



BLTP, JINR

Interaction of topological solitons mediated by fermions

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Thanks to my collaborators:
I. Perapechka and N. Sawado

JHEP 1810 (2018) 081
Phys. Rev. D 99 (2019) 125001

Topological Solitons,
Nonperturbative Gauge Dynamics and Confinement II
20 July 2019, Pisa

Outline

- Warming up: Fermions localized by kinks in 1+1 dim
- Fermions localized by baby Skyrmions in 2+1 dim
- Backreaction of the fermions
- Fermions coupled to chiral planar Skyrmions with DM interaction term
- Fermion exchange interaction between magnetic Skyrmions
- Summary and outlook

Fermions localization on the kink in 1+1 dim

$$L = \frac{1}{2} (\partial_\mu \phi)^2 + i\bar{\psi} \gamma^\mu \partial_\mu \psi + g \phi \bar{\psi} \psi - \frac{1}{2} (\phi^2 - 1)^2$$

R.Jackiw and C.Rebbi
Phys. Rev. D13 3398 (1976)

• Field equations:

$$i\gamma^\mu \partial_\mu \psi = g\phi\psi ; \quad \partial_\mu \partial^\mu \phi = 2\phi(1 - \phi^2) - g\bar{\psi}\psi$$

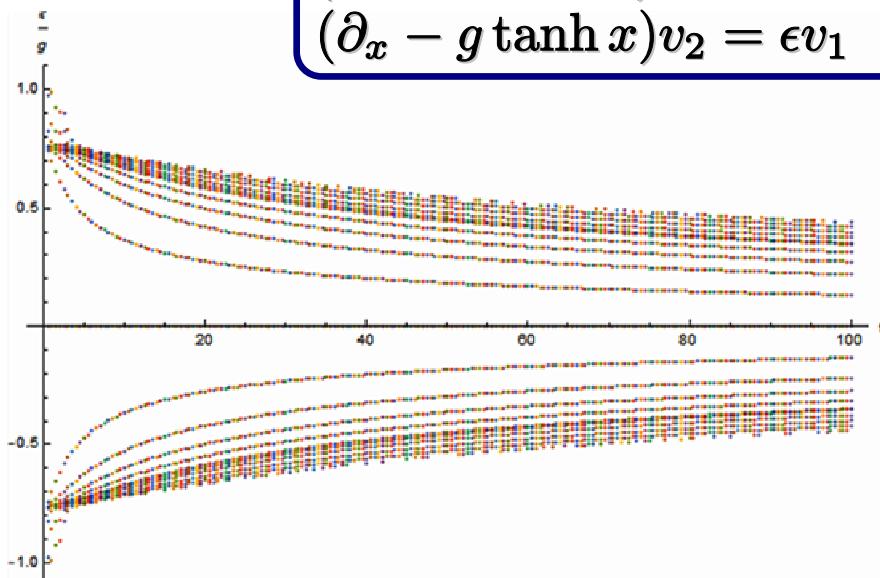
Fixed background ($g \ll 1$):

$$\psi = e^{-i\epsilon t} \begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 \end{pmatrix} \quad \int dx |\bar{\psi}\psi| = 1$$

