Weyl differential geometry and Quantum Mechanics: spin 1/2, EPR and spin statistics from geometry

Enrico Santamato – Francesco De Martini

Napoli, 29/04/2019

Part I

Introduction

Bohm’s interpretation of QM

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + V(\mathbf{r})\psi$$

$$\psi = \sqrt{\rho}\,e^{i\frac{S}{\hbar}} \qquad \rho = |\psi|^2$$

$$-\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\rho\frac{\nabla S}{m}\right)$$

continuity equation

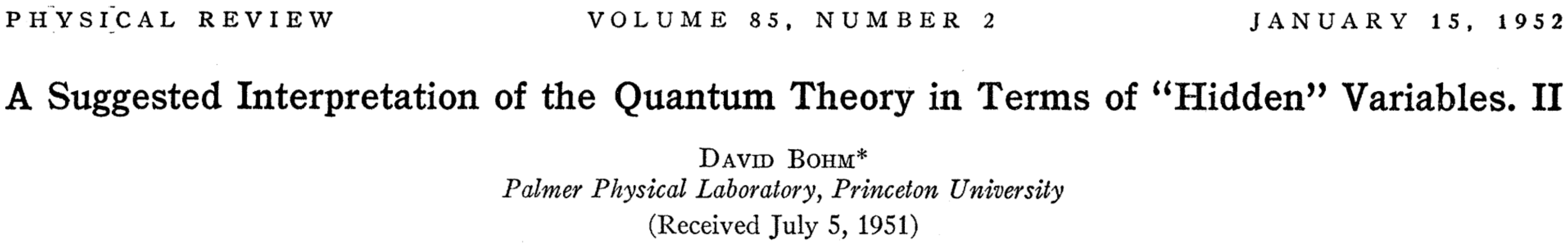
$$-\frac{\partial S}{\partial t} = \frac{1}{2m}(\nabla S)^2 + V(\mathbf{r},t) + Q_B(\mathbf{r},t)$$

H-J equation

$$\boxed{Q(\mathbf{r},t) = -\frac{\hbar^2}{2m}\left[\frac{\nabla^2\rho}{2\rho} - \left(\frac{\nabla\rho}{2\rho}\right)^2\right]}$$

Bohm’s Quantum Potential

Where this **nonlocal potential** comes from?



PHYSICAL REVIEW VOLUME 85, NUMBER 2 JANUARY 15, 1952

A Suggested Interpretation of the Quantum Theory in Terms of “Hidden” Variables. II

DAVID BOHM*
Palmer Physical Laboratory, Princeton University
(Received July 5, 1951)

Bohm's Quantum Potential comes from Geometry

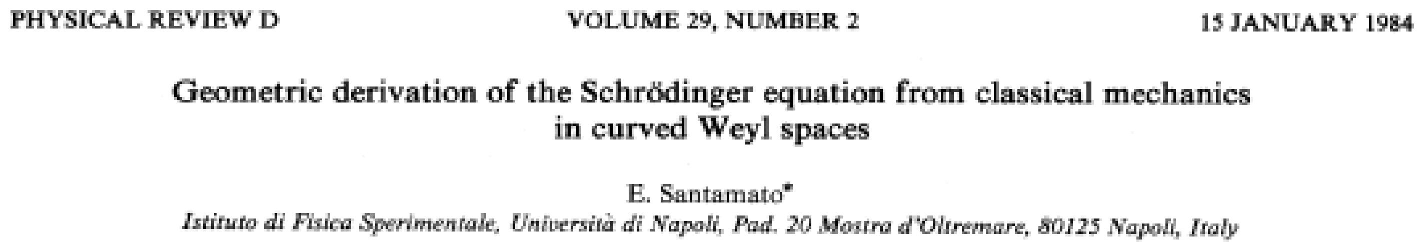
- Schrödinger equation assumes Euclidean space
- Euclidean space has
    - Euclidean metric for vector length

