Some properties of two interacting Sine-Gordon BPS fields in (1+1) dimensions

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[1]based on work carried out with P.Klimas (Florianopolis), L.A. Ferreira (Sao Carlos) and A. Wereszczynski (Krakow)

Introduction

- Two interesting concepts in field theories
  - Integrability
  - BPS properties.
- More has been done on Integrabilities etc.
- But BPS properties are quite interesting too and have important implications.
- Here we will look at some properties and implications of the BPS conditions.

What are the BPS conditions?

- Consider static solutions of a (1+1) field theory with the energy density given by

$$\mathcal{E} = \frac{1}{2}\,\partial_x\phi\,\partial_x\phi + V(\phi);$$

- Rewrite this as (Bogomolny' trick)

$$\mathcal{E} = \frac{1}{2}\left(\partial_x\phi - \sqrt{2V(\phi)}\right)^2 + \frac{1}{\sqrt{2}}\sqrt{V}\partial_x\phi$$

- The last term is clearly $\frac{\partial U}{\partial x}$, for some U

So for the fields which satisfy BPS condition *i.e.*

$$\partial_x \phi = \sqrt{2V(\phi)}$$

- Energy $\int E\, dx \; = \; \int dx \frac{\partial U}{\partial x} \; = \; U(\phi(+\infty)) - U(\phi(-\infty))$.
- So if $U(\phi(\pm\infty))$ are finite
  - the energy is finite
  - this energy is a minimum for the fields which satisfy the BPS condition

Many questions

- How does this generalise to more fields and to higher dimensions?
- Do we just complete the squares, and how do we do this in general?
- What are the properties of such fields (in particular, the condition of finiteness of $U(\phi(\pm\infty)))$?
- What are the implications for the scattering properties of such fields?
- In this talk we will look at the question of more fields in (1+1) dimensions.
- All complications are already visible for 2 fields - so we will really look only at the case of 2 fields.

General case of 2 fields for (1+1) dimensions.

- General discussion (for many fields in (1+1) and higher dimensions) already given in two papers
  - C. Adam, L. A. Ferreira, E. da Hora, A. Wereszczynski and W. J. Zakrzewski, "Some aspects of self-duality and generalised BPS theories," JHEP **1308**, 062 (2013) [arXiv:1305.7239 [hep-th]].
  - C. Adam and F. Santamaria, "The First-Order Euler-Lagrange equations and some of their uses," [arXiv:1609.02154 [hep-th]].
- In this talk we follow the procedure of the first paper.

- Consider a scalar field theory in $(1+1)$ dimensions defined by the Lagrangian $(c=1)$

$$\mathcal{L} = \frac{1}{2}\,\eta_{ab}\,\partial_\mu\varphi_a\,\partial^\mu\varphi_b - V(\varphi_a); \quad a = 1, \ldots N.$$

- The corresponding Euler-Lagrange eqs. are

$$\eta_{ab}\,\partial^2\varphi_b + \frac{\partial\, V}{\partial\,\varphi_a} = 0; \quad (\partial^2 \equiv \partial_t^2 - \partial_x^2)$$

- The static energy is given by

$$E = \int_{-\infty}^{\infty} dx\,\left(\frac{1}{2}\,\eta_{ab}\,\partial_x\varphi_a\,\partial_x\varphi_b + V(\varphi_a)\right).$$

- The static eqs. of motion are given by:

$$\eta_{ab}\,\partial_x^2\varphi_b = \frac{\partial\, V}{\partial\,\varphi_a}.$$

Two Fields.

- Consider

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu\varphi_1)^2 + (\partial_\mu\varphi_2)^2 - \lambda\partial_\mu\varphi_1\,\partial^\mu\varphi_2\right) - V(\varphi_1,\varphi_2),$$

- where $V(\varphi_1,\varphi_2)$ is given by:

$$V \;=\; \frac{2}{4-\lambda^2}\left[M^2 + N^2 + \lambda MN\right]; \quad |\lambda| < 2.$$

- Choose $M = M(\varphi_1)$, $N = N(\varphi_2)$.
- Note that when $\lambda = 0$ we have two real independent real fields, and when $\lambda \neq 0$ this Lagrangian corresponds to η_{ab}

$$\eta = \begin{pmatrix} 1 & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & 1 \end{pmatrix}, \qquad \eta^{-1} = \frac{4}{4-\lambda^2}\begin{pmatrix} 1 & \frac{\lambda}{2} \\ \frac{\lambda}{2} & 1 \end{pmatrix},$$

