perp]$
LT	f_{1LT}			g_{1LT}		$[h_{1LT}], [h_{1LT}^\perp]$
TT	f_{1TT}			g_{1TT}		$[h_{1TT}], [h_{1TT}^\perp]$

Spin-1/2 nucleon (also spin-1 hadrons) (rows U, L, T)

Spin-1 hadrons (rows LL, LT, TT)



Twist-2 collinear PDFs

$[\cdots]$= chiral odd

Hadron \ Quark	U (γ^+) T-even	U (γ^+) T-odd	L ($\gamma^+\gamma_5$) T-even	L ($\gamma^+\gamma_5$) T-odd	T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$) T-even	T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$) T-odd
U	f_1					
L			$g_{1L}\,(g_1)$			
T					$[h_1]$	
LL	$f_{1LL}\,(b_1)$					
LT						*1
TT						

*1 Because of the time-reversal invariance, the collinear PDF $h_{1LT}(x)$ vanishes. However, since the time-reversal invariance cannot be imposed in the fragmentation functions, we should note that the corresponding fragmentation function $H_{1LT}(z)$ should exist as a collinear fragmentation function. (see our PRD paper for the details)

TMDs and PDFs for spin-1 hadrons up to twist 4

Note: Higher-twist effects are sizable at a few GeV^2 Q^2 in tensor-polarized structure functions, W. Cosyn, Yu-Bing Dong, SK, M. Sargsian, PRD 95 (2017) 074036.

SK and Qin-Tao Song, PRD 103 (2021) 014025.

TMD correlation functions for spin-1 hadrons

Spin vector: $S^\mu = S_L \dfrac{P^+}{M}\bar n^\mu - S_L \dfrac{M}{2P^+} n^\mu + S_T^\mu$

Tensor: $T^{\mu\nu} = \dfrac{1}{2}\left[\dfrac{4}{3} S_{LL}\dfrac{(P^+)^2}{M^2}\bar n^\mu \bar n^\nu + \dfrac{P^+}{M}\bar n^{\{\mu}S_{LT}^{\nu\}} - \dfrac{2}{3} S_{LL}\left(\bar n^{\{\mu}n^{\nu\}} - g_T^{\mu\nu}\right) + S_{TT}^{\mu\nu} - \dfrac{M}{2P^+}n^{\{\mu}S_{LT}^{\nu\}} + \dfrac{1}{3} S_{LL}\dfrac{M^2}{(P^+)^2}n^\mu n^\nu\right]$

$$\Phi_{ij}(k,P,T) = \int \frac{d^4\xi}{(2\pi)^4} e^{ik\cdot\xi}\left\langle P,T\right|\bar\psi_j(0)W(0,\xi)\psi_i(\xi)\left|P,T\right\rangle$$

$$W(0,\xi) = P\exp\left[-ig\int_0^\xi d\xi\cdot A(\xi)\right]$$

Tensor part (twist-2): **Bacchetta, Mulders, PRD 62 (2000) 114004**

$$\Phi(k,P,T) = \left(\frac{A_{13}}{M}I + \frac{A_{14}}{M^2}\not P + \frac{A_{15}}{M^2}\not k + \frac{A_{16}}{M^3}\sigma_{\rho\sigma}P^\rho k^\sigma\right)k_\mu k_\nu T^{\mu\nu} + \left[A_{17}\gamma_\nu + \left(\frac{A_{18}}{M}P^\rho + \frac{A_{19}}{M}k^\rho\right)\sigma_{\nu\rho} + \frac{A_{20}}{M^2}\varepsilon_{\tau\nu\rho\sigma}P^\rho k^\sigma\gamma^\tau\gamma_5\right]k_\mu T^{\mu\nu}$$

Tensor part (twist-2, 3, 4): n^μ dependent terms are added for up to twist 4.

[For the spin-1/2 nucleon: Goeke, Metzand, Schlegel, PLB 618 (2005) ,90; Metz, Schweitzer, Teckentrup, PLB 680 (2009) 141.]

Kumano-Song-2021, for the details see PRD 103 (2021) 014025

$$\Phi(k,P,T\,|\,n) = \left(\frac{A_{13}}{M}I + \frac{A_{14}}{M^2}\not P + \frac{A_{15}}{M^2}\not k + \frac{A_{16}}{M^3}\sigma_{\rho\sigma}P^\rho k^\sigma\right)k_\mu k_\nu T^{\mu\nu} + \left[A_{17}\gamma_\nu + \left(\frac{A_{18}}{M}P^\rho + \frac{A_{19}}{M}k^\rho\right)\sigma_{\nu\rho} + \frac{A_{20}}{M^2}\varepsilon_{\tau\nu\rho\sigma}P^\rho k^\sigma\gamma^\tau\gamma_5\right]k_\mu T^{\mu\nu}$$

Bacchetta -Mulders

$$+\left(\frac{B_{21}M}{P\cdot n}k_\mu + \frac{B_{22}M^3}{(P\cdot n)^2}n_\mu\right)n_\nu T^{\mu\nu} + i\gamma_5\varepsilon_{\mu\tau\rho\sigma}P^\rho\left(\frac{B_{23}}{(P\cdot n)M}k^\tau n^\sigma k_\nu + \frac{B_{24}M}{(P\cdot n)^2}k^\tau n^\sigma n_\nu\right)T^{\mu\nu}$$

$$+\left[\frac{B_{25}}{P\cdot n}\not n k_\mu k_\nu + \left(\frac{B_{26}M^2}{(P\cdot n)^2}\not n + \frac{B_{28}}{P\cdot n}\not P + \frac{B_{30}}{P\cdot n}\not k\right)k_\mu n_\nu + \left(\frac{B_{27}M^4}{(P\cdot n)^3}\not n + \frac{B_{29}M^2}{(P\cdot n)^2}\not P + \frac{B_{31}M^2}{(P\cdot n)^2}\not k\right)n_\mu n_\nu + \frac{B_{32}M^2}{P\cdot n}\gamma_\mu n_\nu\right]T^{\mu\nu}$$

$$-\left[\varepsilon_{\mu\tau\rho\sigma}\gamma^\tau P^\rho\left(\frac{B_{34}}{P\cdot n}n^\sigma k_\nu + \frac{B_{33}}{P\cdot n}k^\sigma n_\nu + \frac{B_{35}M^2}{(P\cdot n)^2}n^\sigma n_\nu\right) + \varepsilon_{\lambda\tau\rho\sigma}k^\lambda\gamma^\tau P^\rho n^\sigma\left(\frac{B_{36}}{P\cdot nM^2}k_\mu k_\nu + \frac{B_{37}}{(P\cdot n)^2}k_\mu n_\nu + \frac{B_{38}M^2}{(P\cdot n)^3}n_\mu n_\nu\right)\right]\gamma_5 T^{\mu\nu}$$

$$+\varepsilon_{\mu\tau\rho\sigma}k^\tau P^\rho n^\sigma\left(\frac{B_{39}}{(P\cdot n)^2}k_\nu + \frac{B_{40}M^2}{(P\cdot n)^3}n_\nu\right)\not n\gamma_5 T^{\mu\nu}$$

$$+\sigma_{\rho\sigma}\left[P^\rho k^\sigma\left(\frac{B_{41}}{(P\cdot n)M}k_\mu n_\nu + \frac{B_{42}M}{(P\cdot n)^2}n_\mu n_\nu\right) + P^\rho n^\sigma\left(\frac{B_{43}}{(P\cdot n)M}k_\mu k_\nu + \frac{B_{44}M}{(P\cdot n)^2}k_\mu n_\nu + \frac{B_{45}M^3}{(P\cdot n)^3}n_\mu n_\nu\right)\right]T^{\mu\nu}$$

$$+\sigma_{\rho\sigma}\left[k^\rho n^\sigma\left(\frac{B_{46}}{(P\cdot n)M}k_\mu k_\nu + \frac{B_{47}M}{(P\cdot n)^2}k_\mu n_\nu + \frac{B_{48}M^3}{(P\cdot n)^3}n_\mu n_\nu\right)\right]T^{\mu\nu} + \sigma_{\mu\sigma}\left[n^\sigma\left(\frac{B_{49}M}{P\cdot n}k_\nu + \frac{B_{50}M^3}{(P\cdot n)^2}n_\nu\right) + \left(\frac{B_{51}M}{P\cdot n}P^\sigma + \frac{B_{52}M}{P\cdot n}k^\sigma\right)n_\nu\right]T^{\mu\nu}$$

New terms in our paper (2021)

From this correlation function, new tensor-polarized TMDs are defined in twist-3 and 4 in addition to twist-2 ones.

Terms associated with $n = \dfrac{1}{\sqrt{2}}(1,\ 0,\ 0,\ -1)$

Twist-3 TMDs for spin-1 hadrons

$$\Phi^{[\Gamma]}(x,k_T,T) \equiv \frac{1}{2}\mathrm{Tr}\left[\Phi^{[\Gamma]}(x,k_T,T)\,\Gamma\right] = \frac{1}{2}\mathrm{Tr}\left[\int dk^- \Phi(k,P,T\,|\,n)\,\Gamma\right], \quad F(x,k_T^2) \equiv F'(x,k_T^2) - \frac{k_T^2}{2M^2}F^{\perp}(x,k_T^2)$$

$$\Phi^{[\gamma^i]}(x,k_T,T) = \frac{M}{P^+}\left[f_{LL}^{\perp}(x,k_T^2)\frac{S_{LL}k_T^i}{M} + f'_{LT}(x,k_T^2)S_{LT}^i - f_{LT}^{\perp}(x,k_T^2)\frac{k_T^i S_{LT}\cdot k_T}{M^2} - f'_{TT}(x,k_T^2)\frac{S_{TT}^{ij}k_{Tj}}{M} + f_{TT}^{\perp}(x,k_T^2)\frac{k_T^i k_T\cdot S_{TT}\cdot k_T}{M^3}\right]$$

$$\Phi^{[1]}(x,k_T,T) = \frac{M}{P^+}\left[e_{LL}(x,k_T^2)S_{LL} - e_{LT}^{\perp}(x,k_T^2)\frac{S_{LT}\cdot k_T}{M} + e_{TT}^{\perp}(x,k_T^2)\frac{k_T\cdot S_{TT}\cdot k_T}{M^2}\right]$$

$$\Phi^{[i\gamma_5]}(x,k_T,T) = \frac{M}{P^+}\left[e_{LT}(x,k_T^2)\frac{S_{LT\mu}\varepsilon_T^{\mu\nu}k_{T\nu}}{M} - e_{TT}(x,k_T^2)\frac{S_{TT\mu\rho}k_T^{\rho}\varepsilon_T^{\mu\nu}k_{T\nu}}{M^2}\right]$$

$$\Phi^{[\gamma^i\gamma_5]}(x,k_T,T) = \frac{M}{P^+}\left[-g_{LL}^{\perp}(x,k_T^2)\frac{S_{LL}\varepsilon_T^{ij}k_{Tj}}{M} - g'_{LT}(x,k_T^2)\varepsilon_T^{ij}S_{LTj} + g_{LT}^{\perp}(x,k_T^2)\frac{\varepsilon_T^{ij}k_{Tj}S_{LT}\cdot k_T}{M^2} + g'_{TT}(x,k_T^2)\frac{\varepsilon_T^{ij}S_{TTjl}k_T^l}{M} - g_{TT}^{\perp}(x,k_T^2)\frac{\varepsilon_T^{ij}k_{Tj}k_T\cdot S_{TT}\cdot k_T}{M^3}\right]$$

$$\Phi^{[\sigma^{-+}]}(x,k_T,T) = \frac{M}{P^+}\left[h_{LL}(x,k_T^2)S_{LL} - h_{LT}(x,k_T^2)\frac{S_{LT}\cdot k_T}{M} + h_{TT}(x,k_T^2)\frac{k_T\cdot S_{TT}\cdot k_T}{M^2}\right]$$

$$\Phi^{[\sigma^{ij}]}(x,k_T,T) = \frac{M}{P^+}\left[h_{LT}^{\perp}(x,k_T^2)\frac{S_{LT}^i k_T^j - S_{LT}^j k_T^i}{M} - h_{TT}^{\perp}(x,k_T^2)\frac{S_{TT}^{il}k_{Tl}k_T^j - S_{TT}^{jl}k_{Tl}k_T^i}{M^2}\right]$$

*2, *3 Because of the time-reversal invariance, the collinear PDFs $g_{LT}(x)$ and $h_{LL}(x)$ do not exist. However, the corresponding new collinear fragmentation functions $G_{LT}(z)$ and $H_{LL}(z)$ should exist. (see our PRD paper for the details)

Hadron \ Quark	$\gamma^i, 1, i\gamma_5$		$\gamma^+\gamma_5$		σ^{ij}, σ^{-+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	$f^{\perp}$ $[e]$			$g^{\perp}$		$[h]$
L		$f_L^{\perp}$ $[e_L]$	$g_L^{\perp}$		$[h_L]$	
T		$f_T, f_T^{\perp}$ $[e_T, e_T^{\perp}]$	$g_T, g_T^{\perp}$		$[h_T], [h_T^{\perp}]$	
LL	$f_{LL}^{\perp}$ $[e_{LL}]$			$g_{LL}^{\perp}$		$[h_{LL}]$
LT	$f_{LT}, f_{LT}^{\perp}$ $[e_{LT}, e_{LT}^{\perp}]$			$g_{LT}, g_{LT}^{\perp}$		$[h_{LT}], [h_{LT}^{\perp}]$
TT	$f_{TT}, f_{TT}^{\perp}$ $[e_{TT}, e_{TT}^{\perp}]$			$g_{TT}, g_{TT}^{\perp}$		$[h_{TT}], [h_{TT}^{\perp}]$

Hadron \ Quark	$\gamma^i, 1, i\gamma_5$		$\gamma^+\gamma_5$		σ^{ij}, σ^{-+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	$[e]$					
L					$[h_L]$	
T			g_T			
LL	$[e_{LL}]$					*3
LT	f_{LT}			*2		
TT						