It is in the nature of a metric space to have this parallel transport law

H. Weyl, *Space,Time,Matter*, Dover, 1952

$$\delta\ell = -2\ell d\phi = -2\ell \boldsymbol{\phi}(\mathbf{r})\cdot d\mathbf{r} \qquad \ell = \sqrt{\mathbf{a}\cdot\mathbf{a}}$$

H. Weyl (1919)



PHYSICAL REVIEW D VOLUME 29, NUMBER 2 15 JANUARY 1984

Geometric derivation of the Schrödinger equation from classical mechanics in curved Weyl spaces

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Bohm’s Quantum Potential comes from Geometry

- If $\delta\ell = -2\ell\boldsymbol{\phi}\cdot d\mathbf{r}$ in a parallel transport the 3D space with Euclidean metric acquires a nonzero Weyl scalar curvature

$$R_W = -4\nabla\cdot\boldsymbol{\phi} - 2\boldsymbol{\phi}\cdot\boldsymbol{\phi}$$

- Assume $\boldsymbol{\phi}$ to be a gradient

$$\boldsymbol{\phi} = \nabla(\ln\rho) = \frac{\nabla\rho}{\rho}$$

$$R_W = -8\left[\frac{\nabla^2\rho}{2\rho} - \left(\frac{\nabla\rho}{4\rho}\right)^2\right] = \gamma^{-2}Q_B \qquad \gamma = \frac{\hbar}{4\sqrt{m}}$$

Bohm’s Quantum potential

Geometric formulation of QM in Euclidean-Weyl space

$$-\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\rho\frac{\nabla S}{m}\right)$$

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\nabla S\right)^2 + V + \gamma^2 R_W$$

Weyl curvature

Now ρ has a **geometrical** meaning: it fixes the parallel transport law of vectors in the 3D physical space.

Conclusions (part I)

- Alternatives
  - 3D physical space is Euclidean. QM is not deterministic. ρ has a statistical interpretation (standard).
  - 3D physical space is Euclidean. QM is deterministic but with nonlocal quantum potential. ρ is like a fluid density.
  - Believe in Weyl’s statement

> It is in the nature of any metric space to have a not trivial parallel transport
>
> H. Weyl, *Space*,*Time*,*Matter*, Dover, 1952

  - 3D physical space has Euclidean metric only. QM is deterministic and local. ρ has a **geometric** interpretation.

Part II

The Conformal Quantum Geometrodynamics

(CQG)

Weyl's geometry (*N*-dimensions)

- Two fundamental forms

- A quadratic form (metric)

$$ds^2 = g_{ij}(q)dq^i dq^j$$

- A linear form (parallel transport)

$$\delta\ell = -\ell d\phi = -\ell \phi_i(q)dq^i \qquad \ell = \sqrt{g_{ij}(q)a^i(q)a^j(q)}$$

Weyl’s geometry (*N*-dimensions)

- Weyl’s connections

$$\Gamma^{i}_{jk} = -\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \delta^{i}_{j}\phi_{k} + \delta^{i}_{k}\phi_{j} - g_{jk}\phi^{i} \qquad \left(\Gamma^{i}_{jk} = \Gamma^{i}_{kj}\right)$$

- Weyl’s scalar curvature

$$R_W = R_R + (n-1)\left[\frac{2}{\sqrt{g}}\partial_i\left(\sqrt{g}\,g^{ij}\phi_j\right) - (n-2)g^{ij}\phi_i\phi_j \right]$$

- is

It is in the nature of a metric space to be furnished with these affine connections

H. Weyl, *Space,Time,Matter*, Dover, 1952

Weyl's geometry (*N*-dimensions)

- Assume the Weyl connection integrable

$$\phi_i = \frac{1}{N-2}\partial_i(\ln\rho) = \frac{1}{N-2}\frac{\partial_i\rho}{\rho}$$

$$R_W = R_R - \frac{N-1}{N-2}\left(\frac{2\nabla_k\nabla^k\rho}{\rho} - \frac{\nabla_k\rho\nabla^k\rho}{\rho^2}\right) = R_R + \gamma^{-2}Q_B$$