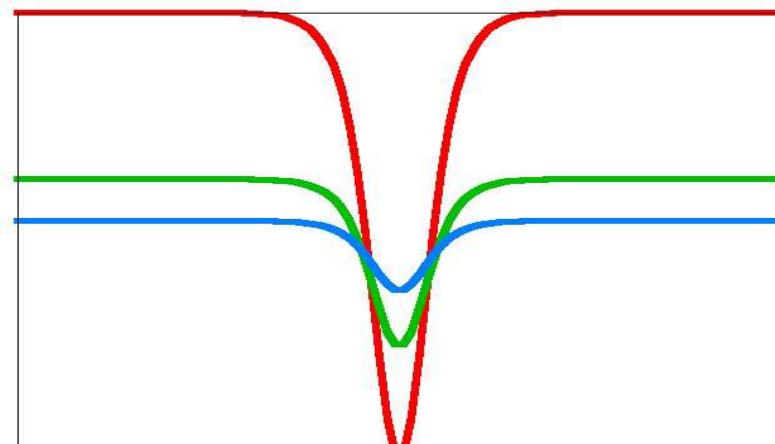
$$\phi_K = \tanh x$$

$$|\epsilon| \leq g$$

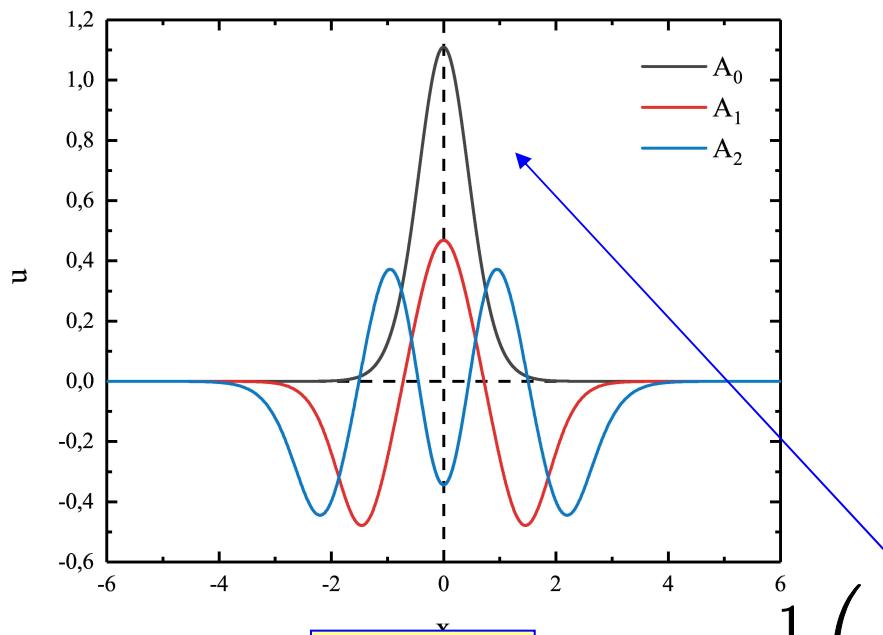
$$\begin{aligned} (\partial_x + g \tanh x)v_1 &= -\epsilon v_2 \\ (\partial_x - g \tanh x)v_2 &= \epsilon v_1 \end{aligned}$$



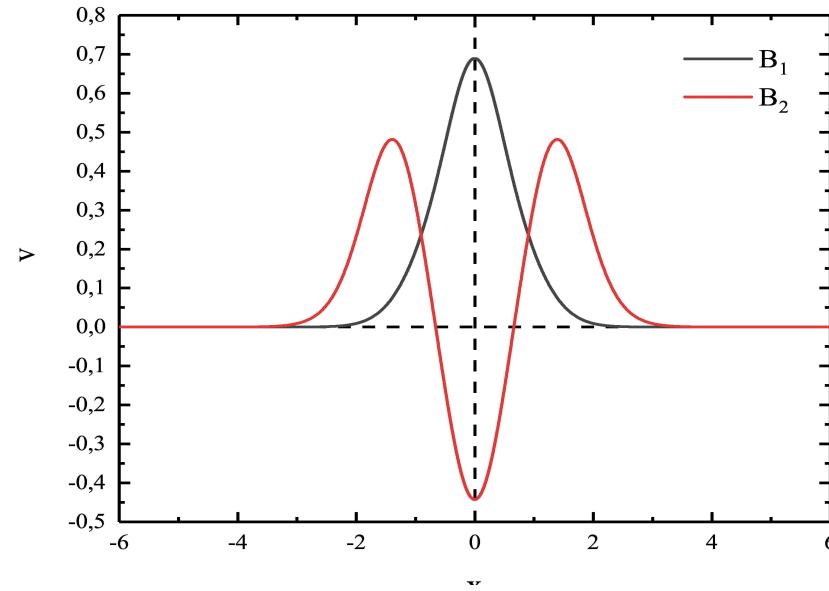
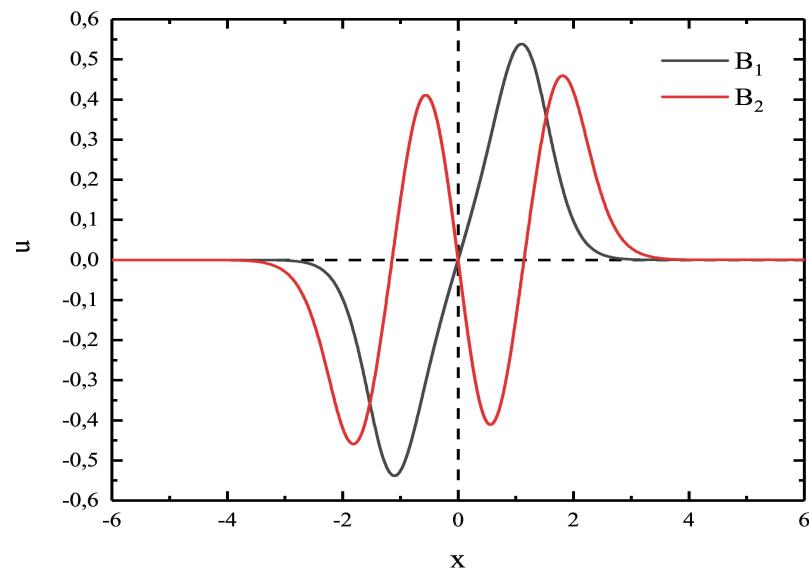
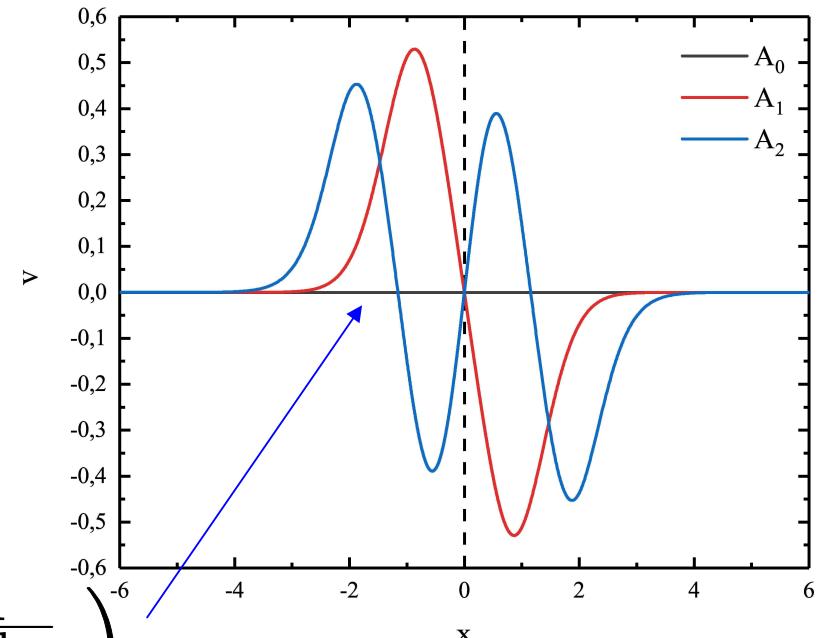
$$\begin{aligned} (-\partial_x^2 + U_\pm(x)) v_{1,2} &= \epsilon^2 v_{1,2} \\ U_\pm(x) &= g^2 - g(g \pm 1) \operatorname{sech}^2 x \end{aligned}$$