New TMDs $[\cdots]$= chiral odd **New collinear PDFs**

Twist-4 TMDs for spin-1 hadrons

may skip

$$\Phi^{[\Gamma]}(x, k_T, T) \equiv \frac{1}{2}\mathrm{Tr}\left[\Phi^{[\Gamma]}(x, k_T, T)\,\Gamma\right] = \frac{1}{2}\mathrm{Tr}\left[\int dk^- \Phi(k, P, T\,|\,n)\,\Gamma\right], \quad F(x, k_T^2) \equiv F'(x, k_T^2) - \frac{k_T^2}{2M^2}F^\perp(x, k_T^2)$$

$$\Phi^{[\gamma^-]}(x, k_T, T) = \frac{M^2}{P^{+2}}\left[f_{3LL}(x, k_T^2) S_{LL} - f_{3LT}(x, k_T^2)\frac{S_{LT}\cdot k_T}{M} + f_{3TT}(x, k_T^2)\frac{k_T\cdot S_{TT}\cdot k_T}{M^2}\right]$$

$$\Phi^{[\gamma^-\gamma_5]}(x, k_T, T) = \frac{M^2}{P^{+2}}\left[g_{3LT}(x, k_T^2)\frac{S_{LT\mu}\varepsilon_T^{\mu\nu}k_{T\nu}}{M} + g_{3TT}(x, k_T^2)\frac{S_{TT\mu\rho}k_T^\rho\varepsilon_T^{\mu\nu}k_{T\nu}}{M^2}\right]$$

$$\Phi^{[\sigma^{i-}]}(x, k_T, T) = \frac{M^2}{P^{+2}}\left[h_{3LL}^\perp(x, k_T^2)\frac{S_{LL}k_T^i}{M} + h'_{3LT}(x, k_T^2)S_{LT}^i - h_{3LT}^\perp(x, k_T^2)\frac{k_T^i S_{LT}\cdot k_T}{M^2} - h'_{3TT}(x, k_T^2)\frac{S_{TT}^{ij}k_{Tj}}{M} + h_{3TT}^\perp(x, k_T^2)\frac{k_T^i k_T\cdot S_{TT}\cdot k_T}{M^3}\right]$$

*4 Because of the time-reversal invariance, $h_{3LT}(x)$ does not exist; however, the corresponding new collinear fragmentation function $H_{3LT}(z)$ should exist because the time-reversal invariance does not have to be imposed.

Quark / Hadron	γ^-		$\gamma^-\gamma_5$		σ^{i-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_3					$[h_3^\perp]$
L			g_{3L}		$[h_{3L}^\perp]$	
T		$f_{3T}^\perp$	g_{3T}		$[h_{3T}], [h_{3T}^\perp]$	
LL	f_{3LL}					$[h_{3LL}^\perp]$
LT	f_{3LT}			g_{3LT}		$[h_{3LT}], [h_{3LT}^\perp]$
TT	f_{3TT}			g_{3TT}		$[h_{3TT}], [h_{3TT}^\perp]$

New TMDs

$[\cdots]$= chiral odd

Quark / Hadron	γ^-		$\gamma^-\gamma_5$		σ^{i-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_3					
L			g_{3L}			
T					$[h_{3T}]$	
LL	f_{3LL}					
LT						*4
TT						

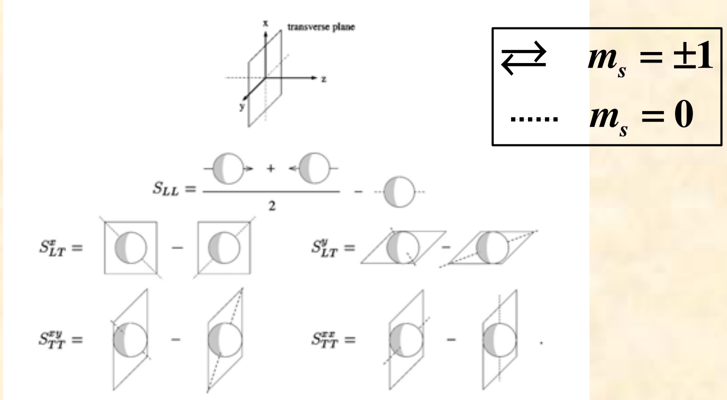
New collinear PDFs

TMDs and their sum rules for spin-1 hadrons

see our PRD paper for the details

Twist-2 TMDs Bacchetta-Mulders, PRD 62 (2000) 114004.

Quark / Hadron	U (γ^+) T-even	U (γ^+) T-odd	L ($\gamma^+\gamma_5$) T-even	L ($\gamma^+\gamma_5$) T-odd	T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$) T-even	T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$) T-odd
U	f_1					$[h_1^\perp]$
L			g_{1L}		$[h_{1L}^\perp]$	
T		$f_{1T}^\perp$	g_{1T}		$[h_1], [h_{1T}^\perp]$	
LL	f_{1LL}					$[h_{1LL}^\perp]$
LT	f_{1LT}			g_{1LT}		$[h_{1LT}], [h_{1LT}^\perp]$
TT	f_{1TT}			g_{1TT}		$[h_{1TT}], [h_{1TT}^\perp]$



Time-reversal invariance in colliear corrlation functions (PDFs)

$$\int d^2k_T \Phi_{\text{T-odd}}(x, k_T^2) = 0$$

Sum rules for the TMDs of spin-1 hadrons

$$\int d^2k_T\, h_{1LT}(x, k_T^2) = 0, \qquad \int d^2k_T\, g_{LT}(x, k_T^2) = 0,$$

$$\int d^2k_T\, h_{LL}(x, k_T^2) = 0, \qquad \int d^2k_T\, h_{3LT}(x, k_T^2) = 0$$

Twist-3 TMDs SK and Qin-Tao Song, PRD 103 (2021) 014025.

Quark / Hadron	$\gamma^i, 1, i\gamma_5$ T-even	$\gamma^i, 1, i\gamma_5$ T-odd	$\gamma^+\gamma_5$ T-even	$\gamma^+\gamma_5$ T-odd	σ^{ij}, σ^{-+} T-even	σ^{ij}, σ^{-+} T-odd
U	$f^\perp$ $[e]$			$g^\perp$		$[h]$
L		$f_L^\perp$ $[e_L]$	$g_L^\perp$		$[h_L]$	
T		$f_T, f_T^\perp$ $[e_T, e_T^\perp]$	$g_T, g_T^\perp$		$[h_T], [h_T^\perp]$	
LL	$f_{LL}^\perp$ $[e_{LL}]$			$g_{LL}^\perp$		$[h_{LL}]$
LT	$f_{LT}, f_{LT}^\perp$ $[e_{LT}, e_{LT}^\perp]$			$g_{LT}, g_{LT}^\perp$		$[h_{LT}], [h_{LT}^\perp]$
TT	$f_{TT}, f_{TT}^\perp$ $[e_{TT}, e_{TT}^\perp]$			$g_{TT}, g_{TT}^\perp$		$[h_{TT}], [h_{TT}^\perp]$

Twist-4 TMDs

Quark / Hadron	γ^- T-even	γ^- T-odd	$\gamma^-\gamma_5$ T-even	$\gamma^-\gamma_5$ T-odd	σ^{i-} T-even	σ^{i-} T-odd
U	f_3					$[h_3^\perp]$
L			g_{3L}		$[h_{3L}^\perp]$	
T		$f_{3T}^\perp$	g_{3T}		$[h_{3T}], [h_{3T}^\perp]$	
LL	f_{3LL}					$[h_{3LL}^\perp]$
LT	f_{3LT}			g_{3LT}		$[h_{3LT}], [h_{3LT}^\perp]$
TT	f_{3TT}			g_{3TT}		$[h_{3TT}], [h_{3TT}^\perp]$

New fragmentation functions (FFs) for spin-1 hadrons

see arXiv:2201.05397

Corresponding fragmentation functions exist for the spin-1 haddrons

simply by changing function names and kinematical variables.

Collinear FFs:
X. Ji, PRD 49, 114 (1994).

TMD distribution functions: $f,\ g,\ h,\ e;\quad x,\ k_T,\ S,\ T,\ M,\ n,\ \gamma^+,\ \sigma^{i+}$

$\Downarrow$

TMD fragmentation functions: $D,\ G,\ H,\ E;\quad z,\ k_T,\ S_h,\ T_h,\ M_h,\ \bar{n},\ \gamma^-,\ \sigma^{i-}$

Collinear FFs, twist 2

Hadron \ Quark	U (γ^+)		L ($\gamma^+\gamma_5$)		T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	D_1					
L			G_{1L}			
T					$[H_1]$	
LL	D_{1LL}					
LT						$[H_{1LT}]$
TT						

Collinear FFs, twist 3

Hadron \ Quark	$\gamma^i, 1, i\gamma_5$		$\gamma^i\gamma_5$		σ^{ij}, σ^{-+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	$[E]$					
L					$[H_L]$	
T			G_T			
LL	$[E_{LL}]$					$[H_{LL}]$
LT	D_{LT}			G_{LT}		
TT						

Collinear FFs, twist 4

Hadron \ Quark	γ^-		$\gamma^-\gamma_5$		σ^{i-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	D_3					
L			G_{3L}			
T					$[H_{3T}]$	
LL	D_{3LL}					
LT						$[H_{3LT}]$
TT						

[ ] = chiral odd

TMD FFs, twist 2

Hadron \ Quark	U (γ^+)		L ($\gamma^+\gamma_5$)		T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	D_1					$[H_1^\perp]$
L			G_{1L}		$[H_{1L}^\perp]$	
T		$D_{1T}^\perp$	G_{1T}		$[H_1], [H_{1T}^\perp]$	
LL	D_{1LL}					$[H_{1LL}^\perp]$
LT	D_{1LT}			G_{1LT}		$[H_{1LT}], [H_{1LT}^\perp]$
TT	D_{1TT}			G_{1TT}		$[H_{1TT}], [H_{1TT}^\perp]$

TMD FFs, twist 3

Hadron \ Quark	$\gamma^i, 1, i\gamma_5$		$\gamma^i\gamma_5$		σ^{ij}, σ^{-+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	$D^\perp$ $[E]$			$G^\perp$		$[H]$
L		$D_L^\perp$ $[E_L]$	$G_L^\perp$		$[H_L]$	
T		$D_T, D_T^\perp$ $[E_T, E_T^\perp]$	$G_T, G_T^\perp$		$[H_T], [H_T^\perp]$	
LL	$D_{LL}^\perp$ $[E_{LL}]$			$G_{LL}^\perp$		$[H_{LL}]$
LT	$D_{LT}, D_{LT}^\perp$ $[E_{LT}, E_{LT}^\perp]$			$G_{LT}, G_{LT}^\perp$		$[H_{LT}], [H_{LT}^\perp]$
TT	$D_{TT}, D_{TT}^\perp$ $[E_{TT}, E_{TT}^\perp]$			$G_{TT}, G_{TT}^\perp$		$[H_{TT}], [H_{TT}^\perp]$

TMD FFs, twist 4

Hadron \ Quark	γ^-		$\gamma^-\gamma_5$		σ^{i-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	D_3					$[H_3^\perp]$
L			G_{3L}		$[H_{3L}^\perp]$	
T		$D_{3T}^\perp$	G_{3T}		$[H_{3T}], [H_{3T}^\perp]$	
LL	D_{3LL}					$[H_{3LL}^\perp]$
LT	D_{3LT}			G_{3LT}		$[H_{3LT}], [H_{3LT}^\perp]$
TT	D_{3TT}			G_{3TT}		$[H_{3TT}], [H_{3TT}^\perp]$

New TMD FFs

PDFs for spin-1 hadrons

Twist-2 PDFs

Hadron \ Quark	U (γ^+) T-even	U (γ^+) T-odd	L ($\gamma^+\gamma_5$) T-even	L ($\gamma^+\gamma_5$) T-odd	T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$) T-even	T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$) T-odd
U	f_1					
L			$g_{1L}\,(g_1)$			
T					$[h_1]$	
LL	$f_{1LL}\,(b_1)$					
LT						*1
TT						

*1: $h_{1LT}(x)$, *2: $g_{LT}(x)$, *3: $h_{LL}(x)$, *4: $h_{3LT}(x)$

Because of the time-reversal invariance, the collinear PDF vanishes. However, since the time-reversal invariance cannot be imposed in the fragmentation functions, we should note that the corresponding fragmentation function should exist as a collinear fragmentation function.

[ ] = chiral odd

Twist-3 PDFs

Hadron \ Quark	$\gamma^i, 1, i\gamma_5$ T-even	$\gamma^i, 1, i\gamma_5$ T-odd	$\gamma^+\gamma_5$ T-even	$\gamma^+\gamma_5$ T-odd	σ^{ij}, σ^{-+} T-even	σ^{ij}, σ^{-+} T-odd
U	$[e]$					
L					$[h_L]$	
T			g_T			
LL	$[e_{LL}]$					*3
LT	f_{LT}			*2		
TT						

Twist-4 PDFs

Hadron \ Quark	γ^- T-even	γ^- T-odd	$\gamma^-\gamma_5$ T-even	$\gamma^-\gamma_5$ T-odd	σ^{i-} T-even	σ^{i-} T-odd
U	f_3					
L			g_{3L}			
T					$[h_{3T}]$	
LL	f_{3LL}					
LT						*4
TT						

New collinear PDFs

Summary on Spin-1 TMDs and PDFs

TMDs of spin-1 hadrons

- TMDs: interdisciplinary field of physics
- We proposed new 30 TMDs and 3 PDFs in twist 3 and 4.
- New sum rules for TMDs.
- New TMD fragmentation functions.