Weyl

Rieman

- ∇_k covariant derivative w.r.t. g_{ij}
- g_{ij} is used to lower/raise the indices

Weyl's gauge invariance

- The Weyl connections are invariant under Weyl's gauge transformations

Conformal transformation

$$g_{ij} \rightarrow \lambda g_{ij}$$

$$\phi_i \rightarrow \phi_i + \frac{\partial_i \lambda}{2\lambda} = \phi_i + \frac{1}{2}\partial_i \ln \lambda$$

$$\rho \rightarrow \lambda^{-\frac{N-2}{2}} \rho$$

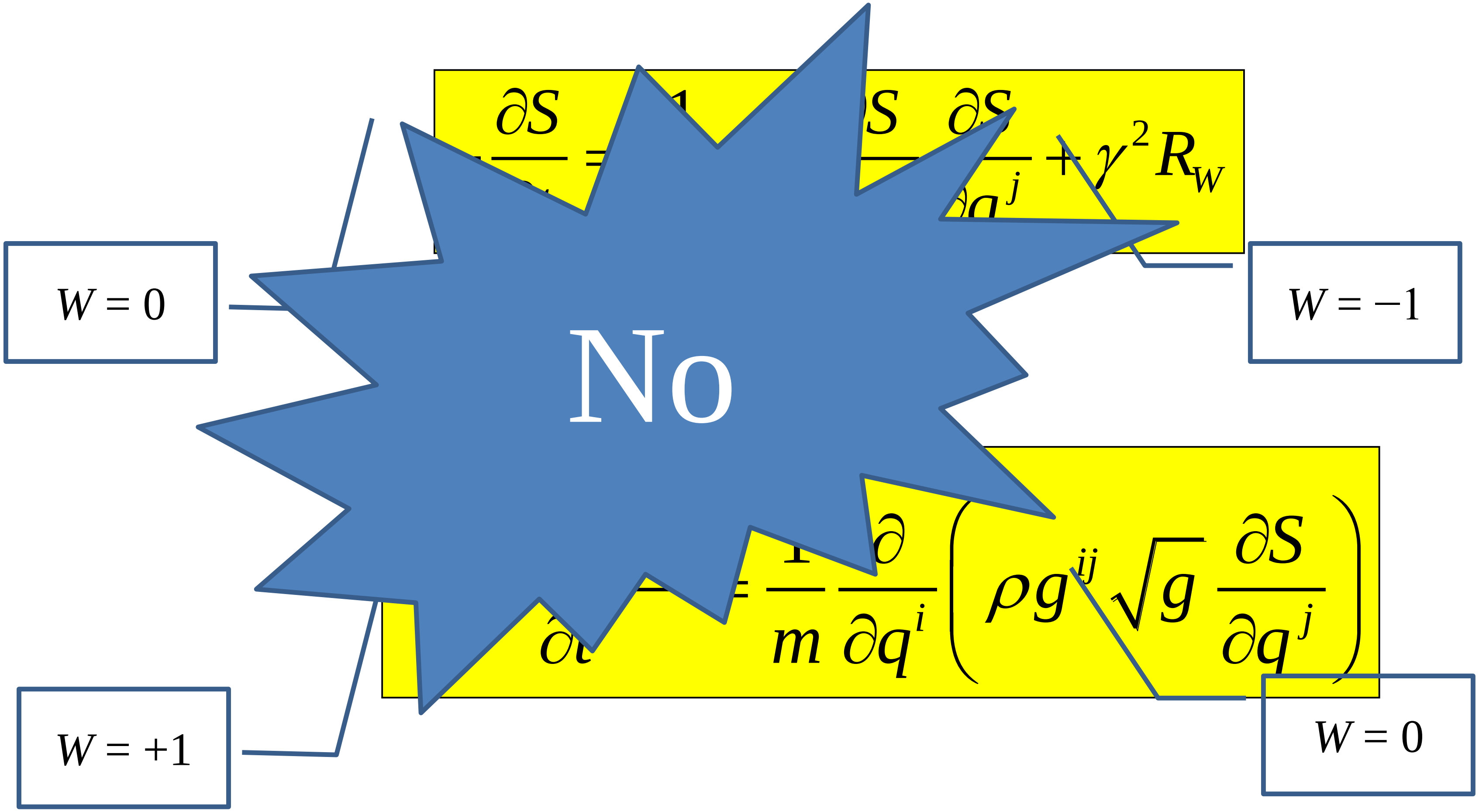
- Tensors T transforms simply under Weyl's gauge if