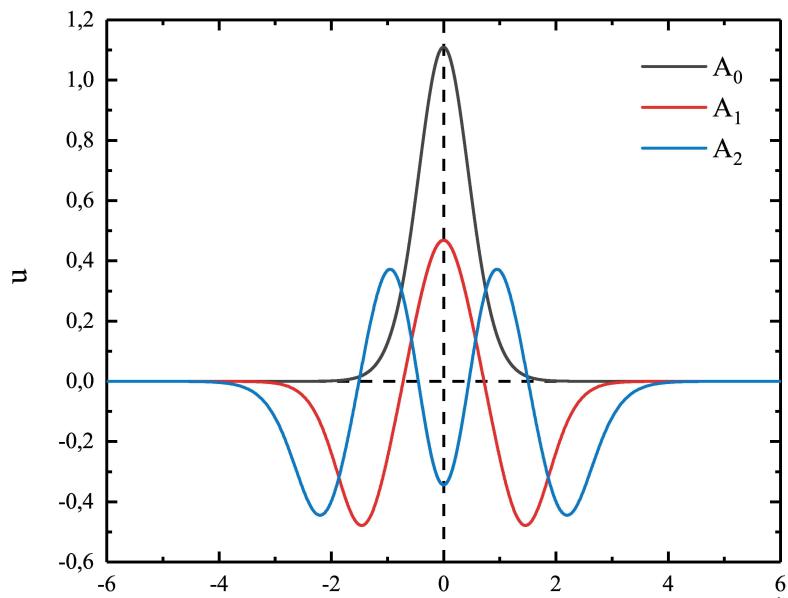
Fermionic modes



$$\psi_0 = \frac{1}{2} \begin{pmatrix} \frac{1}{\cosh x} \\ 0 \end{pmatrix}$$

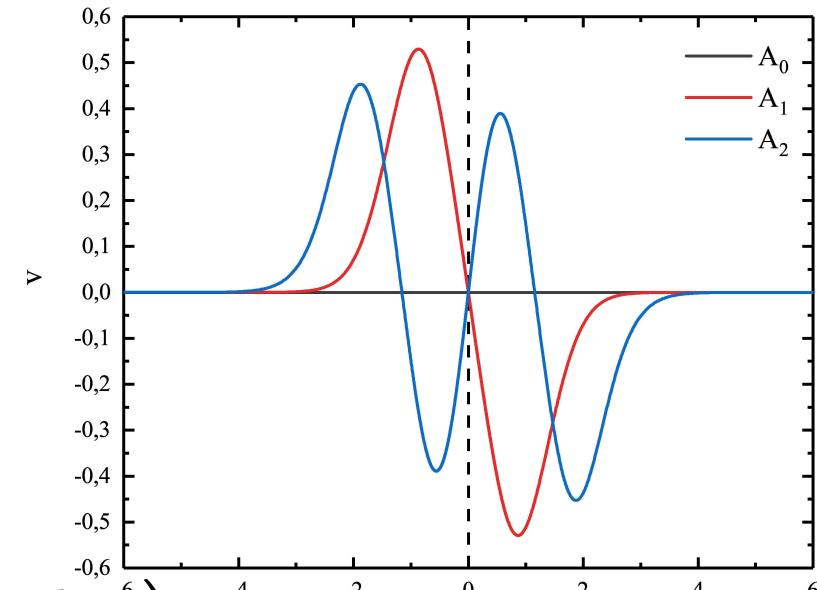
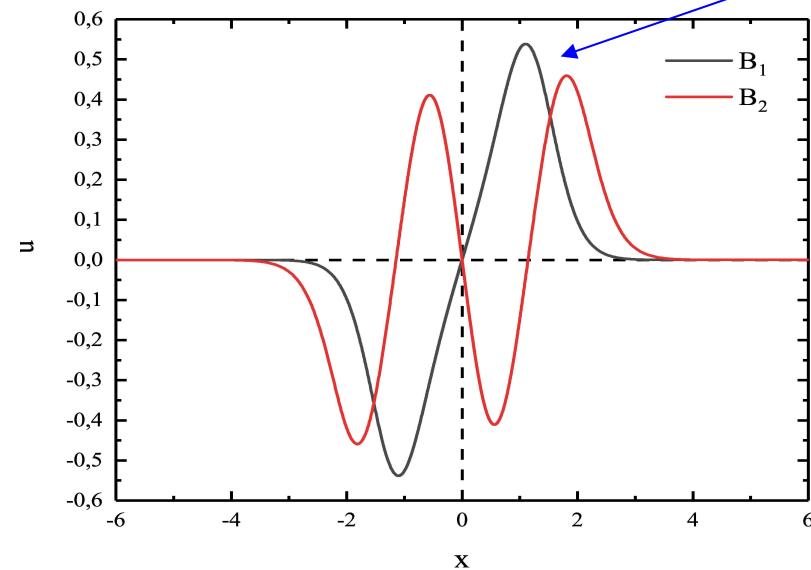