Twist-3 TMD: $f_{LL}^{\perp}, e_{LL}, f_{LT}, f_{LT}^{\perp}, e_{1T}, e_{1T}^{\perp}, f_{TT}, f_{TT}^{\perp}, e_{TT}, e_{TT}^{\perp},$
$g_{LL}^{\perp}, g_{LT}, g_{LT}^{\perp}, g_{TT}, g_{TT}^{\perp}, h_{1L}, h_{LT}, h_{LT}^{\perp}, h_{TT}, h_{TT}^{\perp}$

Twist-4 TMD: $f_{3LL}, f_{3LT}, f_{3TT}, g_{3LT}, f_{3TT}, h_{3LL}^{\perp}, h_{3LT}, h_{3LT}^{\perp}, h_{3TT}, h_{3TT}^{\perp}$

Twist-3 PDF: e_{LL}, f_{LT}

Twist-4 PDF: f_{3LL}

Sum rules: $\int d^2k_T\, g_{LT}(x, k_T^2) = \int d^2k_T\, h_{LL}(x, k_T^2) = \int d^2k_T\, h_{3LL}(x, k_T^2) = 0$

TMD distribution functions: $f, g, h, e;\quad x, k_T, S, T, M, n, \gamma^+, \sigma^{i+}$

$\Downarrow$

TMD fragmentation functions: $D, G, H, E;\quad z, k_T, S_h, T_h, M_h, \bar{n}, \gamma^-, \sigma^{i-}$

Twist-2 relation and sum rule for PDFs of spin-1 hadrons

(analogous to the Wandzura-Wilczek relation and the Burkhardt-Cottingham sum rule)

SK and Qin-Tao Song,
JHEP 09 (2021) 141.

PDFs for spin-1 hadrons

Twist-2 PDFs

Hadron \ Quark	U (γ^+)		L ($\gamma^+\gamma_5$)		T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					
L			$g_{1L}(g_1)$			
T					$[h_1]$	
LL	$f_{1LL}(b_1)$					
LT						*1
TT						

We derived analogous relations to Wandzura-Wilczek relation and Burkhardt-Cottingham sum rule for f_{LT} and f_{1LL}.

SK and Qin-Tao Song (2021)

Twist-3 PDFs

[ ] = chiral odd

Hadron \ Quark	$\gamma^i, 1, i\gamma_5$		$\gamma^+\gamma_5$		σ^{ij}, σ^{-+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	$[e]$					
L					$[h_L]$	
T			g_T			
LL	$[e_{LL}]$					*3
LT	f_{LT}			*2		
TT						

Twist-4 PDFs

Hadron \ Quark	γ^-		$\gamma^-\gamma_5$		σ^{i-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_3					
L			g_{3L}			
T					$[h_{3T}]$	
LL	f_{3LL}					
LT						*4
TT						

Wandzura-Wilczek and Burkhardt-Cottingham relations for g_1 and g_2

Structure functions: $\int \frac{d(P^+\xi^-)}{2\pi} e^{ixP^+\xi^-} \left\langle P,S \right| \bar{\psi}(0)\gamma^\mu \gamma_5 \psi(\xi) \left| P,S \right\rangle_{\xi^+=\vec{\xi}_T=0} = 2M_N \left[g_{1L}(x)\bar{n}^\mu S\cdot n + g_T(x) S_T^\mu + g_{3L}(x)\frac{M_N^2}{(P^+)^2} n^\mu S\cdot n \right]$

$$S^\mu = S_L \frac{P^+}{M}\bar{n}^\mu - S_L \frac{M_N}{2P^+} n^\mu + S_T^\mu, \quad P^\mu = P^+\bar{n}^\mu + \frac{M_N^2}{2P^+} n^\mu, \quad S\cdot n = S_L \frac{P^+}{M_N}$$

$$g_1(x) = \frac{1}{2}\left[g_{1L}(x) + g_{1L}(-x) \right], \quad g_1(x) + g_2(x) = \frac{1}{2}\left[g_T(x) + g_T(-x) \right]$$

J. Kodaira and K. Tanaka,
Prog. Theor. Phys. 101 (1999) 191.

Operators: $R^{\sigma\{\mu_1 \cdots \mu_{n-1}\}} = i^{n-1}\bar{\psi}\gamma^\sigma \gamma_5 D^{\{\mu_1} \cdots D^{\mu_{n-1}\}}\psi = R^{\{\sigma\mu_1 \cdots \mu_{n-1}\}} + R^{[\sigma\{\mu_1] \cdots \mu_{n-1}\}} = \text{twist 2} + \text{twist 3}$

$$R^{\{\sigma\mu_1 \cdots \mu_{n-1}\}} = \frac{1}{n}\left[S^\sigma P^{\{\mu_1} P^{\mu_2} \cdots P^{\mu_{n-1}\}} + S^{\mu_1} P^{\{\sigma} P^{\mu_2} \cdots P^{\mu_{n-1}\}} + S^{\mu_2} P^{\{\mu_1} P^{\sigma} \cdots P^{\mu_{n-1}\}} + \cdots \right]$$

$$R^{[\sigma\{\mu_1] \cdots \mu_{n-1}\}} = \frac{1}{n}\left[(n-1) S^\sigma P^{\{\mu_1} P^{\mu_2} \cdots P^{\mu_{n-1}\}} - S^{\mu_1} P^{\{\sigma} P^{\mu_2} \cdots P^{\mu_{n-1}\}} - S^{\mu_2} P^{\{\mu_1} P^{\sigma} \cdots P^{\mu_{n-1}\}} - \cdots \right]$$

$$\left\langle P,S \right| R^{\{\sigma\mu_1 \cdots \mu_{n-1}\}} \left| P,S \right\rangle = \frac{2}{n} a_n M_N \left[S^\sigma P^{\mu_1} \cdots P^{\mu_{n-1}} + P^{\mu_1} S^\sigma \cdots P^{\mu_{n-1}} + \cdots \right]$$

$$\left\langle P,S \right| R^{[\sigma\{\mu_1] \cdots \mu_{n-1}\}} \left| P,S \right\rangle = \frac{2}{n} d_n M_N \left[(S^\sigma P^{\mu_1} - P^\sigma S^{\mu_1}) P^{\mu_2} \cdots P^{\mu_{n-1}} + (S^\sigma P^{\mu_2} - P^\sigma S^{\mu_2}) P^{\mu_1} \cdots P^{\mu_{n-1}} + \cdots \right]$$

$$\frac{1}{2M_N (P^+)^{n-1}} n_{\mu_1} \cdots n_{\mu_{n-1}} \left\langle P,S \right| R^{\sigma\{\mu_1 \cdots \mu_{n-1}\}} \left| P,S \right\rangle = \bar{n}^\sigma (S\cdot n)\int_{-1}^1 dx x^{n-1} g_{1L}(x) + S_T^\sigma \int_{-1}^1 dx x^{n-1} g_T(x)$$

$$= \frac{1}{2M_N (P^+)^{n-1}} n_{\mu_1} \cdots n_{\mu_{n-1}} \left\langle P,S \right| R^{\{\sigma\mu_1 \cdots \mu_{n-1}\}} \left| P,S \right\rangle + \frac{1}{2M_N (P^+)^{n-1}} n_{\mu_1} \cdots n_{\mu_{n-1}} \left\langle P,S \right| R^{[\sigma\{\mu_1] \cdots \mu_{n-1}\}} \left| P,S \right\rangle$$

$$\to \int_{-1}^1 dx x^{n-1} g_{1L}(x) = a_n, \quad \int_{-1}^1 dx x^{n-1} g_T(x) = \frac{1}{n} a_n + \frac{n-1}{n} d_n$$

$$\to \int_0^1 dx x^{n-1} g_1(x) = \int_{-1}^1 dx x^{n-1} \frac{1}{2} g_{1L}(x) = \frac{1}{2} a_n, \quad \int_0^1 dx x^{n-1} [g_1(x) + g_2(x)] = \int_{-1}^1 dx x^{n-1} \frac{1}{2} g_T(x) = \frac{1}{2n} a_n + \frac{n-1}{2n} d_n$$

$$\to \int_0^1 dx x^{n-1} g_2(x) = \int_0^1 dx x^{n-1} \left[-g_1(x) + \int_x^1 \frac{dy}{y} g_1(y) \right] + \frac{n-1}{2n} d_n$$

If we write $g_2(x) = g_2^{WW}(x) + \bar{g}_2(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(y) + \bar{g}_2(x)$

$\to g_2^{WW}(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(y)$ **(Wandzura-Wilczek relation),** $\int_0^1 dx x^{n-1} \bar{g}_2(x) = \frac{n-1}{2n} d_n$

$\to \int_0^1 dx\, g_2(x) = 0$ **(Burkhardt-Cottingham sum rule)**

Note: Twist-3 operators $R^{[\sigma\{\mu_1] \cdots \mu_{n-1}\}}$ are obtained by the Tayler expansion of $\xi_\mu \bar{\psi}(0)(\partial^\mu \gamma^\sigma - \partial^\sigma \gamma^\mu)\gamma_5 \psi(\xi)$, which needs to be investigated in details for finding the details of twist-3 terms.

Twist-2 relation and sum rule

- Twist-3 matrix element in terms of tensor-polarized PDFs

$$\left\langle P,T\left|\bar{\psi}(0)(\partial^\mu\gamma^\alpha-\partial^\alpha\gamma^\mu)\psi(\xi)\right|P,T\right\rangle = 2MS_{LT}^\alpha\int_-^1 dx\, e^{-ixP^+\xi^-}\left[-\frac{3}{2}f_{1LL}(x)+f_{LT}(x)-\frac{d}{dx}\{xf_{LT}(x)\}\right]$$

see Appendix I for the details

- Twist-3 operator in terms of gluon field tensor

$$\xi_\mu\left[\bar{\psi}(0)(\gamma^\alpha\partial^\mu-\gamma^\mu\partial^\alpha)\psi(\xi)\right]=g\int_0^1 dt\,\bar{\psi}(0)\left\{i\left(t-\frac{1}{2}\right)G^{\alpha\mu}(t\xi)-\frac{1}{2}\gamma_5\tilde{G}^{\alpha\mu}(t\xi)\right\}\xi_\mu\not{\xi}\psi(\xi)$$

Note: Twist-3 operators $R^{[\sigma\{\mu_1]\cdots\mu_{n-1}\}}$ are obtained by the Tayler expansion of $\xi_\mu\bar{\psi}(0)(\partial^\mu\gamma^\sigma-\partial^\sigma\gamma^\mu)\psi(\xi)$, which needs to be investigated in details for finding the details of twist-3 terms.

- Matrix element of field tensor in terms of twist-3 multiparton distribution functions

$$\int\frac{d(P\cdot\xi)}{2\pi}e^{ix_1P\cdot\xi}\langle P,T|\,g\int_0^1 dt\,\bar{\psi}(0)\left\{i\left(t-\frac{1}{2}\right)G^{\mu\nu}(t\xi)-\frac{1}{2}\gamma_5\tilde{G}^{\mu\nu}(t\xi)\right\}\xi_\mu\not{\xi}\psi(\xi)|P,T\rangle_{\xi^+=\vec{\xi}_T=0}$$

$$=-2MS_{LT}^\nu\mathcal{P}\int_0^1 dx_2\frac{1}{x_1-x_2}\left[\frac{\partial}{\partial x_1}\{F_{G,LT}(x_1,x_2)+G_{G,LT}(x_1,x_2)\}+\frac{\partial}{\partial x_2}\{F_{G,LT}(x_2,x_1)+G_{G,LT}(x_2,x_1)\}\right]$$

$$x\frac{df_{LT}(x)}{dx}=-\frac{3}{2}f_{1LL}(x)-f_{LT}^{(HT)}(x),$$ Higher-twist: $$f_{LT}^{(HT)}(x)=-\mathcal{P}\int_0^1 dy\frac{1}{x-y}\left[\frac{\partial}{\partial x}\{F_{G,LT}(x,y)+G_{G,LT}(x,y)\}+\frac{\partial}{\partial y}\{F_{G,LT}(y,x)+G_{G,LT}(y,x)\}\right]$$

$$\to f_{LT}(x)=\frac{3}{2}\int_x^{\varepsilon(x)}\frac{dy}{y}f_{1LL}(y)+\int_x^{\varepsilon(x)}\frac{dy}{y}f_{LT}^{(HT)}(y),\quad \varepsilon(x)=\frac{i}{\pi}P\int_{-\infty}^{\infty}dy\frac{1}{y}e^{-ixy}=\begin{cases}+1 & x>0\\ -1 & x<0\end{cases}$$

Define $f^+(x)=f(x)+\bar{f}(x)=f(x)-f(-x),\quad f=f_{1LL},\ f_{LT},\ f_{LT}^{(HT)},\quad x>0$

$$\to f_{LT}^+(x)=\frac{3}{2}\int_x^1\frac{dy}{y}f_{1LL}^+(y)+\int_x^1\frac{dy}{y}f_{LT}^{(HT)+}(y)$$ $\to$ Twist-2 relation: $$f_{LT}^+(x)=\frac{3}{2}\int_x^1\frac{dy}{y}f_{1LL}^+(y)$$

If we define $f_{2LT}(x)=\frac{2}{3}f_{LT}(x)-f_{1LL}(x)$,

$$f_{2LT}^+(x)=-f_{1LL}^+(x)+\int_x^1\frac{dy}{y}f_{1LL}^+(y)+\frac{2}{3}\int_x^1\frac{dy}{y}f_{LT}^{(HT)+}(y)$$ $\to$ Twist-2 relation: $f_{2LT}^+(x)=-f_{1LL}^+(x)+\int_x^1\frac{dy}{y}f_{1LL}^+(y)$, Wandzura-Wilczek like

$\to$ Sum rule: $\int_0^1 dx\ f_{2LT}^+(x)=0$, Burkhardt-Cottingham like

If the parton-model sum rule without the tensor-polarized antiquark distributions $\int_0^1 dx\, f_{1LL}^+(x)=\frac{2}{3}\int_0^1 dx\ b_1^+(x)=0$ is valid, $\to$ Sum rule: $\int_0^1 dx\ f_{LT}^+(x)=0$

$$\int dx\, b_1^D(x)=\lim_{t\to 0}-\frac{5}{12}\frac{t}{M^2}F_Q(t)+\sum_i e_i^2\int dx\,\delta_T\bar{q}_i(x)$$
$=0$?