$$\boxed{T \rightarrow \lambda^{w(T)} T}$$

$w(T)$ Weyl's weigth

- Examples: $w(g_{ij}) = 1$, $w(R_W) = -1$, $w(\rho) = -(N-2)/2$, and ϕ_i does not transform simply.

Are the CQG equations Weyl invariant?



$W = 0$

$W = -1$

No

$W = +1$

$W = 0$

Relativistic CQG

$$g^{\mu\nu}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\nu}}+\gamma^{2}R_{W}+m^{2}c^{2}=0 \qquad g_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$$

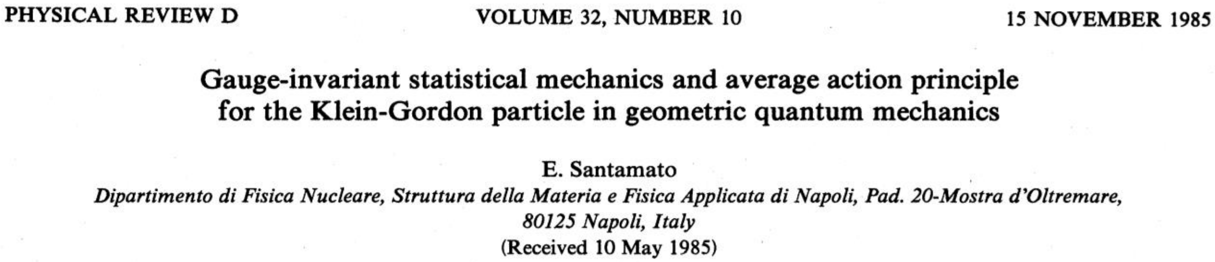
$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{g}\,\rho\, g^{\mu\nu}\frac{\partial S}{\partial x^{\nu}}\right)=0$$

$$W(m^2) = -1 \ !?$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{g}\,g^{\mu\nu}\frac{\partial \psi}{\partial x^{\nu}}\right)=\cancel{R_{R}}+\frac{m^{2}c^{2}}{\hbar^{2}}\psi$$

Klein-Gordon equation

Weyl invariance is hidden



PHYSICAL REVIEW D VOLUME 32, NUMBER 10 15 NOVEMBER 1985

Gauge-invariant statistical mechanics and average action principle for the Klein-Gordon particle in geometric quantum mechanics

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Relativistic CQG

- Gauge invariant action principle: $\delta I = 0$

$$I = \int \rho \left(g^{\mu\nu} \frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x^{\nu}} + \gamma^2 R_W + m^2 c^2 \right) \sqrt{g}\, d^4 x$$

$$g^{\mu\nu} \frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x^{\nu}} + \gamma^2 R_W + m^2 c^2 = 0$$

Variation w.r.t. ρ

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{g}\, \rho g^{\mu\nu} \frac{\partial S}{\partial x^{\nu}} \right) = 0$$

Variation w.r.t. S

- Variation w.r.t. $g^{\mu\nu}$ leads to incompatible field equations unless $m = 0$