Fermionic modes

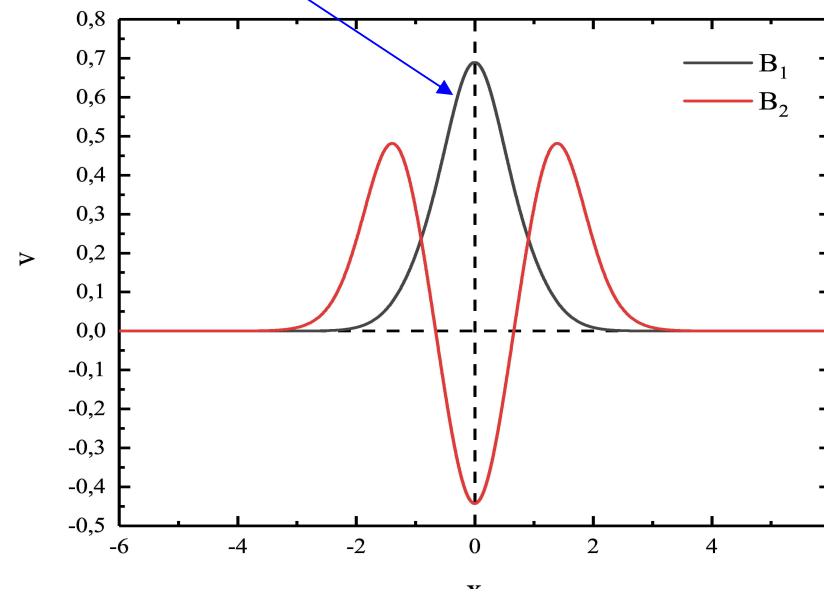


B₁ mode

$$\psi_1 = \frac{1}{2} \left(\begin{array}{c} \sqrt{3} \tanh x \\ \cosh x \\ \hline 1 \\ \cosh^2 x \end{array} \right)$$



$$g = 2, \quad {}^x\varepsilon = \pm \sqrt{3}/2$$



● **N=1 SUSY kink**

$$L = \frac{1}{2} (\partial_\mu \phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi + g\phi\bar{\psi}\psi - \frac{1}{2} (\phi^2 - 1)^2$$



$$L_{N=1} = (\partial_\mu \phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi + F^2 + 2FW - W'\bar{\psi}\psi$$

F - auxiliary field: $F = -W$



$$L_{N=1} = (\partial_\mu \phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi - W'\bar{\psi}\psi - W^2$$

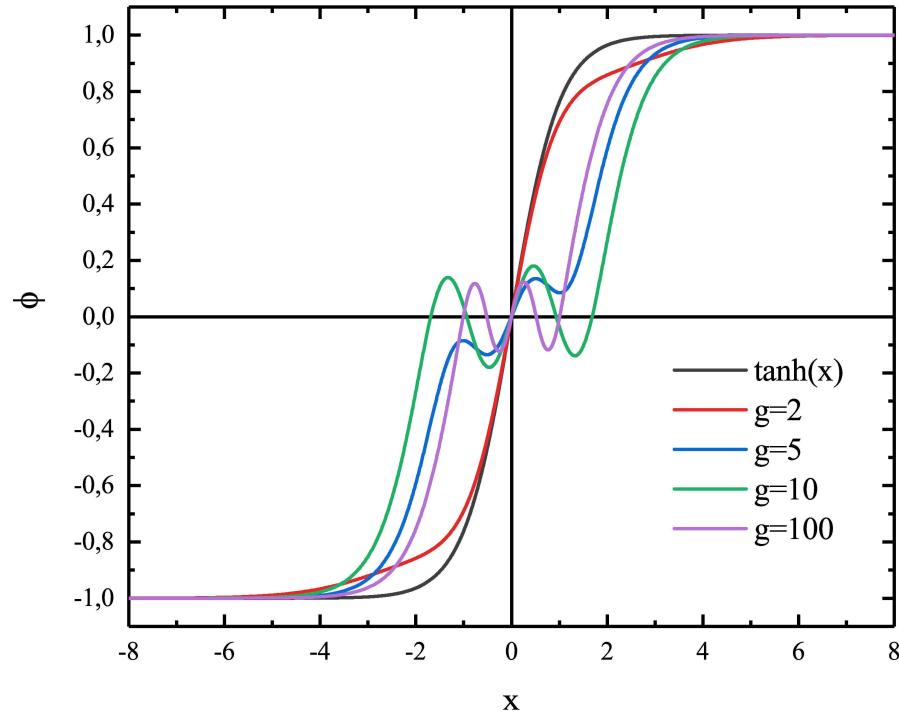
$$W[\phi] = \frac{1}{\sqrt{2}} (\phi^2 - 1)$$