F. E. Close and SK, PRD 42 (1990) 2377.

Summary on the twist-2 relation and sum rule

$g_2(x)=-g_1(x)+\int_x^1\frac{dy}{y}g_1(y)$ (Wandzura-Wilczek relation), $\int_0^1 dx\, g_2(x)=0$ (Burkhardt-Cottingham sum rule)

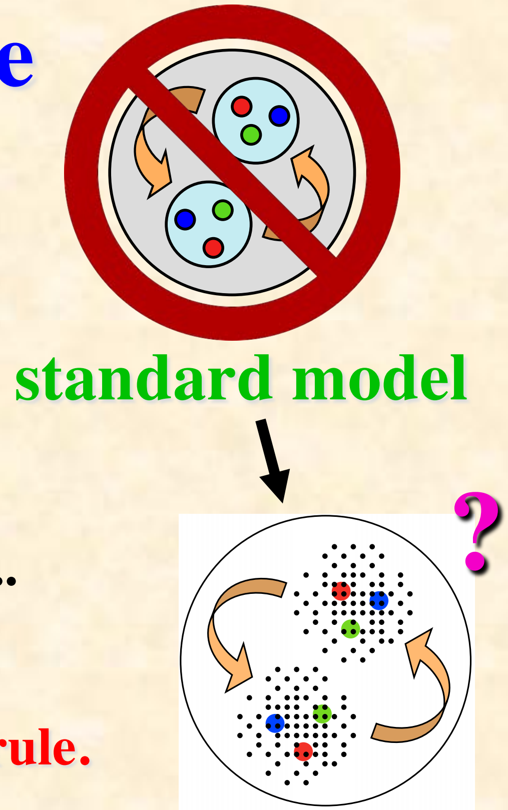
For tensor-polarized spin-1 hadrons, we obtained

$f_{2LT}^+(x)=-f_{1LL}^+(x)+\int_x^1\frac{dy}{y}f_{1LL}^+(y)$, $\int_0^1 dx\ f_{2LT}^+(x)=0$, $f_{2LT}(x)\equiv\frac{2}{3}f_{LT}(x)-f_{1LL}(x)$

$\int_0^1 dx\ f_{LT}^+(x)=0$ if $\int_0^1 dx\, f_{1LL}^+(x)=\frac{2}{3}\int_0^1 dx\ b_1^+(x)=0$

Existence of multiparton distribution functions: $F_{G,LT}(x_1,x_2)$, $G_{G,LT}(x_1,x_2)$, $H_{G,LL}^\perp(x_1,x_2)$, $H_{G,TT}(x_1,x_2)$

Summary on Twist-2 relation and sum rule



Spin-1 structure functions of the deuteron (new spin structure)

- **tensor structure in quark-gluon degrees of freedom**
- **b_1, gluon transversity, new TMDs**
- **new signature beyond "standard" hadron physics?**
- **experiments: JLab (approved), Fermilab (to be proposed), ... , NICA (in progress), LHCspin (~2028), AMBER?, EIC, EicC, ...**

We derived twist-2 relation and sum rule analogous to Wandzura-Wilczek relation and Burkardt-Cottingham sum rule.

We showed the existence of tensor-polarized multiparton distribution functions.

For spin-1/2 nucleons,

$$g_2(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(y) \text{ (Wandzura-Wilczek relation)}, \qquad \int_0^1 dx\, g_2(x) = 0 \text{ (Burkhardt-Cottingham sum rule)}$$

For tensor-polarized spin-1 hadrons, we obtained

$$f_{2LT}^{+}(x) = -f_{1LL}^{+}(x) + \int_x^1 \frac{dy}{y} f_{1LL}^{+}(y), \qquad \int_0^1 dx\, f_{2LT}^{+}(x) = 0, \qquad f_{2LT}(x) \equiv \frac{2}{3} f_{LT}(x) - f_{1LL}(x)$$

$$\int_0^1 dx\, f_{LT}^{+}(x) = 0 \text{ if } \int_0^1 dx\, f_{1LL}^{+}(x) = \frac{2}{3}\int_0^1 dx\, b_1^{+}(x) = 0$$

Existence of multiparton distribution functions: $F_{G,LT}(x_1,x_2)$, $G_{G,LT}(x_1,x_2)$, $H_{G,LL}^{\perp}(x_1,x_2)$, $H_{G,TT}(x_1,x_2)$

Relations from equation of motion for PDFs of spin-1 hadrons

(Equation-of-motion and Lorentz-invariance relations)

SK and Qin-Tao Song,
PLB 826 (2022) 136908.

Relations from equation of motion and Lorentz-invariance relation for spin-1 hadrons

Lorentz invariance = frame independence of twist-3 observables

see Appendix II
for works on spin-1/2 nucleon

We explain derivations on relations from equation of motion for quarks

- $x f_{LT}(x) - \int_{-1}^{+1} dy \left[F_{D,LT}(x,y) + G_{D,LT}(x,y) \right] = 0, \;\; x f_{LT}(x) - f_{1LT}^{(1)}(x) - \mathcal{P}\int_{-1}^{+1} dy \frac{F_{G,LT}(x,y) + G_{G,LT}(x,y)}{x-y} = 0$
- $x e_{LL}(x) - 2\int_{-1}^{+1} dy H_{D,LL}^{\perp}(x,y) - \frac{m}{M} f_{1LL}(x) = 0, \quad x e_{LL}(x) - 2\mathcal{P}\int_{-1}^{+1} dy \frac{H_{G,LL}^{\perp}(x,y)}{x-y} - \frac{m}{M} f_{1LL}(x) = 0$

and the Lorentz-invariance relation

- $\frac{d f_{1LT}^{(1)}(x)}{dx} - f_{LT}(x) + \frac{3}{2} f_{1LL}(x) - 2\mathcal{P}\int_{-1}^{+1} dy \frac{F_{G,LT}(x,y)}{(x-y)^2} = 0,$ transverse-momemtum moment of TMD: $f^{(1)}(x) = \int d^2 k_T \frac{\vec{k}_T^2}{2M^2} f(x, k_T^2)$

Twist-2 PDFs

Quark / Hadron	U (γ^+)		L ($\gamma^+\gamma_5$)		T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					
L			$g_{1L}(g_1)$			
T					$[h_1]$	
LL	$f_{1LL}(b_1)$					
LT						
TT						

Twist-3 PDFs

Quark / Hadron	$\gamma^i, 1, i\gamma_5$		$\gamma^+\gamma_5$		σ^{ij}, σ^{-+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	$[e]$					
L					$[h_L]$	
T			g_T			
LL	$[e_{LL}]$					
LT	f_{LT}					*1
TT						

Twist-3 TMDs

Quark / Hadron	U (γ^+)		L ($\gamma^+\gamma_5$)		T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					$[h_1^{\perp}]$
L			g_{1L}		$[h_{1L}^{\perp}]$	
T		$f_{1T}^{\perp}$	g_{1T}		$[h_1], [h_{1T}^{\perp}]$	
LL	f_{1LL}					$[h_{1LL}^{\perp}]$
LT	f_{1LT}			g_{1LT}		$[h_{1LT}], [h_{1LT}^{\perp}]$
TT	f_{1TT}			g_{1TT}		$[h_{1TT}], [h_{1TT}^{\perp}]$

[ ] = chiral odd

Equation of motion for quarks I

$0=(iD_\mu\gamma^\mu-m)\psi=(iD^+\gamma^-+iD^-\gamma^++iD_\alpha\gamma^\alpha-m)\psi,\qquad \alpha=1,2$ (transverse index)

$i\sigma^{+\alpha}\cdot$(this equation of motion), use $\sigma^{\alpha\mu}\gamma^\rho=i(g^{\rho\mu}\gamma^\alpha-g^{\rho\alpha}\gamma^\mu)-\varepsilon^{\alpha\mu\rho\sigma}\gamma_\sigma\gamma_5,\quad \varepsilon^{0123}=+1$

$$\to\left[i(\gamma^+D^\alpha-\gamma^\alpha D^+)+i\varepsilon_T^{\alpha\mu}\gamma_\mu\gamma_5 iD^+-i\varepsilon_T^{\alpha\mu}\gamma^+\gamma_5 iD_{T\mu}+im\sigma^{+\alpha}\right]\psi=0$$

$$0=\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left\langle P,T\left|\bar\psi(0)\left[\gamma^+iD^\alpha-\gamma^\alpha iD^++i\varepsilon_T^{\alpha\mu}\gamma_\mu\gamma_5 iD^+-i\varepsilon_T^{\alpha\mu}\gamma^+\gamma_5 iD_{T\mu}+im\sigma^{+\alpha}\right]\psi(\xi^-)\right|P,T\right\rangle$$

$$\to\ 0=P^+\mathrm{Tr}\left[\Phi_D^\alpha(x,P,T)\gamma^+\right]-P^+\mathrm{Tr}\left[\Phi_D^+(x,P,T)\gamma^\alpha\right]+i\varepsilon_T^{\alpha\mu}P^+\mathrm{Tr}\left[\Phi_D^+(x,P,T)\gamma_\mu\gamma_5\right]-i\varepsilon_T^{\alpha\mu}P^+\mathrm{Tr}\left[\Phi_{D\mu}(x,P,T)\gamma^+\gamma_5\right]+im\mathrm{Tr}\left[\Phi(x,P,T)\sigma^{+\alpha}\right]$$

"Equation of motion" expressed by multiparton correlation functions, P: momentum, T: tensor polarization

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Multiparton correlation functions: $(\Phi_X^\mu)_{ij}(y,x,P,T)=\int\frac{d\xi_1^-}{2\pi}\frac{d\xi_2^-}{2\pi}e^{iyP^+\xi_1^-}e^{i(x-y)P^+\xi_2^-}\left\langle P,T\left|\bar\psi_j(0)Y^\mu(\xi_2^-)W(0,\xi^-)\psi_i(\xi_1^-)\right|P,T\right\rangle$

$$X\ (Y^\mu)=G\ (gG^{+\mu}),\ A\ (gA^\mu),\ D\ (iD^\mu),\ D^\mu=\partial^\mu-igA^\mu,\quad W(0,\xi^-)=P\exp\left[-ig\int_0^{\xi^-}d\xi^-A^+(\xi)\right]_{\xi^+=\vec\xi_T=0}$$

Collinear correlation function: $(\Phi_D^\mu)_{ij}(x,P,T)\equiv\int_{-1}^{+1}dy(\Phi_D^\mu)_{ij}(y,x,P,T)=\frac{1}{P^+}\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left\langle P,T\left|\bar\psi_j(0)iD^\mu(\xi^-)W(0,\xi^-)\psi_i(\xi^-)\right|P,T\right\rangle$

$$\Phi_D^\alpha(x,P,T)=\int_{-1}^{+1}dy\Phi_D^\alpha(y,x,P,T),\quad \Phi_D^\alpha(y,x,P,T)=\frac{M}{2P^+}\left[S_{LT}^\alpha F_{D,LT}(y,x)+i\varepsilon_T^{\alpha\mu}S_{LT,\mu}\gamma_5 G_{D,LT}(y,x)+S_{LL}\gamma^\alpha H_{D,LL}^\perp(y,x)+S_{TT}^{\alpha\mu}\gamma_\mu H_{D,TT}(y,x)\right]\bar n$$

Expression in terms of multiparton distribution functions.

$$F_{D,LT}(x,y)=F_{D,LT}(y,x),\ G_{D,LT}(x,y)=-G_{D,LT}(y,x),\ H_{D,LL}^\perp(x,y)=-H_{D,LL}^\perp(y,x),\ H_{D,TT}(x,y)=-H_{D,TT}(y,x)$$

$$\Phi_D^+(x,P,T)=\int_{-1}^{+1}dy\Phi_D^+(y,x,P,T)=\int_{-1}^{+1}dy\delta(y-x)y\Phi(y,P,T)=x\Phi(x,P,T)$$

$$\Phi_{ij}(x,P,T)=\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left\langle P,T\left|\bar\psi_j(0)W(0,\xi^-)\psi_i(\xi)\right|P,T\right\rangle_{\xi^+=\vec\xi_T=0}=\frac12\left[S_{LL}\bar n f_{1LL}(x)+\frac{M}{P^+}S_{LL}e_{LL}(x)+\frac{M}{P^+}\not S_{LT}f_{LT}(x)+\frac{M^2}{(P^+)^2}S_{LL}\bar n f_{3LL}(x)\right]_{ij}$$

Refs. SK, Qin-Tao Song, PRD 103 (2021) 014025; JHEP 09 (2021) 141; PLB 826 (2022) 136908.

$$\to\ xf_{LT}(x)-\int_{-1}^{+1}dy\left[F_{D,LT}(x,y)+G_{D,LT}(x,y)\right]=0$$

$$\Phi_G^\alpha(x,P,T)=\frac{M}{2}i\left[S_{LT}^\alpha F_{G,LT}(x_1,x_2)+i\varepsilon_T^{\alpha\mu}S_{LT,\mu}\gamma_5 G_{G,LT}(x_1,x_2)+S_{LL}\gamma^\alpha H_{G,LL}^\perp(x_1,x_2)+S_{TT}^{\alpha\mu}\gamma_\mu H_{G,TT}(x_1,x_2)\right]\bar n$$

$$F_{D,LT}(x,y)=\delta(x-y)f_{1LT}^{(1)}(x)+\mathcal{P}\left(\frac{1}{x-y}\right)F_{G,LT}(x,y),\ G_{D,LT}(x,y)=\mathcal{P}\left(\frac{1}{x-y}\right)G_{G,LT}(x,y),\ f^{(1)}(x)=\int d^2k_T\frac{\vec k_T^2}{2M^2}f(x,k_T^2)$$

see Appendix III for the details

$$\to\ xf_{LT}(x)-f_{1LT}^{(1)}(x)-\mathcal{P}\int_{-1}^{+1}dy\frac{F_{G,LT}(x,y)+G_{G,LT}(x,y)}{x-y}=0,$$ transverse-momemtum moment of TMD: $f^{(1)}(x)=\int d^2k_T\frac{\vec k_T^2}{2M^2}f(x,k_T^2)$

Equation of motion for quarks II

$0=(iD_\mu\gamma^\mu - m)\psi = (iD^+\gamma^- + iD^-\gamma^+ + iD_\alpha\gamma^\alpha - m)\psi,$ $\alpha = 1,2$ (transverse index)