Extended gravity in (4+*K*)-D

$$I = \int \rho \left(g^{ij} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} + \gamma^2 R_W \right) \sqrt{g}\, d^N q$$

Gauge invariant action

$$g_{ij} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{\alpha\beta} \end{pmatrix}$$

Space-like extra coordinates

$\gamma_{\alpha\beta}$ is the metric of a homogeneous space (e.g. a group G)

The constant Riemann curvature R_R of $\gamma_{\alpha\beta}$ plays role of mass

Field equations in (4+*K*)-D

$$\nabla_k S \nabla^k S + \gamma^2 R_W = 0$$

$$\nabla_k j^k = \nabla_k \left(\rho \nabla^k S \right) = 0$$

Weyl curvature

Weyl gauge invariance is manifest

$$\Psi = \sqrt{\rho} e^{\frac{iS}{\hbar}}$$

$$\nabla_\mu \nabla^\mu \Psi + \nabla_\alpha \nabla^\alpha \Psi = \boxed{\nabla_i \nabla^i \Psi = \gamma^2 R_R \Psi}$$

Riemann curvature

$$\Psi = \psi(x)\phi(\alpha)$$

$$\nabla_\alpha \nabla^\alpha \phi = -k^2 \phi$$

$$\nabla_\mu \nabla^\mu \psi = \left(\gamma^2 R_R + k^2 \right) \psi = \frac{m^2 c^2}{\hbar^2} \psi$$

Weyl gauge invariance is hidden

Klein-Gordon

Conclusions (part II)

- The relativistic QM is Weyl gauge-invariant
- Only gauge-invariant quantities have an objective physical meaning, e.g. the phase action *S* and the current vector density $j^i = \rho g^{ij} \partial_i S \sqrt{g}$
- $\rho = |\Psi|^2$ is not Weyl-gauge invariant. It has no objective physical meaning.
- Born's statistical interpretation of $\rho = |\Psi|^2$ has no objective physical meaning
- The current density j^i is Weyl invariant and has objective physical meaning (count rate in particle detectors)

Conclusions (part II)

- The CQG is local?
- Assume space $M_4 \times G$ in (4+*K*)-D

De Broglie-Bohm **nonlocal** pilot wave

$$R_W = R_G + \gamma^{-2} Q_B$$

$$Q_B(\mathbf{r},t) = -\left[\frac{\nabla^2 \rho}{2\rho} - \left(\frac{\nabla \rho}{2\rho}\right)^2\right] \qquad \rho = |\psi|^2$$

$$I = \int \rho \left(g^{ij} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} + \gamma^2 R_W \right) \sqrt{g}\, d^N q$$

➡ Gauge invariant Klein-Gordon

- Pass to the gauge where $\rho = 1$

New metric

“Gravitational” action of the scalar field *S*

$$I = \int \left(\bar{g}^{ij} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} + \gamma^2 \bar{R}_{\mathrm{R}} \right) \sqrt{\bar{g}}\, d^N q \qquad \bar{g}_{ij} = \rho^{\frac{2}{2+K}} g_{ij}$$

Part III

Spin 1/2

Dirac equation

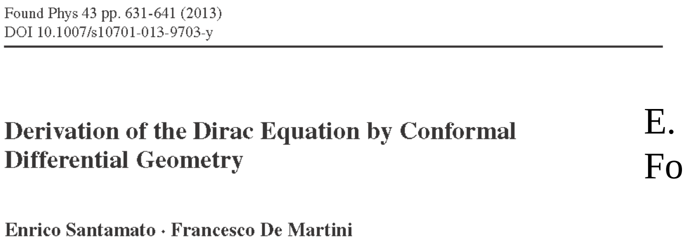
- Spin degrees of freedom: the six Euler angles of the Lorentz group SO(3,1)
- Space $M_4 \times \mathrm{SO}(3,1)$ in 10-D. Coordinates $q^i = \{x^\mu, y^\alpha\} = \{x^\mu, \lambda\theta^\alpha\}$