● **SUSY transformations:** $\delta\phi = \eta\psi; \quad \delta\psi = \eta(\gamma^\mu \partial_\mu \phi - W)$

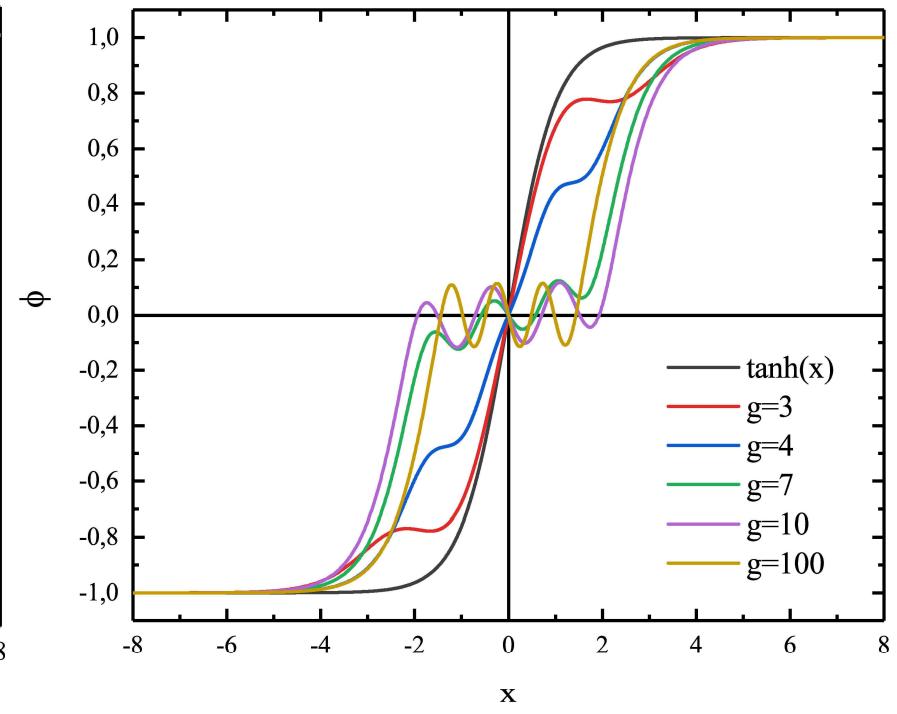
Fermionic zero mode of the kink:
Grassmann-valued deformation of the bosonic field

$$\psi_0 = \frac{1}{2} \begin{pmatrix} \frac{1}{\cosh x} \\ 0 \end{pmatrix}$$

Backreaction of the fermions



Kink + A_1 mode



Kink + B_1 mode

Scale symmetry of the massless model: $\varepsilon_0 = 0$ for any values of the coupling g

Baby Skyrme model

(J. Verbaarschot (1986), L.Bogolubskaya & I Bogolubsky (1989)
 R.A. Leese et al (1990)

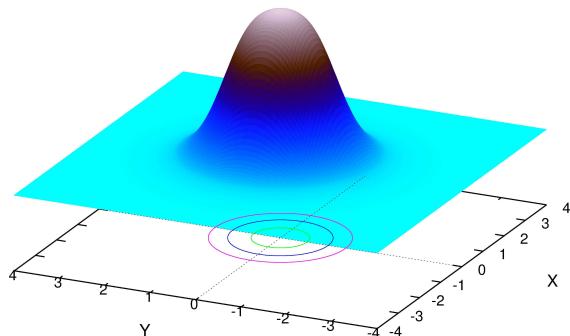
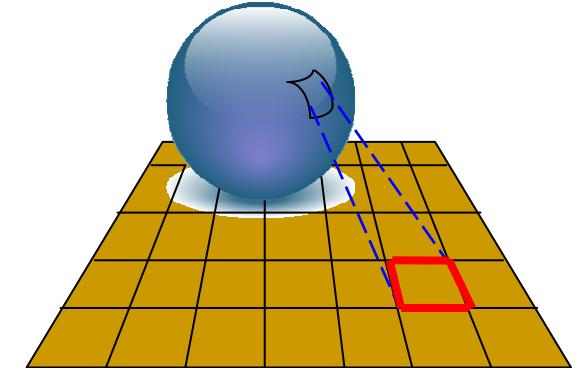
$$\phi = (\phi^1, \phi^2, \phi^3); \quad \phi^a \cdot \phi^a = 1; \quad \phi : S^2 \rightarrow S^2$$

$$Q = \frac{1}{4\pi} \int d^2x \ \epsilon_{abc} \epsilon_{ij} \phi^a \partial_i \phi^b \partial_j \phi^c = 1$$