$\gamma^+\cdot$(this equation of motion), $(iD^+ - i\sigma^{+\mu} iD_\mu - m\gamma^+)\psi = 0$

$$0=\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left\langle P,T\,\middle|\,\bar\psi(0)\left[iD^+ - i\sigma^{+\mu}iD_\mu - m\gamma^+\right]\psi(\xi^-)\,\middle|\,P,T\right\rangle$$

$$\to\ 0=P^+\mathrm{Tr}\left[\Phi_D^+(x,P,T)\right]-P^+ i\mathrm{Tr}\left[\Phi_D^+(x,P,T)\sigma^{+-}\right]-P^+ i\mathrm{Tr}\left[\Phi_{D\alpha}(x,P,T)\sigma^{+\alpha}\right]-m\mathrm{Tr}\left[\Phi(x,P,T)\gamma^+\right]$$

$$\Phi_D^\alpha(x,P,T)=\int_{-1}^{+1}dy\,\Phi_D^\alpha(y,x,P,T),\ \ \Phi_D^\alpha(y,x,P,T)=\frac{M}{2P^+}\left[S_{LT}^\alpha F_{D,LT}(y,x)+i\varepsilon_T^{\alpha\mu}S_{LT,\mu}\gamma_5 G_{D,LT}(y,x)+S_{LL}\gamma^\alpha H_{D,LL}^\perp(y,x)+S_{TT}^{\alpha\mu}\gamma_\mu H_{D,TT}(y,x)\right]\bar{n}\!\!\!/$$

$$F_{D,LT}(x,y)=F_{D,LT}(y,x),\ G_{D,LT}(x,y)=-G_{D,LT}(y,x),\ H_{D,LL}^\perp(x,y)=-H_{D,LL}^\perp(y,x),\ H_{D,TT}(x,y)=-H_{D,TT}(y,x)$$

$$\Phi_D^+(x,P,T)=x\Phi(x,P,T),\quad \Phi(x,P,T)=\frac{1}{2}\left[S_{LL}\bar{n}\!\!\!/ f_{1LL}(x)+\frac{M}{P^+}S_{LL}e_{LL}(x)+\frac{M}{P^+}S\!\!\!/_{LT}f_{LT}(x)+\frac{M^2}{(P^+)^2}S_{LL}\bar{n}\!\!\!/ f_{3LL}(x)\right]$$

$$\to\ xe_{LL}(x)-2\int_{-1}^{+1}dy H_{D,LL}^\perp(x,y)-\frac{m}{M}f_{1LL}(x)=0,\quad H_{D,LL}^\perp(x,y)=\mathcal{P}\left(\frac{1}{x-y}\right)H_{G,LL}^\perp(x,y)$$

$$\to\ xe_{LL}(x)-2\mathcal{P}\int_{-1}^{+1}dy\frac{H_{G,LL}^\perp(x,y)}{x-y}-\frac{m}{M}f_{1LL}(x)=0$$

Lorentz-invariance relation for tensor-polarized PDFs

$$xf_{LT}(x)-f_{1LT}^{(1)}(x)-\mathcal{P}\int_{-1}^{+1}dy\frac{F_{G,LT}(x,y)+G_{G,LT}(x,y)}{x-y}=0\qquad (1)\quad \text{from equation of motion I}$$

$$x\frac{df_{LT}(x)}{dx}=-\frac{3}{2}f_{1LL}(x)+\mathcal{P}\int_{-1}^{+1}dy\frac{1}{x-y}\left[\frac{\partial}{\partial x}\left\{F_{G,LT}(x,y)+G_{G,LT}(x,y)\right\}+\frac{\partial}{\partial y}\left\{F_{G,LT}(y,x)+G_{G,LT}(y,x)\right\}\right]\qquad (2)\quad \text{from Wandzura-Wilczek-type studies}$$

$\frac{d}{dx}$ of (1) and use (2):

$$f_{LT}(x)+x\frac{df_{LT}(x)}{dx}-\frac{f_{1LT}^{(1)}(x)}{dx}-\mathcal{P}\int_{-1}^{+1}dy\left[\frac{F_{G,LT}(y,x)-G_{G,LT}(y,x)}{(y-x)^2}+\frac{1}{y-x}\left\{\frac{\partial F_{G,LT}(y,x)}{\partial x}-\frac{\partial G_{G,LT}(y,x)}{\partial x}\right\}\right]=0$$

$$=f_{LT}(x)-\frac{3}{2}f_{1LL}(x)+\mathcal{P}\int_{-1}^{+1}dy\frac{1}{x-y}\left[\frac{\partial}{\partial x}\left\{F_{G,LT}(x,y)+G_{G,LT}(x,y)\right\}+\frac{\partial}{\partial y}\left\{F_{G,LT}(y,x)+G_{G,LT}(y,x)\right\}\right]$$

$$-\frac{df_{1LT}^{(1)}(x)}{dx}-\mathcal{P}\int_{-1}^{+1}dy\left[\frac{F_{G,LT}(y,x)-G_{G,LT}(y,x)}{(y-x)^2}+\frac{1}{y-x}\left\{\frac{\partial F_{G,LT}(y,x)}{\partial x}-\frac{\partial G_{G,LT}(y,x)}{\partial x}\right\}\right]$$

$$=f_{LT}(x)-\frac{df_{1LT}^{(1)}(x)}{dx}-\frac{3}{2}f_{1LL}(x)+2\mathcal{P}\int_{-1}^{+1}dy\frac{F_{G,LT}(x,y)}{(x-y)^2},\qquad F_{G,LT}(y,x)=-F_{G,LT}(x,y),\ \ G_{G,LT}(y,x)=G_{G,LT}(x,y)$$

$$\frac{df_{1LT}^{(1)}(x)}{dx}-f_{LT}(x)+\frac{3}{2}f_{1LL}(x)-2\mathcal{P}\int_{-1}^{+1}dy\frac{F_{G,LT}(x,y)}{(x-y)^2}=0$$

Summary on Relations from equation of motion and a Lorentz-invariance relation

- **We derived relations among tensor-polarized PDFs and multiparton distribution functions by using the equation of motion for quarks and also showed a Lorentz-invariance relation.**

Relations from equation of motion for quarks

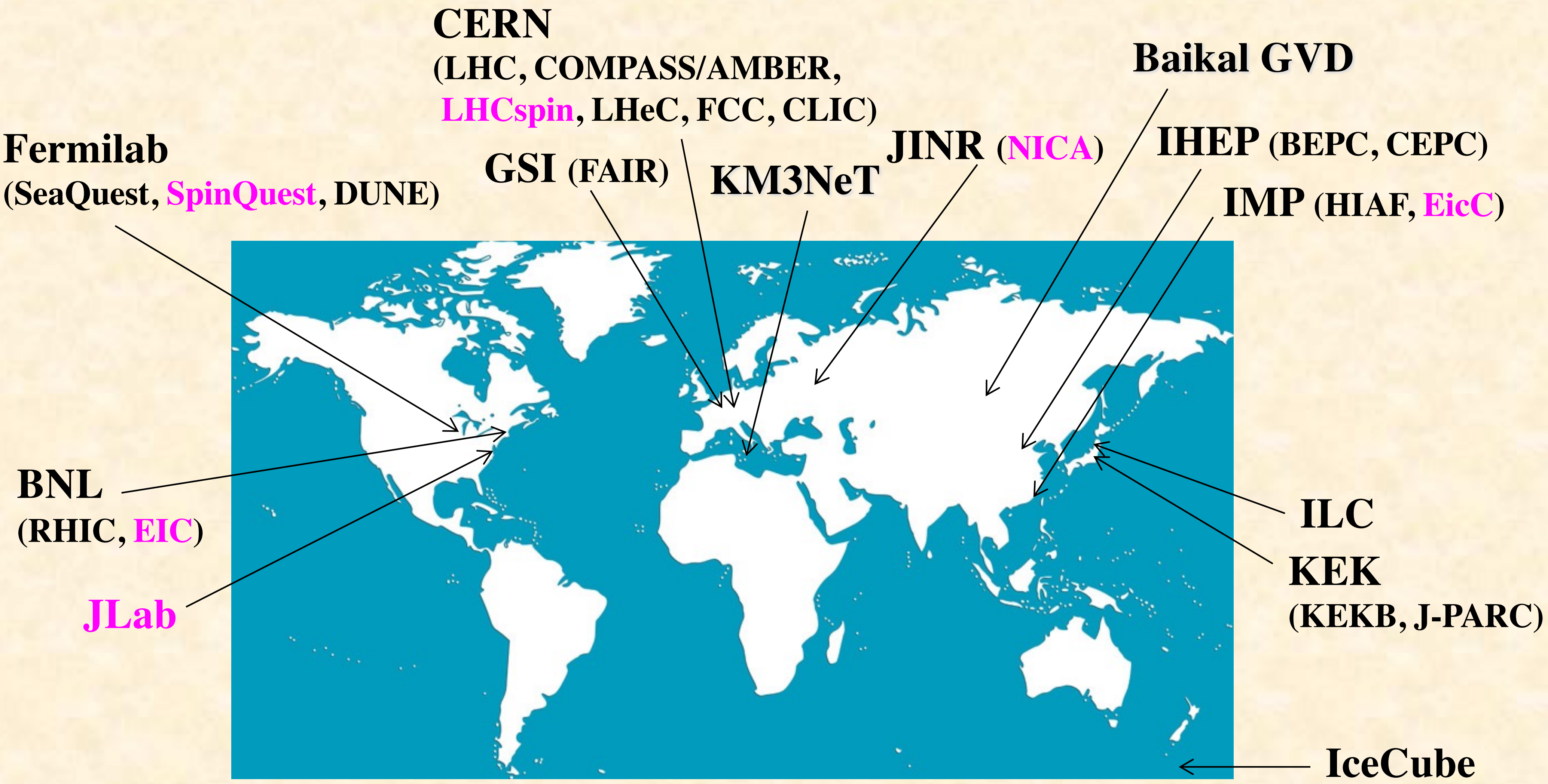
- $x f_{LT}(x) - \int_{-1}^{+1} dy \left[F_{D,LT}(x,y) + G_{D,LT}(x,y) \right] = 0, \quad x f_{LT}(x) - f_{1LT}^{(1)}(x) - \mathcal{P}\int_{-1}^{+1} dy \frac{F_{G,LT}(x,y) + G_{G,LT}(x,y)}{x-y} = 0$
- $x e_{LL}(x) - 2\int_{-1}^{+1} dy H_{D,LL}^{\perp}(x,y) - \frac{m}{M} f_{1LL}(x) = 0, \quad x e_{LL}(x) - 2\mathcal{P}\int_{-1}^{+1} dy \frac{H_{G,LL}^{\perp}(x,y)}{x-y} - \frac{m}{M} f_{1LL}(x) = 0$

Lorentz-invariance relation

- $\frac{d f_{1LT}^{(1)}(x)}{dx} - f_{LT}(x) + \frac{3}{2} f_{1LL}(x) - 2\mathcal{P}\int_{-1}^{+1} dy \frac{F_{G,LT}(x,y)}{(x-y)^2} = 0$

Future prospects and summary

High-energy hadron physics experiments



Facilities on spin-1 hadron structure functions including future possibilities.

JLab PAC-38 (Aug. 22-26, 2011) proposal, PR12-11-110

The Deuteron Tensor Structure Function b_1

A Proposal to Jefferson Lab PAC-38.
(Update to LOI-11-003)

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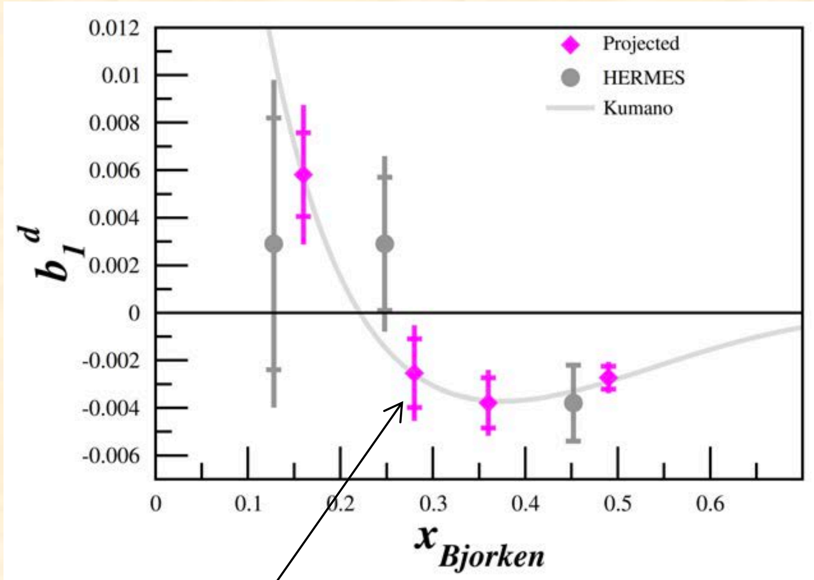

Expected errors by JLab

Approved!

A Letter of Intent to Jefferson Lab PAC 44, June 6, 2016

Search for Exotic Gluonic States in the Nucleus

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Experimental possibility at Fermilab in 2020's

Polarized fixed-target experiments at the Main Injector, Proton beam = 120 GeV

Fermilab-E1039

Drell-Yan experiment with a polarized proton target

Co-Spokespersons: A. Klein, X. Jiang, Los Alamos National Laboratory

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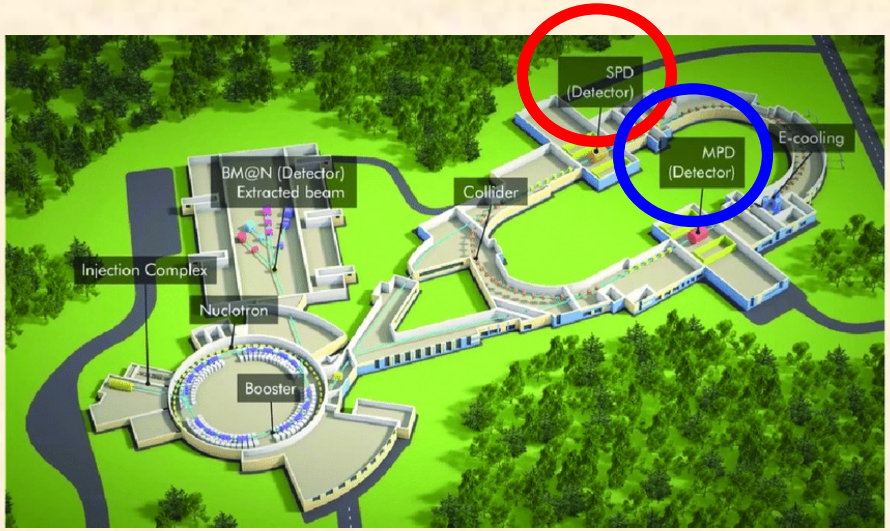
Fermilab experimentalists are interested in the gluon transversity by replacing the E1039 proton target for the deuteron one. (Spokesperson of E1039: D. Keller) However, there was no theoretical formalism until our work.