$$I = \int \rho \left[g^{ij} \left(\frac{\partial S}{\partial q^i} - \frac{e}{c} A_i \right) \left(\frac{\partial S}{\partial q^j} - \frac{e}{c} A_j \right) + \hbar^2 \xi^2 R_W \right] \sqrt{g g_{SO(3,1)}}\, d^N q$$

$$A_i = \{A_\mu, A_\alpha)\} = \{A_\mu, \xi^r_\alpha(y) A_r(x))\} \quad A_r = -\{\mathbf{E}(x), \mathbf{H}(x)\}$$

$$\psi = \sqrt{\rho}\, e^{\frac{iS}{\hbar}}$$

$$\text{Klein-Gordon } (4+N)\text{-D} \qquad g^{ij} \left[\left(-i\hbar \nabla_i - \frac{e}{c} A_i \right) \left(-i\hbar \nabla_j - \frac{e}{c} A_j \right) \psi \right] + \hbar^2 \xi^2 R_R$$



Found Phys 43 pp. 631-641 (2013)
DOI 10.1007/s10701-013-9703-y

Derivation of the Dirac Equation by Conformal Differential Geometry

Enrico Santamato · Francesco De Martini

E. Santamato and F. De Martini,
Found. Phys. **43**, 631 (2013)

Dirac equation

$$\psi_s(x,y)=\sum_{p=-1/2}^{1/2} D^{(0,1/2)}(\Lambda^{-1}(y))_p^s\,\psi_L{}^p(x)+D^{(1/2,0)}(\Lambda^{-1}(y))_p^s\,\psi_R{}^p(x)$$

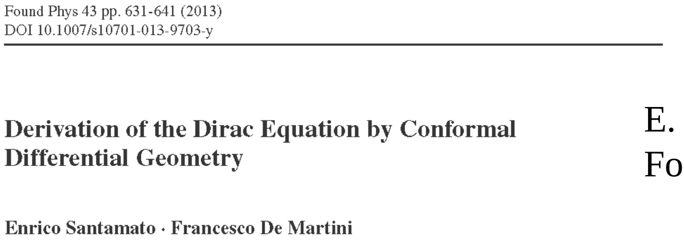
$$\Psi_D(x)=\begin{pmatrix}\psi_L^{1/2}(x)\\ \psi_L^{-1/2}(x)\\ \psi_R^{1/2}(x)\\ \psi_R^{-1/2}(x)\end{pmatrix}$$

Dirac's spinor

$$\hat{D}_+\hat{D}_-\psi_D=\hat{D}_-\hat{D}_+\psi_D=0$$

$$\hat{D}_\pm=\gamma^\mu\left(-i\hbar\partial_\mu-\frac{e}{c}A_\mu\right)\pm m$$

$$m^2=\frac{\hbar^2}{2c^2\lambda^2}\left(3+4\xi^2\right)+o\left(F_{\mu\nu}F^{\mu\nu}\right)$$



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Derivation of the Dirac Equation by Conformal Differential Geometry

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EPR paradoxes

- Because the CQG is deterministic and complete the riddle of EPR paradoxes is automatically solved