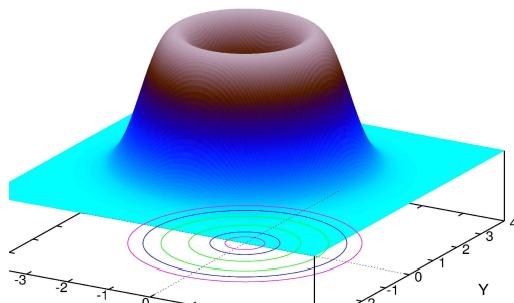
Derrick's scaling theorem: Skyrme term provides a scale but cannot stabilise the soliton: potential term is necessary

$$\mathcal{L}_{Sk} = \frac{\kappa_2}{2} \left(\partial_\mu \vec{\phi} \right)^2 - \frac{\kappa_4}{4} \left(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi} \right)^2 - \kappa_0 U(\phi)$$

$$E \geq \pm 4\pi Q \quad \text{equality is possible if } \kappa = 4 \text{ and } \kappa_0 = 0$$



Q=1



Q=2

● **Symmetric ansatz:**

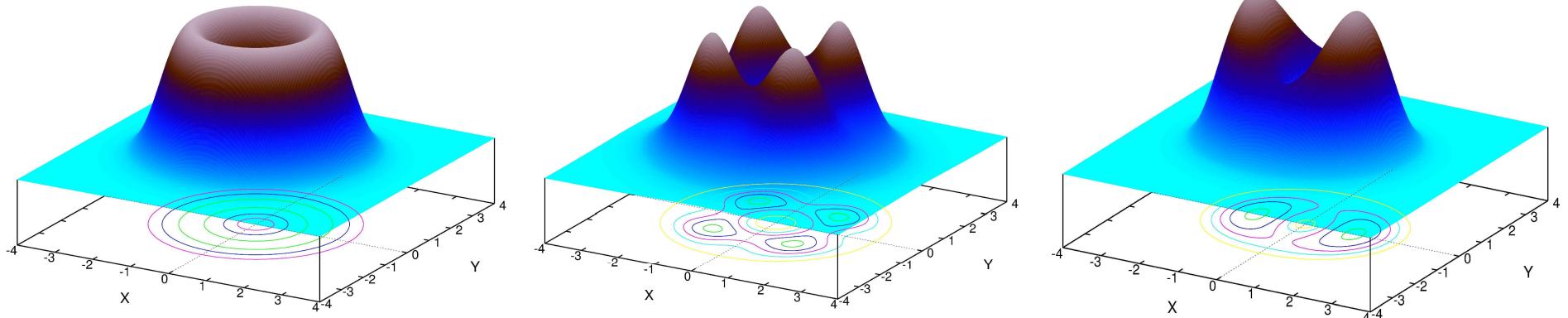
$$\begin{aligned} \phi^1 &= \sin f(r) \cos(Q\varphi - \delta); \\ \phi^2 &= \sin f(r) \sin(Q\varphi - \delta); \\ \phi^3 &= \cos f(r) \end{aligned}$$