- Moreover, the potential V

$$V \;=\; 2\eta^{-1}_{ab}G_aG_b, \quad \text{where} \quad G_1 = M, \quad G_2 = N.$$

Another way to think about this.
Consider $\mathcal{L}'$

$$\mathcal{L}' = \frac{1}{2}\left((\partial_\mu\varphi_1 - F_1)^2 + (\partial_\mu\varphi_2 - F_2)^2 - \lambda(\partial_\mu\varphi_1 - F_1)\left(\partial^\mu\varphi_2 - F_2\right)\right)$$

$$= \mathcal{L} - \partial_\mu\varphi_1(F_1 - \frac{\lambda}{2}F_2) - \partial_\mu\varphi_2(F_2 - \frac{\lambda}{2}F_1).$$

For topology we take

$$F_1 - \frac{\lambda}{2}F_2 = M(\varphi_1) = \partial_{\varphi_1}U, \quad F_2 - \frac{\lambda}{2}F_1 = N(\varphi_2) = \partial_{\varphi_2}U.$$

Then with V given by

$$V = \frac{1}{2}\left(F_1^2 + F_2^2 - \lambda F_1 F_2\right) = \frac{2}{4-\lambda^2}\left[M^2 + N^2 + \lambda MN\right]$$

the fields (for $|\lambda| < 2$) satisfy the BPS conditions.

The Sine-Gordon case

Take $M = 4\sin(\varphi_1)$ and $N = 4\epsilon\sin(\varphi_2)$

- The pre-potential then becomes

$$U = -4\big(\cos(\varphi_1) + \epsilon\cos(\varphi_2)\big)$$

- The energy

$$E = -4\left(\cos(\varphi_1) + \epsilon\cos(\varphi_2)\right)_{-\infty}^{\infty}.$$

- Choose $\epsilon = 1$ and boundary conditions (for kink in each field) $\varphi_i(x = -\infty) = 0$; $\varphi_i(x = \infty) = \pi$,
- The energy then becomes $E = 16$.
- The same is true if $\epsilon = -1$ and we take boundary conditions of the kink-antikink system.

BPS conditions for two Sine Gordon fields.

- First we looked at the solitons of the model involving 2 Sine Gordon fields.
- Have to solve the equations:

$$\partial_x \varphi_1 = \frac{1}{4-\lambda^2}\Big(2\lambda\epsilon\sin(\varphi_2) + 4\sin(\varphi_1)\Big) = F_1,$$

$$\partial_x \varphi_2 = \frac{1}{4-\lambda^2}\Big(2\lambda\sin(\varphi_1) + 4\epsilon\sin(\varphi_2)\Big) = F_2,$$

- Equations are hard to solve. Need to solve them numerically.
- We also introduced a modified gradient flow of the prepotential U

$$\vec{\nabla} U = \frac{4}{4-\lambda^2}\Big(2\frac{\partial U}{\partial \varphi_1}+\lambda\frac{\partial U}{\partial \varphi_2}, \lambda\frac{\partial U}{\partial \varphi_1}+2\frac{\partial U}{\partial \varphi_2}\Big) = (F_1, F_2).$$

- This gradient flow will play an important role in our discussion.

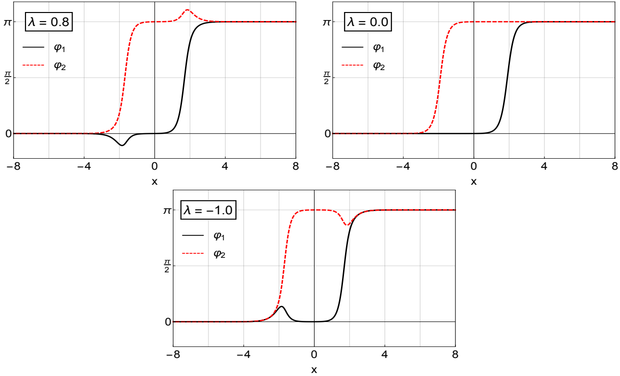
Numerical support.

- First we solved the BPS equations. As the equations are first order, their solutions depend on the values of φ_1 and φ_2 fields at one specific value of x, which in our simulations was always $x = 0$.
- Of course, the BPS equations do not 'know about the topology' they are just responsible for the evolution to the 'nearest' vacuum. Hence, need to choose the values of φ_1 and φ_2 at $x = 0$ well.
- We solved them by first moving forward in x and then going backwards in x and then joining them together. We performed many such simulations, varying both the simulation step dx and how far we were moving in x. For small values of dx ($dx < 0.00001$) the results were essentially the same.
- In the plots, that we include here, we present the results obtained for $dx = 0.000002$.