The Transverse Structure of the Deuteron with Drell-Yan

D. Keller[1]
[1] *University of Virginia, Charlottesville, VA 22904*

New proposal for a Fermilab-PAC in 2022.

Nuclotron-based Ion Collider fAcility (NICA)



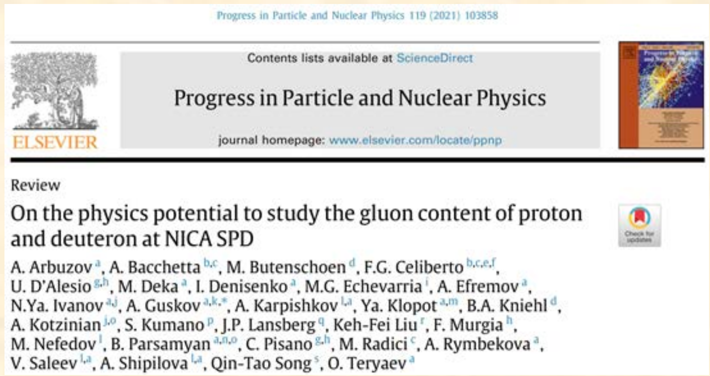
SPD (Spin Physics Detector for physics with polarized beams)
MPD (MultiPurpose Detector for heavy ion physics)

$$\vec{p}+\vec{p}: \quad \sqrt{s_{pp}} = 12 \sim 27 \text{ GeV}$$

$$\vec{d}+\vec{d}: \quad \sqrt{s_{NN}} = 4 \sim 14 \text{ GeV}$$

$\vec{p}+\vec{d}$ is also possilbe.

On the physics potential to study the gluon content of proton and deuteron at NICA SPD, A. Arbuzov *et al.* (NICA project), Nucl. Part. Phys. 119 (2021) 103858.



Progress in Particle and Nuclear Physics 119 (2021) 103858
Contents lists available at ScienceDirect
Progress in Particle and Nuclear Physics
journal homepage: www.elsevier.com/locate/ppnp

Review

On the physics potential to study the gluon content of proton and deuteron at NICA SPD

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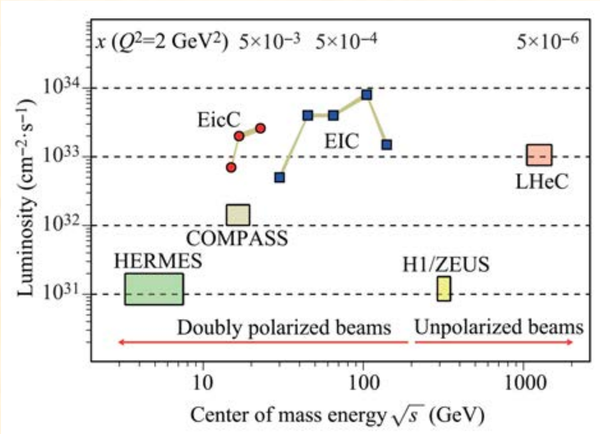
Unique opportunity in high-energy spin physics, especially on the deuteron spin physics.
→ Theoretical formalisms need to be developed.

Electron-ion collider projects in the world



CERN

BNL

EIC-US

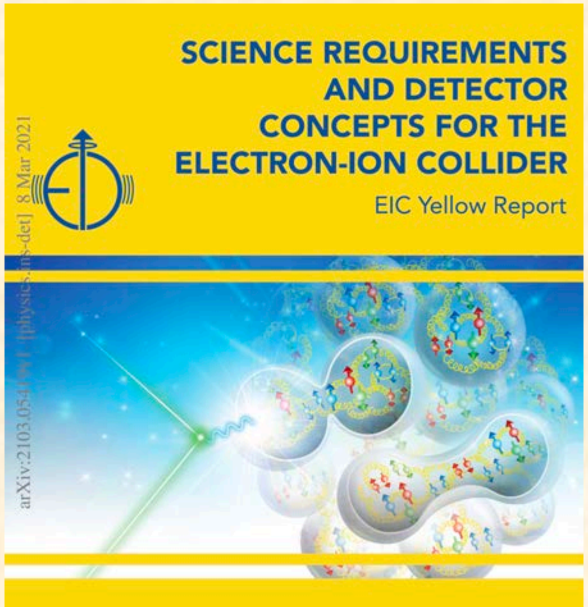
R. Abdul Khalek *et al.*, arXiv:2103.05419.

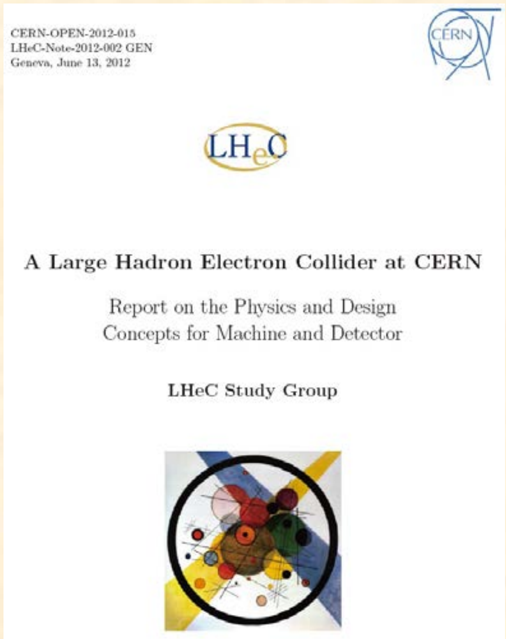
LHeC

J. L. Abelleira Fernandez *et al.*, J. Phys. G: Nucl. Part. Phys. 39 (2012) 075001.

Institute of Modern Physics, High Intensity Heavy Ion Accelerator Facility (HIAF)
→ Electron-ion collider in China (EicC)

D. P. Anderle *et al.*, Front. Phys. 16 (2021) 64701.



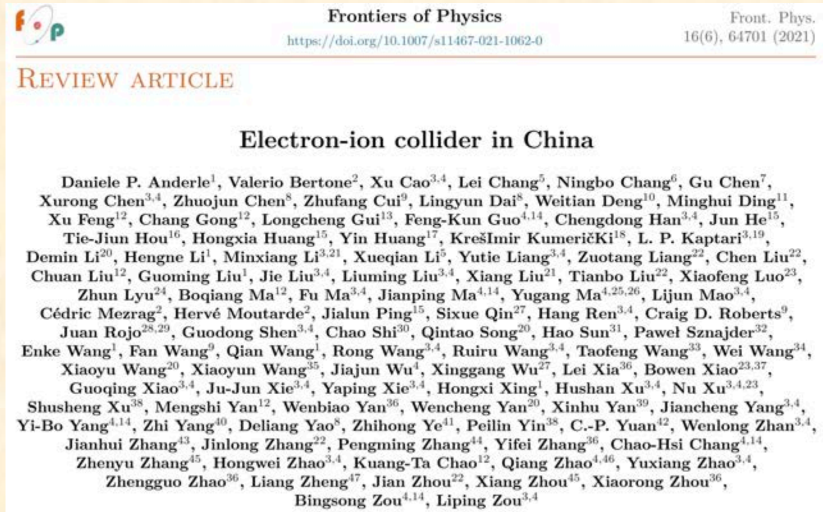


CERN-OPEN-2012-015
LHeC-Note-2012-002 GEN
Geneva, June 13, 2012

A Large Hadron Electron Collider at CERN

Report on the Physics and Design Concepts for Machine and Detector

LHeC Study Group



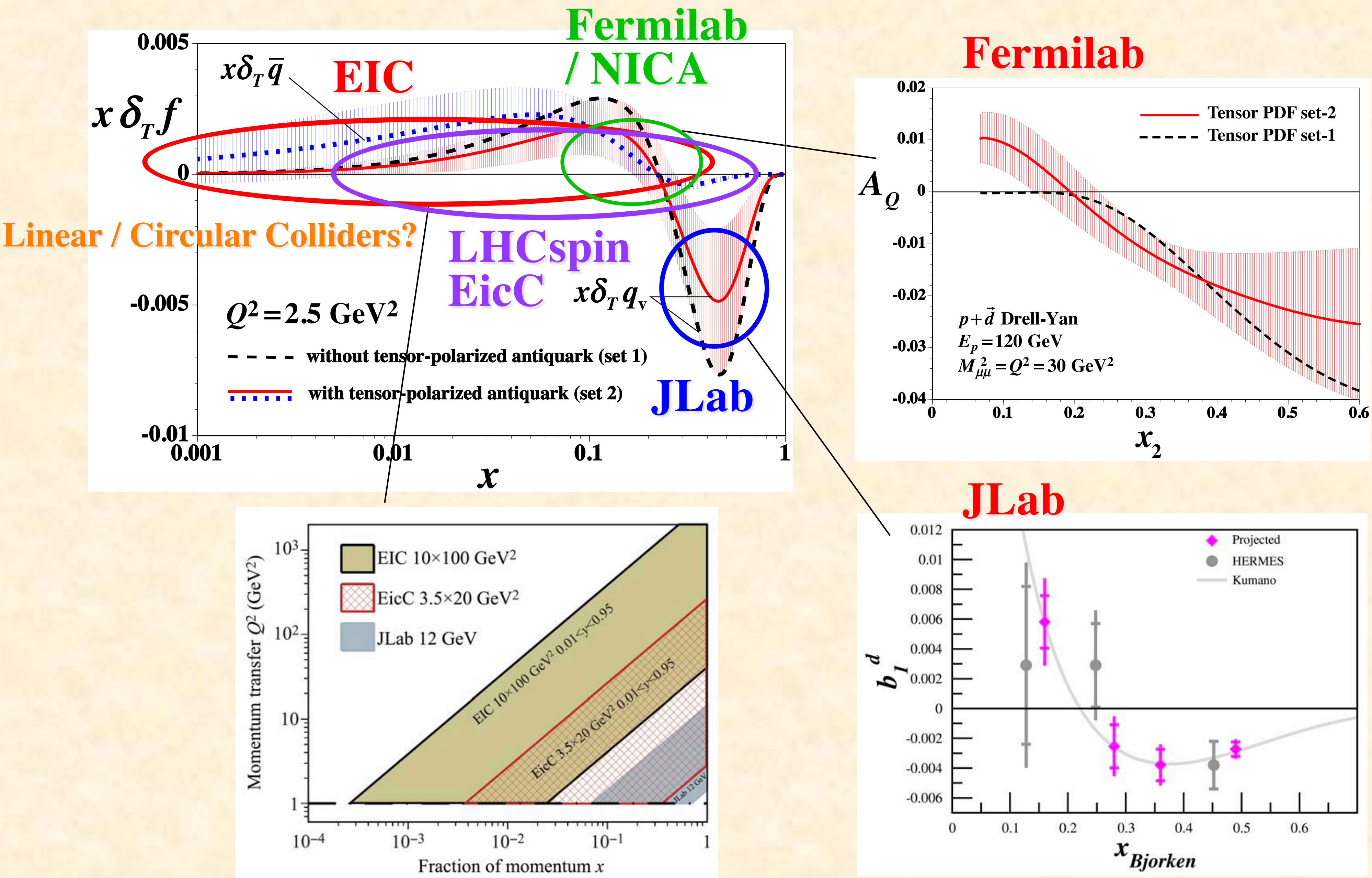
Frontiers of Physics
https://doi.org/10.1007/s11467-021-1062-0
Front. Phys. 16(6), 64701 (2021)

REVIEW ARTICLE

Electron-ion collider in China

Daniele P. Anderle[1], Valerio Bertone[2], Xu Cao[3,4], Lei Chang[5], Ningbo Chang[6], Gu Chen[7], Xurong Chen[3,4], Zhuojun Chen[8], Zhufang Cui[9], Lingyun Dai[8], Weitian Deng[10], Minghui Ding[11], Xu Feng[12], Chang Gong[12], Longcheng Gui[13], Feng-Kun Guo[4,14], Chengdong Han[3,4], Jun He[15], Tie-Jiun Hou[16], Hongxia Huang[15], Yin Huang[17], KrešImir KumeričKi[18], L. P. Kaptari[3,19], Demin Li[20], Hengne Li[1], Minxiang Li[3,21], Xueqian Li[5], Yutie Liang[3,4], Zuotang Liang[22], Chen Liu[22], Chuan Liu[12], Guoming Liu[1], Jie Liu[3,4], Liuming Liu[3,4], Xiang Liu[21], Tianbo Liu[22], Xiaofeng Luo[23], Zhun Lyu[24], Boqiang Ma[12], Fu Ma[3,4], Jianping Ma[4,14], Yugang Ma[4,25,26], Lijun Mao[3,4], Cédric Mezrag[2], Hervé Moutarde[2], Jialun Ping[15], Sixue Qin[27], Hang Ren[3,4], Craig D. Roberts[9], Juan Rojo[28,29], Guodong Shen[3,4], Chao Shi[30], Qintao Song[20], Hao Sun[31], Paweł Sznajder[32], Enke Wang[1], Fan Wang[9], Qian Wang[1], Rong Wang[3,4], Ruiru Wang[3,4], Taofeng Wang[33], Wei Wang[34], Xiaoyu Wang[20], Xiaoyun Wang[35], Jiajun Wu[4], Xinggang Wu[27], Lei Xia[36], Bowen Xiao[23,37], Guoqing Xiao[3,4], Ju-Jun Xie[3,4], Yaping Xie[3,4], Hongxi Xing[1], Hushan Xu[3,4], Nu Xu[3,4,23], Shusheng Xu[38], Mengshi Yan[12], Wenbiao Yan[36], Wencheng Yan[20], Xinhu Yan[39], Jiancheng Yang[3,4], Yi-Bo Yang[4,14], Zhi Yang[40], Deliang Yao[8], Zhihong Ye[41], Peilin Yin[38], C.-P. Yuan[42], Wenlong Zhan[3,4], Jianhui Zhang[43], Jinlong Zhang[22], Pengming Zhang[44], Yifei Zhang[36], Chao-Hsi Chang[4,14], Zhenyu Zhang[45], Hongwei Zhao[3,4], Kuang-Ta Chao[12], Qiang Zhao[4,46], Yuxiang Zhao[3,4], Zhengguo Zhao[36], Liang Zheng[47], Jian Zhou[22], Xiang Zhou[45], Xiaorong Zhou[36], Bingsong Zou[4,14], Liping Zou[3,4]

x regions of b_1 in 2020's and 2030's



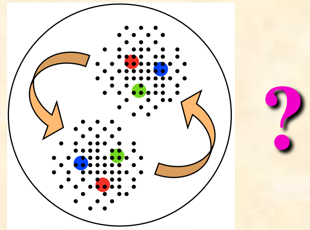
Summary

Spin-1 structure functions of the deuteron (additional spin structure to nucleon spin)

- Tensor structure in quark-gluon degrees of freedom
- Tensor-polarized structure function b_1 and PDFs, gluon transversity

  Experiments at JLab, Fermilab, NICA, LHCspin/AMBER, EIC/EicC, • • •
- New signature beyond "standard" hadron physics?

standard model



- TMDs up to twist 4
- Higher-twist effects could be sizable at Q^2 of a few GeV^2

  → Our relations (WW-like, BC-like, from eq. of motion, Lorentz invariance) could become valuable for future experimental analyses.
- Not discussed: GPDs, GDAs (Generalized Distribution amplitudes = timelike GPDs), • • •

There are various experimental projects on the polarized spin-1 deuteron in 2020's and 2030', and "exotic" hadron structure could be found by focusing on the spin-1 nature.