Baby Skyrme model

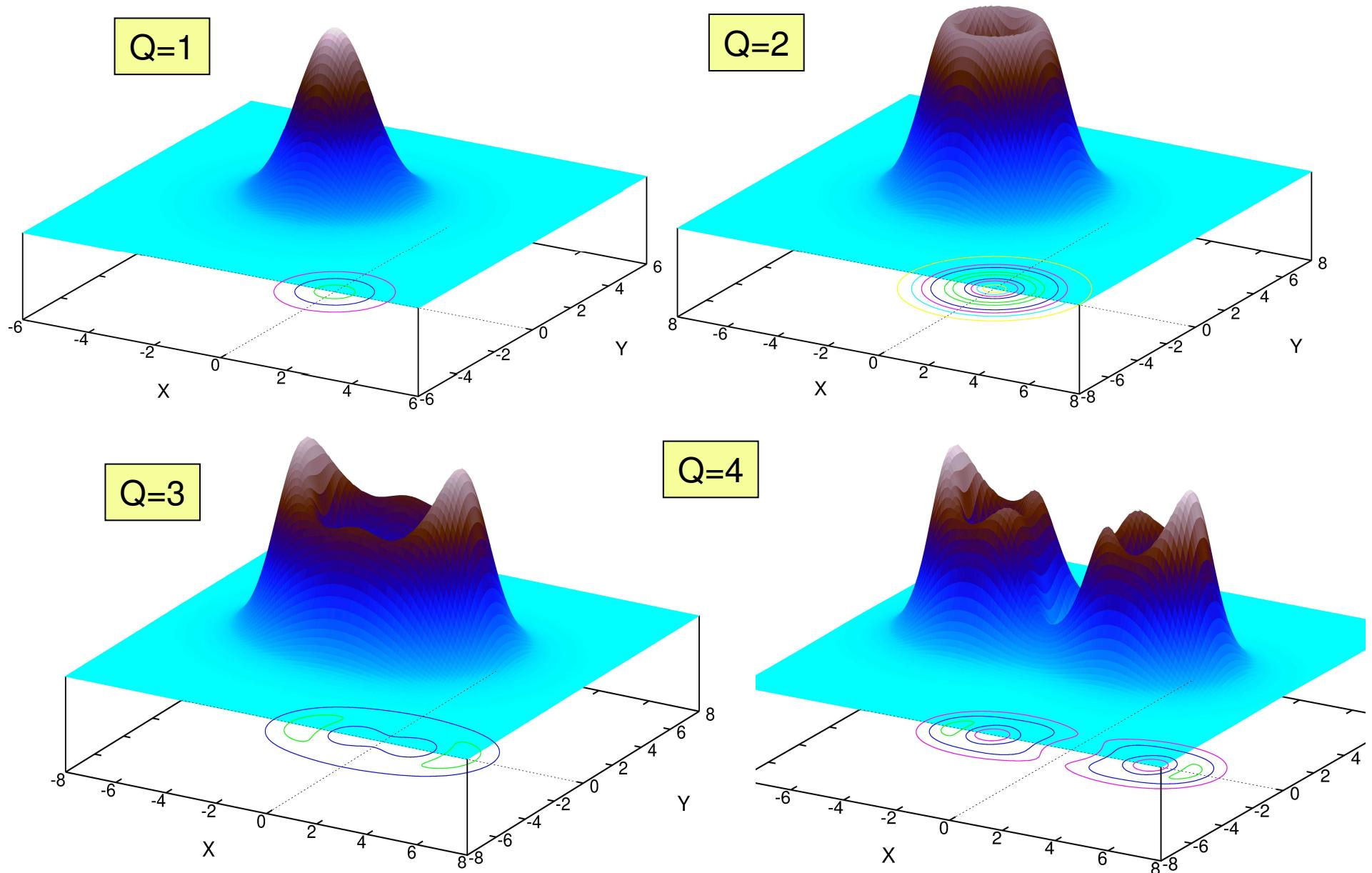
Potential of the baby Skyrme model: potential term $U(\Phi)$ may be chosen almost arbitrarily, however must vanish at infinity for a given vacuum field value in order to ensure existence of the finite energy solutions: $\phi_{(0)}^a = (0, 0, 1)$

Several potential terms have been studied in great detail:

- “Old” model, with $U(\phi) = m^2(1 - \phi_3)$  **weak attraction**
- Holomorphic model, with $U(\phi) = m^2(1 - \phi_3)^4$  **weak repulsion**
- “Double vacuum” model, with $U(\phi) = m^2(1 - \phi_3^2)$  **strong attraction**



Baby Skyrme model: solitons



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Durham University bags £7m to explore 'magnetic skyrmion' storage

Quantum mechanics could dramatically improve data storage capacities

Graeme Bell @graemebe

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Science

Break out the Elder scrolls characters seek storage

Durham Uni-based adventure efforts

By Chris Mellor 5 Aug 2016 at 14:51

Aug 9, 2013

Skyrmion spin control could revolutionise electronics

Researchers at the University of Durham have succeeded in controlling tiny magnetic whirlpools, known as "skyrmions" for the first time. This is important for future high-density and nanodigital electronic devices, as it can transfer speeds and processing power.

- Nacre-like graphene composite is stronger and tougher
- Thermoresponsive polymer helps graphene fold into 3D shapes
- Light polarization modulated rapidly by gold nanorods
- Scanning tunnelling microscope creates all-graphene p-n junctions
- Quantum Čerenkov effect

Durham University News

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Research

Nanosize magnetic whirlpools could be the future of data storage (2 August 2016)

The use of nanoscale magnetic whirlpools, known as magnetic skyrmions, to create novel and efficient ways to store data will be explored in a new £7M [research programme](#) led by Durham University.

Skyrmions, which are a new quantum mechanical state of matter, could be used to make our day-to-day gadgets, such as mobile phones and laptops, much smaller and cheaper whilst using less energy and generating less heat.

It is hoped better and more in-depth knowledge of skyrmions could address society's ever-increasing demands for processing and storing large amounts of data and improve current hard drive technology.