- Then we used the obtained BPS fields as initial conditions for longer simulations of the full equations of motion corresponding to our cases. The time evolution was simulated by the 4th order Runge Kutta method.
- Our simulations were performed with absorbing boundary conditions but, in fact, the variations of the fields at the boundaries were so small so the absorption was infinitesimal.
- Of course, as our BPS fields were static solutions, we did not expect any evolution and ideed this was essentially confirmed by our simulations. In fact, we did not see any **significant** changes of the fields.
- Thus our simulations confirmed that the BPS solutions were indeed the static solutions of the full field equations and that these solutions were stable.

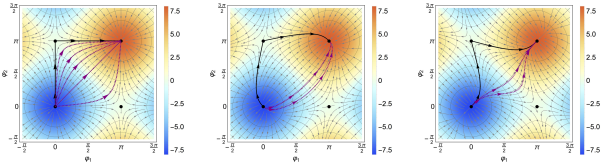
First results for two kinks

- Here we present plots of fields φ_1 and φ_2 for 3 values of λ, namely 0.8, 0.0 and -1.0.



- Note 'bumps' and their reversal for different signs of λ.
- The height of the bumps depends on λ (vanishes for $\lambda = 0$).

- Note that the bumps are very well explained by U $(\lambda = 0, +0.8 \text{ and } -1.0)$



- This explanation is based on the observation that

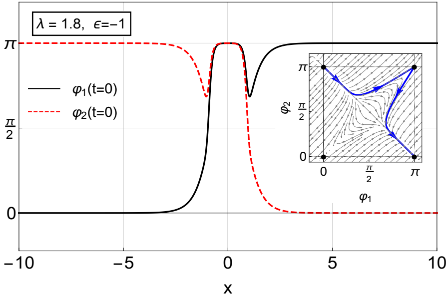
$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \varphi_1} F_1 + \frac{\partial U}{\partial \varphi_1} F_2$$

$$= \frac{4}{4-\lambda^2}\left[\left(\frac{\partial U}{\partial \varphi_1}\right)^2 + \lambda \frac{\partial U}{\partial \varphi_1}\frac{\partial U}{\partial \varphi_2} + \left(\frac{\partial U}{\partial \varphi_2}\right)^2\right] = 2V$$

- For $|\lambda| < 2$ the expression above is strictly positive and as x changes U changes too.
- Thus the curves in the plots coincide with the flow: $F_a \equiv +(\eta^{-1})_{ab}\frac{\partial U}{\partial \varphi_b}$.

Kink-antikink system

- We have looked also at the kink-antikink systems.
- Here we present the plots of the system consisting of a kink (φ_2) and an antikink (φ_2) for $\lambda = 1.8$.



- The subfigure shows the BPS curve in the space of fields.

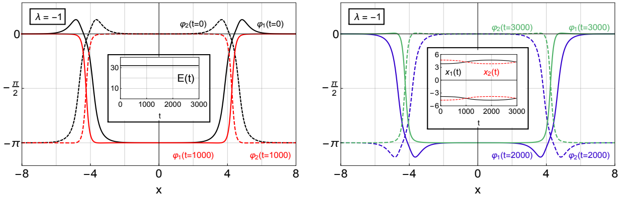
General comments

- Results are clearly very interesting; we have two kinks (in two different fields, which interact through λ) but which do not move. The same is true for a system of a kink and antikink.
- The Flow $\vec{F} = \eta^{-1}\nabla_{\vec{\varphi}} U$ is key here. As system evolves (in x) it follows the flow lines.
- Are these structures really stable? What would happen if we use the static solutions of the BPS equations as initial conditions for full simulations?
- What would happen if we send them (the structures) towards each other?

Some comments

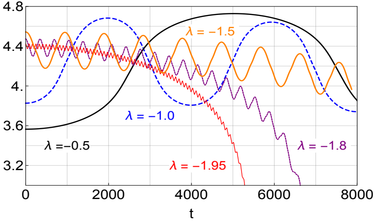
- We have performed many simulations of various system.
- We sent solitons towards each other with various velocities and found many interesting results.
- They showed many interesting properties: See - **https://arxiv.org/pdf/1803.08985**.
- We have also constructed multisoliton initial conditions (these are **NOT** solutions) and studied the interactions of the solitons (work in preparation). No time to discuss here.
- Interesting results were found for 2 static breathers in each field.