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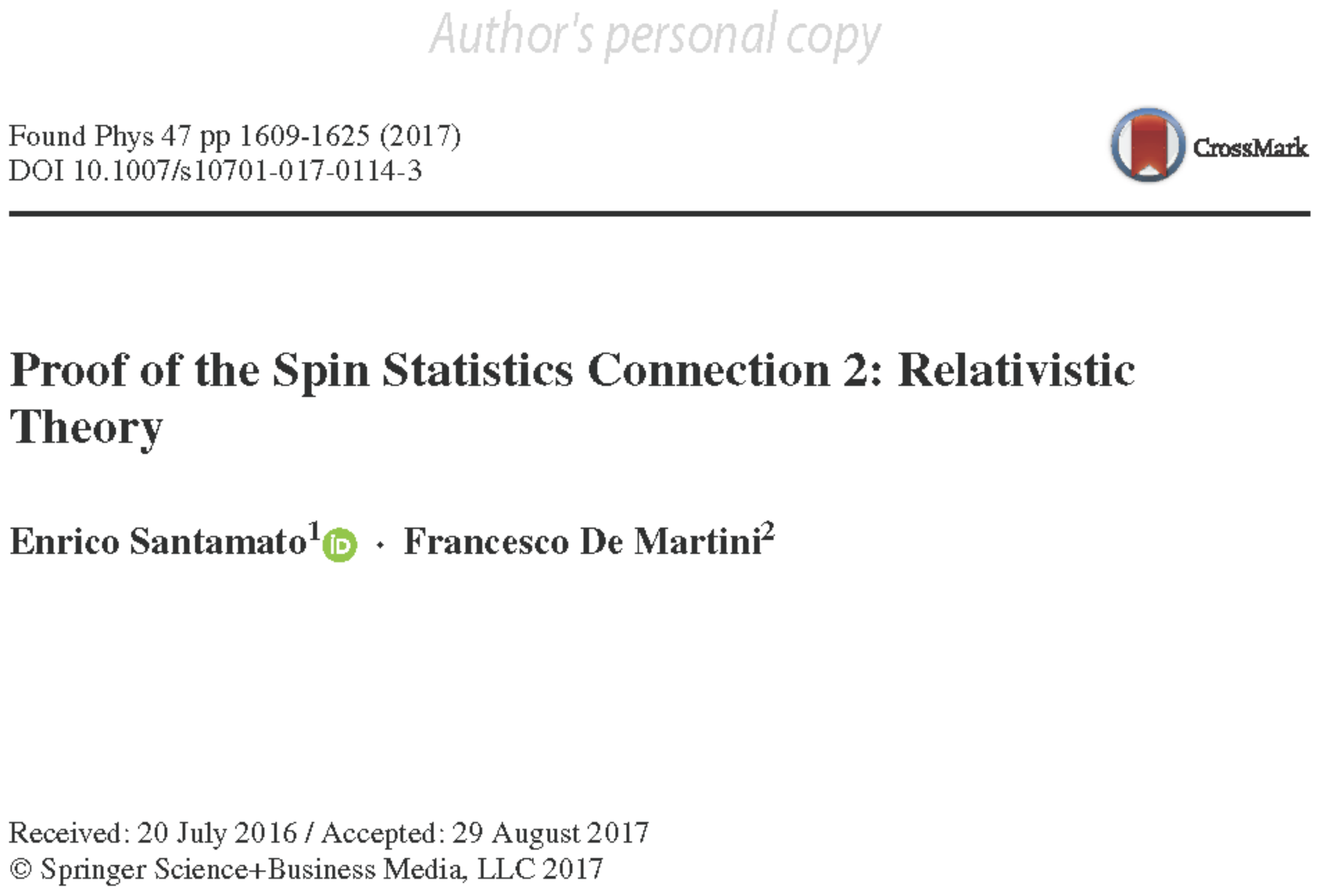
Spin statistics



Found Phys 47 pp 1609-1625 (2017)
DOI 10.1007/s10701-017-0114-3

Proof of the Spin Statistics Connection 2: Relativistic Theory

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Received: 20 July 2016 / Accepted: 29 August 2017

Spin statistics

- In SO(3,1) use Euler angles $y^\alpha = \{\alpha,\beta,\gamma,\varphi,\theta,\chi\}$.
- The H-J the continuity equations of the CQG in (4+6)-D do not contain γ (γ is ignorable)

$$\boxed{\begin{array}{l} S = \hbar s\gamma + S_0(q) \qquad s > 0 \\ \psi_s(x,y) = e^{is\gamma}\Phi(z) \end{array}}$$

s = intrinsic helicity

- The helicity s is strictly constant
- From the harmonic expansion above we see that s = 1/2 for the spin 1/2.
- For general spin, the harmonic expansion in SO(3,1) yields s integer or half-integer.