Appendix

Collinear PDFs for spin-1 hadrons

Appendix I

Tensor polarization: $T^{\mu\nu} = \frac{1}{2}\left[\frac{4}{3}S_{LL}\frac{(P^+)^2}{M^2}\bar{n}^\mu\bar{n}^\nu - \frac{2}{3}S_{LL}\left(\bar{n}^\mu n^\nu + \bar{n}^\nu n^\mu - g_T^{\mu\nu}\right) + \frac{1}{3}S_{LL}\frac{M^2}{(P^+)^2}n^\mu n^\nu + \frac{P^+}{M}(\bar{n}^\mu S_{LT}^\nu + \bar{n}^\nu S_{LT}^\mu) - \frac{M}{2P^+}(n^\mu S_{LT}^\nu + n^\nu S_{LT}^\mu) + S_{TT}^{\mu\nu}\right]$

Collinear correlation function: $\Phi(x,P,T) = \frac{1}{2}\left[f_{1LL}(x)S_{LL}\bar{n\!\!\!/} + \frac{M}{P^+}e_{LL}(x)S_{LL} + \frac{M}{P^+}f_{LT}(x)S_{LT}\right]$, **up to twist-3**

Matrix element of vector operator: $\left\langle P,T\middle|\bar{\psi}(0)\gamma^\mu\psi(\xi^-)\middle|P,T\right\rangle = \int_{-1}^{1}dx\,e^{-ixP^+\xi^-}P^+\mathrm{Tr}\left[\Phi_{ij}(x,P,T)(\gamma^\mu)_{ji}\right] = \int_{-}^{1}dx\,e^{-ixP^+\xi^-}2P^+\left[f_{1LL}(x)S_{LL}\bar{n}^\mu + \frac{M}{P^+}S_{LT}^\mu f_{LT}(x)\right]$

$\alpha = 1,2 =$ **transverse:** $\left\langle P,T\middle|\bar{\psi}(0)(\partial^\mu\gamma^\alpha - \partial^\alpha\gamma^\mu)\psi(\xi)\middle|P,T\right\rangle = 2MS_{LT}^\alpha\int_{-}^{1}dx\,e^{-ixP^+\xi^-}\left[-\frac{3}{2}f_{1LL}(x) + f_{LT}(x) - \frac{d}{dx}\{xf_{LT}(x)\}\right]$

Note: Twist-3 operators $R^{[\sigma\{\mu_1]\cdots\mu_{n-1}\}}$ are obtained by the Tayler expansion of $\xi_\mu\bar{\psi}(0)(\partial^\mu\gamma^\sigma - \partial^\sigma\gamma^\mu)\psi(\xi)$, which needs to be investigated in details for finding the details of twist-3 terms.

$$\bar{\psi}(0)(\partial^\mu\gamma^\alpha - \partial^\alpha\gamma^\mu)\psi(\xi) = \bar{\psi}(0)(D^\mu\gamma^\alpha - D^\alpha\gamma^\mu)\psi(\xi) - \bar{\psi}(0)\gamma^\mu\psi(\xi)ig\int_0^1 dt\,t\,\xi_\rho G^{\rho\alpha}(t\xi)$$

In the Fock-Schwinger gauge: $\xi_\mu A^\mu(\xi) = 0$, we have $A^\nu(\xi) = \int_0^1 dt\,t\xi_\mu G^{\mu\nu}(t\xi)$, $G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$

$$\bar{\psi}(0)(D^\mu\gamma^\alpha - D^\alpha\gamma^\mu)\psi(\xi) = -\frac{i}{2}\bar{\psi}(0)\sigma^{\alpha\mu}\vec{D\!\!\!/}\psi(\xi) - \frac{i}{2}\bar{\psi}(0)\overleftarrow{D\!\!\!/}\sigma^{\alpha\mu}\psi(\xi) - \frac{1}{2}g\int_0^1 dt\,\xi_\nu G^{\rho\nu}(t\xi)\bar{\psi}(0)\gamma_\rho\sigma^{\alpha\mu}\psi(\xi) + \frac{i}{2}\bar{\partial}_\rho\{\bar{q}(0)W(0,\,\xi)\gamma^\rho\sigma^{\alpha\mu}q(\xi)\}$$

$$-\frac{1}{2}\xi_\mu g\int_0^1 dt\,\xi_\nu G^{\rho\nu}(t\xi)\bar{\psi}(0)\gamma_\rho\sigma^{\alpha\mu}\psi(\xi) = \frac{1}{2}g\int_0^1 dt\,\bar{\psi}(0)\left\{-i\xi_\mu G^{\alpha\mu}(t\xi)\xi\!\!\!/ + \xi_\mu\widetilde{G}^{\alpha\mu}(tx)\xi\!\!\!/\gamma_5 - \left[\xi^2\widetilde{G}^{\alpha\sigma}(t\xi) - \xi_\mu x^\alpha\widetilde{G}^{\mu\sigma}(tx)\right]\gamma_\sigma\gamma_5\right\}\psi(x)$$

$$\bar{\partial}_\rho\{\bar{q}(0)W(0,\,\xi)\gamma^\rho\sigma^{\alpha\mu}q(\xi)\} = \bar{\psi}(0)\overleftarrow{D\!\!\!/}\sigma^{\alpha\mu}\psi(\xi) + \bar{\psi}(0)\vec{D\!\!\!/}\sigma^{\alpha\mu}\psi(\xi) - ig\int_0^1 dt\,\xi^\nu G_{\rho\nu}(t\xi)\bar{\psi}(0)\gamma^\rho\sigma^{\alpha\mu}\psi(\xi)$$

$$\xi_\mu\left[\bar{\psi}(0)(\partial^\mu\gamma^\alpha - \partial^\alpha\gamma^\mu)\psi(\xi)\right] = g\int_0^1 dt\,\bar{\psi}(0)\left\{i\left(t - \frac{1}{2}\right)G^{\alpha\mu}(t\xi) - \frac{1}{2}\widetilde{G}^{\alpha\mu}(t\xi)\gamma_5\right\}\xi_\mu\xi\!\!\!/\psi(x) + \frac{1}{2}g\int_0^1 dt\,\bar{\psi}(0)\left[\xi_\mu\xi^\alpha\widetilde{G}^{\mu\sigma}(t\xi) - \xi^2\widetilde{G}^{\alpha\sigma}(t\xi)\right]\gamma_\sigma\gamma_5\psi(\xi)$$

$$-\frac{i}{2}\xi_\mu\bar{\psi}(0)\sigma^{\alpha\mu}(\vec{D\!\!\!/} - m)\psi(\xi) - \frac{i}{2}\xi_\mu\bar{\psi}(0)(\overleftarrow{D\!\!\!/} + m)\sigma^{\alpha\mu}\psi(\xi) + \frac{i}{2}\xi_\mu\bar{\partial}_\rho\{\bar{\psi}(0)W(0,\,\xi)\gamma^\rho\sigma^{\alpha\mu}\psi(\xi)\}$$

$$\simeq g\int_0^1 dt\,\bar{\psi}(0)\left\{i\left(t - \frac{1}{2}\right)G^{\alpha\mu}(t\xi) - \frac{1}{2}\widetilde{G}^{\alpha\mu}(t\xi)\gamma_5\right\}\xi_\mu\xi\!\!\!/\psi(x)$$

Multiparton correlation function: $(\Phi_G^\nu)_{ij}(x_1,x_2) = \int\frac{d\xi_1^-}{2\pi}\frac{d\xi_2^-}{2\pi}e^{ix_1P^+\xi_1^-}e^{i(x_2-x_1)P^+\xi_2^-}\left\langle P,T\middle|\bar{\psi}_j(0)\,gG^{+\nu}(\xi_2^-)\psi_i(\xi_1^-)\middle|P,T\right\rangle$

Express Φ_G^ν in terms of possible Lorentz vectors and multiparton distribution functions with the conditions Hermiticity, parity invariance, and time-reversal invariance

$$\Phi_G^\nu(x_1,x_2) = \frac{M}{2}\left[iS_{LT}^\nu F_{G,LT}(x_1,x_2) - \varepsilon_\perp^{\alpha\mu}S_{LT\mu}\gamma_5 G_{G,LT}(x_1,x_2) + iS_{LL}\gamma^\alpha H_{G,LL}^\perp(x_1,x_2) + iS_{TT}^{\alpha\mu}\gamma_\mu H_{G,TT}(x_1,x_2)\right]\bar{n\!\!\!/}$$

$(\Phi_G^\nu)_{ij}(n\!\!\!/)_{ji}$: $\quad S_{LT}^\nu F_{G,LT}(x_1,x_2) = -\frac{i}{2M}g\int\frac{d\xi_1^-}{2\pi}\frac{d\xi_2^-}{2\pi}e^{ix_1P^+\xi_1^-}e^{i(x_2-x_1)P^+\xi_2^-}\left\langle P,T\middle|\bar{\psi}(0)\,n\!\!\!/ n_\mu G^{\mu\nu}(\xi_2^-)\psi(\xi_1^-)\middle|P,T\right\rangle$

$(\Phi_G^\nu)_{ij}(i\gamma_5 n\!\!\!/)_{ji}$: $\quad S_{LT}^\nu G_{G,LT}(x_1,x_2) = \frac{i}{2M}g\int\frac{d\xi_1^-}{2\pi}\frac{d\xi_2^-}{2\pi}e^{ix_1P^+\xi_1^-}e^{i(x_2-x_1)P^+\xi_2^-}\left\langle P,T\middle|\bar{\psi}(0)\,i\gamma_5 n\!\!\!/ n_\mu\tilde{G}^{\mu\nu}(\xi_2^-)\psi(\xi_1^-)\middle|P,T\right\rangle$

$$\int\frac{d(P\cdot\xi)}{2\pi}e^{ix_1P\cdot\xi}\langle P,T|g\int_0^1 dt\,\bar{\psi}(0)\left\{i\left(t - \frac{1}{2}\right)G^{\mu\nu}(t\xi) - \frac{1}{2}\gamma_5\tilde{G}^{\mu\nu}(t\xi)\right\}\xi_\mu\xi\!\!\!/\psi(\xi)|P,T\rangle_{\xi^+=\vec{\xi}_T=0} = -2MS_{LT}^\nu\mathcal{P}\int_0^1 dx_2\frac{1}{x_1 - x_2}\left[\frac{\partial}{\partial x_1}\{F_{G,LT}(x_1,x_2) + G_{G,LT}(x_1,x_2)\} + \frac{\partial}{\partial x_2}\{F_{G,LT}(x_2,x_1) + G_{G,LT}(x_2,x_1)\}\right]$$

Relations from equation of motion and Lorentz-invariance relation for spin-1/2 nucleons

Appendix II

References on related works

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We may miss some of your works.