Two breathers



- Plots of the fields at various values of time, obtained in a simulation for $\lambda = -1.0$. The inserts show the total energy and the trajectories of the solitons.
- Overall all solitons are approaching the centre (*i.e. overall attraction*)

The dependence on λ



- Plots of trajectories $x_1(t)$, obtained for different values of $\lambda < 0$.
- Note that we have oscillation for negative λ of each pair of kink and anti-kink.

Extra comments

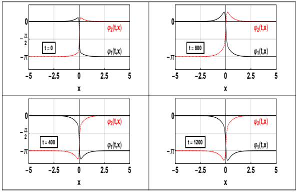
- These oscillations are very interesting.
- Can show that for $\lambda > 0$ the solitons gradually move away from each other.
- For two kinks - it is other way round (as a function of λ).
- Explanation: extra minimal energy gets converted to change the positions of the solitons.
- This is a general phenomenon.
- In fact all ideas show that the oscillations are unrelated to extra solitons.

Oscillations and BPS conditions

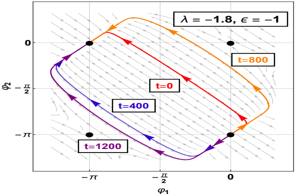
- So consider a system of one kink in each field, or a kink and an antikink in each field, respoectively.
- Such a system is provided by a solution of our BPS equations, which we obtained numerically.
- Moreover, such a system has zero modes (as we can choose the values of each field at one point, say $x = 0$, arbitrarily)
- But any system studied numerically has small (or extremely small) numerical errors, so our system has them too.
- 'Normal' numerical errors will preserve overall momentum (as system is at rest), so for a one field soliton nothing much can happen. This effectively *eliminates* one zero mode.
- For a two (or more) field systems the solitons in each field can move towards each other or away from each - while preserving overall momentum. And this is what we see in our system.

BPS solutions revisited

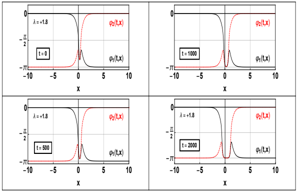
- So we have looked at the BPS solution of one kink and one antikink for various values of λ.
- For $\lambda > 0$ we see **extremely** slow repulsion and for $\lambda < 0$ **extremely** slow attraction.
- First attraction $(\lambda = -1.8)$

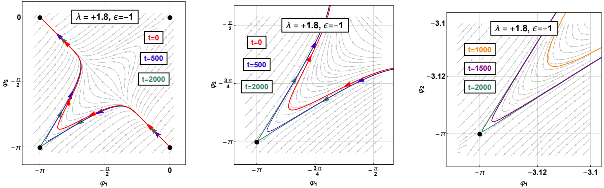


- The same in the plot of movement of the fields (*i.e.* using the flow field.



- Next we looked for positive λ, $(\lambda = 1.8)$
- This time we see repulsion.



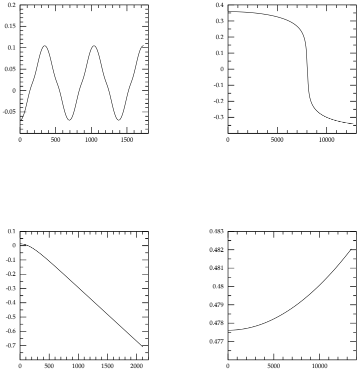


Extra comments

- Note that these oscillations are very slow.
- They are really induced by the small numerical errors.
- The energy is **extremely** well conserved.
- The same phenomena are also true for the systems involving two kinks.
- The only difference that for two kinks the evolution is much slower.
- In both cases as $|\lambda|$ gets closer to 2 the evolution gets faster.

Comparison

- Here we compare the kink/kink system for $|\lambda| = 1.95$ with kink/antikink one at for $|\lambda| = 1.8$



Top two plots are attraction cases, lower ones are repulsion ones.

Summary and some Conclusions

- Have made some progress in understanding the BPS conditions.
- In simple cases, nothing very spectacular and we get **almost** stable soliton solutions.
- When one considers more complicated fields consisting of 'relatively' isolated solitons - the system behaves like a system of Sine Gordon fields.
- However, the interactions **depend on** λ and for larger values of this parameter - behaviour changes.
- We have spotted a **numerically induced** small effect which allows the ostensibly **static** solutions to move (due to more fields).
- Most of our work is still somewhat preliminary and so far only in $(1+1)$ dimensions.
- A lot still remains to be done

Thank you for listening.