Spin statistics

Single particle

$$\psi_s(q) = \psi_s(x,y) = e^{is\gamma} \sum_{p=-s}^{s} D^{(0,s)}(B^{-1}(z))^{s}_{p} \psi_L{}^{p}(x) + D^{(s,0)}(B^{-1}(z))^{s}_{p} \psi_R{}^{p}(x)$$

$$z = (\alpha, \beta, \phi, \theta, \chi) \qquad B(z) = \mathrm{SO}(3,1) / \mathrm{SO}(2)$$

Because $s > 0$, the rotation of γ must be always counterclockwise!

Two identical particles

$$\psi_s(q_1, q_2) = e^{is(\gamma_1+\gamma_2)} \sum_{p_1, p_2=-s}^{s} D^{(0,s)}(B^{-1}(z_1))^{s}_{p_1} D^{(0,s)}(B^{-1}(z_2))^{s}_{p_2} \Phi_{p_1, p_2}(x_1, x_2) + \text{ dotted terms}$$

Spin statistics

- The H-J and continuity equations require

$$\boxed{S(q_1,q_2)=S(q_2,q_1)}$$

- Because $\rho(q_1,q_2) = \rho(q_2,q_1)$ we have also

$$\boxed{\psi_s(q_1,q_2)=\psi_s(q_2,q_1)}$$

- Because the rotation of γ_1 and γ_2 must be counterclockwise, the exchange of γ_1 and γ_2 introduces a factor $(-1)^{2s}$ in front of ψ_s
- The exchange of the other coordinates introduces no factor

Spin statistics

$$\psi_s(q_2,q_1)=(-1)^{2s}\left[e^{is(\gamma_1+\gamma_2)}\sum_{p_1,p_2=-s}^{s}D^{(0,s)}(B^{-1}(z_2))^s_{p_1}D^{(0,s)}(B^{-1}(z_1))^s_{p_2}\Phi_{p_1,p_2}(x_2,x_1)+\text{ dotted terms}\right]=$$

$$=(-1)^{2s}\left[e^{is(\gamma_1+\gamma_2)}\sum_{p_1,p_2=-s}^{s}D^{(0,s)}(B^{-1}(z_2))^s_{p_2}D^{(0,s)}(B^{-1}(z_1))^s_{p1}\Phi_{p_2,p_1}(x_2,x_1)+\text{ dotted terms}\right]$$

Comparison with

$$\psi_s(q_1,q_2)=e^{is(\gamma_1+\gamma_2)}\sum_{p_1,p_2=-s}^{s}D^{(0,s)}(B^{-1}(z_1))^s_{p_1}D^{(0,s)}(B^{-1}(z_2))^s_{p_2}\Phi_{p_1,p_2}(x_1,x_2)+\text{ dotted terms}$$

yields

$$\Phi_{p_1,p_2}(x_1,x_2)=(-1)^{2s}\Phi_{p_2,p_1}(x_2,x_1)$$

EOP

Perspectives

- Extend the space to SU(2)×U(1)×SO(3,1) to include Weinberg's quantum model of leptons
- Introduce gauge Yang-Mills fields
- Introduce Higgs field to provide masses
- Apply Weyl's conformal symmetry to quantum field theory
- Study gravitational and cosmological implications

Conclusions

- Weyl's geometry and symmetry shed new light on Quantum Mechanics
- The statistical Born's interpretation of $|\psi|^2$ has no objective physical meaning and should be replaced by a geometrical interpretation
- When the CQG is applied to a particle with spin, a new conserved quantity appears: the particle "helicity"
- EPR and other QM paradoxes are automatically solved, because the CQG is deterministic and complete
- The spin-statistic connection can be obtained for any spin exploiting symmetry consideration only