Relations among multiparton distribution functions

Appendix III

Multiparton correlation functions

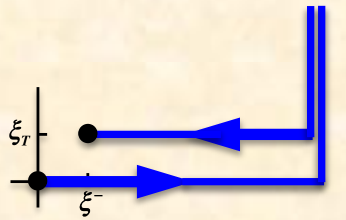
$$(\Phi^{\mu}_{X})_{ij}(x_1,x_2,P,T)=\int\frac{d\xi_1^-}{2\pi}\frac{d\xi_2^-}{2\pi}e^{ix_1P^+\xi_1^-}e^{i(x_2-x_1)P^+\xi_2^-}\left\langle P,T\right|\bar{\psi}_j(0)Y^{\mu}(\xi_2^-)\psi_i(\xi_1^-)\left|P,T\right\rangle,\qquad X\ (Y^\mu)=G\ (gG^{+\mu}),\ A\ (gA^\mu),\ D\ (iD^\mu),\ D^\mu=\partial^\mu-igA^\mu$$

$$\Phi^{\alpha}_{D}(x_1,x_2,P,T)=\frac{M}{2P^+}\left[S_{LT}^{\alpha}F_{D,LT}(x_1,x_2)+i\varepsilon_T^{\alpha\mu}S_{LT,\mu}\gamma_5G_{D,LT}(x_1,x_2)+S_{LL}\gamma^{\alpha}H^{\perp}_{D,LL}(x_1,x_2)+S_{TT}^{\alpha\mu}\gamma_\mu H_{D,TT}(x_1,x_2)\right]\bar{n}$$

$$\Phi^{\alpha}_{G}(x_1,x_2,P,T)=\frac{M}{2}i\left[S_{LT}^{\alpha}F_{G,LT}(x_1,x_2)+i\varepsilon_T^{\alpha\mu}S_{LT,\mu}\gamma_5G_{G,LT}(x_1,x_2)+S_{LL}\gamma^{\alpha}H^{\perp}_{G,LL}(x_1,x_2)+S_{TT}^{\alpha\mu}\gamma_\mu H_{G,TT}(x_1,x_2)\right]\bar{n}$$

---

k_T-weighted correlation function



$$(\Phi^{\alpha}_{\partial})_{ij}(x,P,T)=\int d^2k_Tk_T^{\alpha}\Phi^{[C]}_{ij}(x,k_T,P,T)=\int d^2k_Tk_T^{\alpha}\int\frac{d\xi^-d^2\xi_T}{(2\pi)^3}e^{ixP^+\xi^--i\vec{k}_T\cdot\vec{\xi}_T}\left\langle P,T\right|\bar{\psi}_j(0)W^{[C]}(0,\xi)\psi_i(\xi)\left|P,T\right\rangle_{k^+=xP^+,\xi^+=0}=\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left\langle P,T\right|\bar{\psi}_j(0)i\partial_T^{\alpha}(\xi)W^{[C]}(0,\xi)\psi_i(\xi)\left|P,T\right\rangle_{k^+=xP^+,\xi^+=\vec{\xi}_T=0}$$

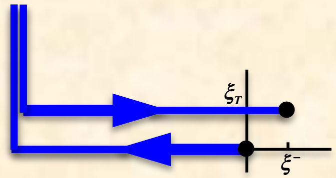
$$W^{[\pm]}(0,\xi)=U^-[0,\pm\infty^-]U^T[0_T,\infty_T]U^T[\infty_T,\vec{\xi}_T]U^-[\pm\infty^-,\xi^-]$$

$$\partial_T^{\alpha}(\xi)W^{[\pm]}(0,\xi)_{\xi^+=\vec{\xi}_T=0}=U^-[0,\xi^-]D_T^{\alpha}(\xi)_{\xi^+=\vec{\xi}_T=0}+igU^-[0,\pm\infty^-]\int_{\pm\infty^-}^{\xi^-}d\eta^-U^-[\pm\infty^-,\eta^-]G^{+\alpha}(\eta^-)U^-[\eta^-,\xi^-],\ \ \alpha=\text{transverse}=1,2$$

$$(\Phi^{[\pm]\alpha}_{\partial})_{ij}(x,P,T)=\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left\langle P,T\right|\bar{\psi}_j(0)i\partial_T^{\mu}(\xi)W^{[C]}(0,\xi)\psi_i(\xi)\left|P,T\right\rangle_{k^+=xP^+,\xi^+=\vec{\xi}_T=0}=\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left[\left\langle P,T\right|\bar{\psi}_j(0)iD_T^{\alpha}(\xi)\psi_i(\xi)\left|P,T\right\rangle-\left\langle P,T\right|\bar{\psi}_j(0)\int_{\pm\infty^-}^{\xi^-}d\eta^-gG^{+\alpha}(\eta^-)\psi_i(\xi)\left|P,T\right\rangle\right]_{k^+=xP^+,\xi^+=\vec{\xi}_T=0}$$

$$G^{+\alpha}=\partial_-A^{\alpha}(y^-)\quad\text{for }A^+=0\qquad\to\int_{\pm\infty^-}^{\xi^-}d\eta^-G^{+\alpha}(\eta^-)=A^{\alpha}(\xi^-)-A^{\alpha}(\pm\infty),\ \ iD_T^{\alpha}(\xi^-)=i\partial_T^{\alpha}(\xi^-)+gA_T^{\alpha}(\xi^-)$$

$$=\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left[\left\langle P,T\right|\bar{\psi}_j(0)i\partial_T^{\alpha}(\xi^-)\psi_i(\xi)\left|P,T\right\rangle+\left\langle P,T\right|\bar{\psi}_j(0)gA^{\alpha}(\xi^-=\pm\infty)\psi_i(\xi)\left|P,T\right\rangle\right]_{k^+=xP^+,\xi^+=\vec{\xi}_T=0}$$



Average of $\Phi^{[+]\alpha}_{\partial}$ and $\Phi^{[-]\alpha}_{\partial}$:

$$(\tilde{\Phi}^{\alpha})_{ij}(x,P,T)=\frac{(\Phi^{[+]\alpha}_{\partial})_{ij}(x,P,T)+(\Phi^{[-]\alpha}_{\partial})_{ij}(x,P,T)}{2}=\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left[\left\langle P,T\right|\bar{\psi}_j(0)i\partial_T^{\alpha}(\xi^-)\psi_i(\xi)\left|P,T\right\rangle+\left\langle P,T\right|\bar{\psi}_j(0)g\left\{\frac{A^{\alpha}(\infty^-)+A^{\alpha}(-\infty^-)}{2}\right\}\psi_i(\xi)\left|P,T\right\rangle\right]$$

$$\Phi(x,k_T,T)_{\text{twist-2}}=\frac{1}{2}\Bigg[f_{1LL}(x,k_T^2)S_{LL}\bar{n}-f_{1LT}(x,k_T^2)\frac{k_T\cdot S_{LT}}{M}\bar{n}+f_{1TT}(x,k_T^2)\frac{k_{T\mu}\cdot S_{TT}^{\mu\nu}\cdot k_{T\nu}}{M^2}\bar{n}+g_{1\_}(x,k_T^2)\frac{\varepsilon^{\text{Song}}_{T,\mu\nu}S_{LT}^{\mu}k_T^{\nu}}{M}\gamma_5\bar{n}+g_{1TT}(x,k_T^2)\frac{\varepsilon^{\text{Song}}_{T,\mu\nu}S_{TT}^{\mu\rho}k_T^{\nu}k_{T,\rho}}{M^2}\gamma_5\bar{n}$$

$$+h_{1LL}^{\perp}(x,k_T^2)S_{LL}\frac{k_T^i}{M}\sigma^{i\mu}\bar{n}_\mu+h_{1LT}(x,k_T^2)S_{LT}^i\sigma^{i\mu}\bar{n}_\mu+h_{1LT}^{\perp}(x,k_T^2)\frac{S_{LT}^jk_T^ik_T^j-S_{LT}^i\vec{k}_T^2/2}{M^2}\sigma^{i\mu}\bar{n}_\mu+h_{1TT}(x,k_T^2)\frac{S_{TT}^{ij}k_T^j}{M}\sigma^{i\mu}\bar{n}_\mu+h_{1TT}^{\perp}(x,k_T^2)\frac{(S_{TT}^{lj}k_T^ik_T^l-S_{TT}^{ij}\vec{k}_T^2/2)k_T^j}{M^3}\sigma^{i\mu}\bar{n}_\mu\Bigg]$$

$$\Phi^{\mu}_{\partial}(x,T)_{\text{twist-2}}=\int d^2k_Tk_T^{\mu}\Phi(x,k_T,T)_{\text{twist-2}}=\frac{M}{2}\left[f_{1LT}^{(1)}(x)S_{LT}^{\alpha}\bar{n}+g_{1LT}^{(1)}(x)\varepsilon_T^{\alpha\mu}S_{LT,\mu}\gamma_5\bar{n}-h_{1LL}^{\perp(1)}(x)S_{LL}\sigma^{\alpha\mu}\bar{n}_\mu+h_{1TT}^{\prime(1)}(x)S_{TT}^{\alpha\beta}\sigma_{\beta\mu}\bar{n}^{\mu}\right]$$

$$=\frac{M}{4}\left[f_{1LT}^{[+](1)}(x)+f_{1LT}^{[-](1)}(x)\right]S_{LT}^{\alpha}\bar{n}_{ij}\equiv\frac{M}{2}f_{1LT}^{(1)}(x)S_{LT}^{\alpha}\bar{n}_{ij},$$

$\tilde{\Phi}^{\alpha}$ = T-even, only $f_{1LT}^{(1)}(x)$ is T-even, transverse-momemtum moments of TMDs: $f^{(1)}(x)=\int d^2k_T\frac{\vec{k}_T^2}{2M^2}f(x,k_T^2)$

Lightcone gauge $A^+=0$, $G^{+\alpha}=\partial_-A^{\alpha}(y^-)$ $\to A^{\alpha}(y^-)=-\int_{y^-}^{\infty}dz^-G^{+\alpha}(z^-)+A^{\alpha}(\infty)=-\int_{-\infty}^{\infty}dz^-\theta(z^--y^-)G^{+\alpha}(z^-)+A^{\alpha}(\infty)$, $A^{\alpha}(y^-)=\int_{-\infty}^{y^-}dz^-G^{+\alpha}(z^-)+A^{\alpha}(-\infty)=\int_{-\infty}^{\infty}dz^-\theta(y^--z^-)G^{+\alpha}(z^-)+A^{\alpha}(-\infty)$

$$A^{\alpha}(y^-)=\frac{A^{\alpha}(\infty)+A^{\alpha}(-\infty)}{2}-\frac{1}{2}\int_{-\infty}^{\infty}dz^-\varepsilon(z^--y^-)G^{+\alpha}(z^-),\ \ \varepsilon(z^--y^-)=\theta(z^--y^-)-\theta(y^--z^-)$$

$$(\Phi^{\alpha}_{A})_{ij}(x_1,x_2,P,T)=\int\frac{d\xi_1^-}{2\pi}\frac{d\xi_2^-}{2\pi}e^{ix_1P^+\xi_1^-}e^{i(x_2-x_1)P^+\xi_2^-}\left\langle P,T\right|\bar{\psi}_j(0)gA^{\alpha}(\xi_2^-)\psi_i(\xi_1^-)\left|P,T\right\rangle$$

$$=\int\frac{d\xi_1^-}{2\pi}\frac{d\xi_2^-}{2\pi}e^{ix_1P^+\xi_1^-}e^{i(x_2-x_1)P^+\xi_2^-}\left\langle P,T\right|\bar{\psi}_j(0)g\left[\frac{A^{\alpha}(\infty^-)+A^{\alpha}(-\infty^-)}{2}-\frac{1}{2}\int_{-\infty^-}^{\infty^-}d\eta^-\varepsilon(\eta^--\xi_2^-)G^{+\alpha}(\eta^-)\right]\psi_i(\xi_1^-)\left|P,T\right\rangle,$$

$$(\Phi^{\alpha}_{A(\pm\infty^-)})_{ij}(x_1,P,T)=\frac{1}{P^+}\int\frac{d\xi_1^-}{2\pi}e^{ix_1P^+\xi_1^-}\left\langle P,T\right|\bar{\psi}_j(0)gA^{\alpha}(\xi_2^-=\pm\infty^-)\psi_i(\xi_1^-)\left|P,T\right\rangle$$

$$\varepsilon(\eta^--\xi_2^-)=\frac{i}{\pi}\mathcal{P}\int_{-\infty}^{\infty}d\omega\frac{1}{\omega}e^{-i\omega(\eta^--\xi_2^-)},\quad\int\frac{d\xi_1^-}{2\pi}\frac{d\xi_2^-}{2\pi}e^{ix_1P^+\xi_1^-}e^{i(x_2-x_1)P^+\xi_2^-}\int_{-\infty^-}^{\infty^-}d\eta^-\varepsilon(\eta^--\xi_2^-)=2i\mathcal{P}\frac{1}{(x_1-x_2)P^+}\int\frac{d\xi_1^-}{2\pi}e^{ix_1P^+\xi_1^-}\int_{-\infty^-}^{\infty^-}\frac{d\eta^-}{2\pi}e^{i(x_2-x_1)P^+\eta^-}$$

$$=\delta(x_1-x_2)\frac{(\Phi^{\alpha}_{A(+\infty^-)})_{ij}(x_1,P,T)+(\Phi^{\alpha}_{A(-\infty^-)})_{ij}(x_1,P,T)}{2}-\frac{1}{2}2i\mathcal{P}\frac{1}{(x_1-x_2)P^+}\int\frac{d\xi_1^-}{2\pi}\frac{d\xi_2^-}{2\pi}e^{ix_1P^+\xi_1^-}e^{i(x_2-x_1)P^+\xi_2^-}\left\langle P,T\right|\bar{\psi}_j(0)gG^{+\alpha}(\xi_2^-)\psi_i(\xi_1^-)\left|P,T\right\rangle$$

$$=\delta(x_1-x_2)\frac{(\Phi^{\alpha}_{A(+\infty^-)})_{ij}(x_1,P,T)+(\Phi^{\alpha}_{A(-\infty^-)})_{ij}(x_1,P,T)}{2}-\mathcal{P}\frac{i}{(x_1-x_2)P^+}(\Phi^{\alpha}_{G})_{ij}(x_1,x_2,P,T)$$

$$\Phi^{\alpha}_{D}(x_1,x_2,P,T)=\delta(x_2-x_1)\frac{1}{P^+}\tilde{\Phi}^{\alpha}(x,P,T)-\mathcal{P}\frac{i}{(x_1-x_2)P^+}\Phi^{\alpha}_{G}(x_1,x_2,P,T)$$

$$\to F_{D,LT}(x_1,x_2)=\delta(x_1-x_2)f_{1LT}^{(1)}(x_1)+\mathcal{P}\left(\frac{1}{x_1-x_2}\right)F_{G,LT}(x_1,x_2),\ \ G_{D,LT}(x_1,x_2)=\mathcal{P}\left(\frac{1}{x_1-x_2}\right)G_{G,LT}(x_1,x_2),\ \ H^{\perp}_{D,LL}(x_1,x_2)=\mathcal{P}\left(\frac{1}{x_1-x_2}\right)H^{\perp}_{G,LL}(x_1,x_2),\ \ H_{D,TT}(x_1,x_2)=\mathcal{P}\left(\frac{1}{x_1-x_2}\right)H_{G,TT}(x_1,x_2)$$

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