Revolutionise data storage

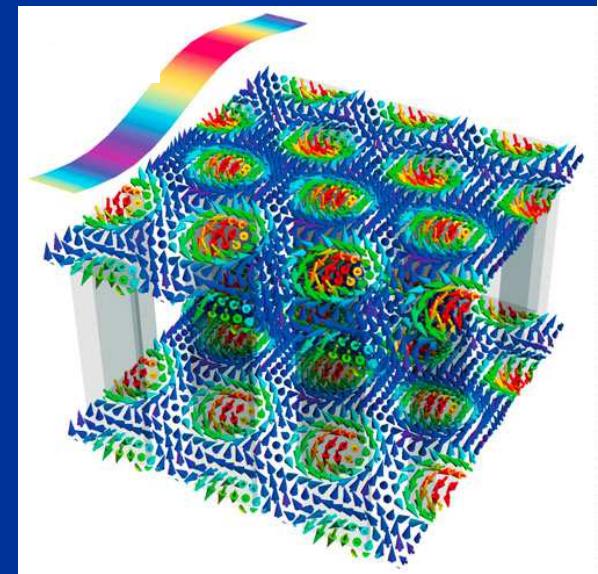
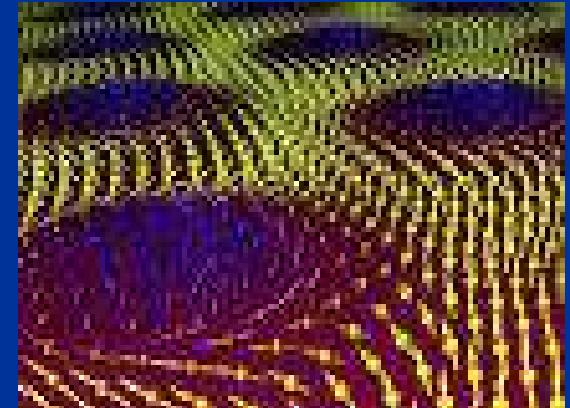
Scientists first predicted the existence of skyrmions in 1962 but they were only discovered experimentally in magnetic materials in 2009.

The UK team, funded by the [Engineering and Physical Sciences Research Council](#) (EPSRC), now aims to make a step change in our understanding of skyrmions with the goal of producing a new type of demonstrator device in partnership with industry.

Skyrmions, tiny swirling patterns in magnetic fields, can be created, manipulated and controlled in certain magnetic materials. Inside a skyrmion, magnetic moments point in different directions in a self-organised vortex.

Baby Skyrme model: Applications

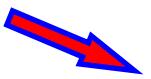
- A Heisenberg-type model of interacting spins
- A model of the topological quantum Hall effect
- Elementary excitations in quantum Hall magnets
- Soft matter (Liquid crystals)
- Models of condensed matter systems with intrinsic and induced chirality
- Chiral magnetic structures with DM interaction
- Applications in future development of data storage technologies?



Fermion-Skyrmion system in 2+1 dim

$$\mathcal{L} = \mathcal{L}_{Sk} + \mathcal{L}_f$$

$$\mathcal{L}_{Sk} = \frac{\kappa_2}{2} \left(\partial_\mu \vec{\phi} \right)^2 - \frac{\kappa_4}{4} \left(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi} \right)^2 - \kappa_0 U(\phi)$$



$$\mathcal{L}_f = \bar{\Psi} \left(i\gamma^\mu \partial_\mu - g \vec{\tau} \cdot \vec{\phi} - m \right) \Psi$$

● **Rescaling:**

$$r \rightarrow \sqrt{\frac{\kappa_4}{\kappa_2}} r, \quad \Psi \rightarrow \sqrt{\frac{\kappa_2}{\kappa_4}} \Psi, \quad \kappa_0 \rightarrow \frac{\kappa_2^2}{\kappa_4} \kappa_0, \quad g \rightarrow \sqrt{\frac{\kappa_2}{\kappa_4}} g, \quad m \rightarrow \sqrt{\frac{\kappa_2}{\kappa_4}} m$$

$$\sqrt{\kappa_2 \kappa_4} = 1, \quad \kappa_0, \quad m, \quad g$$

Stationary modes: $\Psi = \psi(r, \theta) e^{-i\varepsilon t}$

● **Fermionic density:** $\rho = \bar{\psi} \hat{\gamma}_3 \psi = \psi^\dagger \psi, \quad \int d^2x \psi^\dagger \psi = 1$

● **Hamiltonian:**

$$H = \int d^2x \psi^\dagger \mathcal{H} \psi + \int d^2x \left(\frac{1}{2} \left(\partial_k \vec{\phi} \right)^2 + \frac{1}{4} \left(\partial_k \vec{\phi} \times \partial_n \vec{\phi} \right)^2 + \kappa_0 U \right)$$

● **Field equations:**

Scalar current

$$j_\mu = \vec{\phi} \times \partial_\mu \vec{\phi} + \partial_\nu \vec{\phi} (\partial^\nu \vec{\phi} \cdot (\vec{\phi} \times \partial_\mu \vec{\phi}))$$

$$\mathcal{H}\psi \equiv \hat{\gamma}_3 \left(-i\hat{\gamma}_k \partial_k + g\vec{\tau} \cdot \vec{\phi} + m \right) \psi = \varepsilon \psi$$

$$\partial_\mu j^\mu = \kappa_0 \vec{\phi}_\infty \times \vec{\phi} + g\vec{\phi} \times (\psi^\dagger \vec{\tau} \psi)$$

Spin-Isospin fermions

● **Rotationally invariant configuration:**

$Q=n:$

$$\begin{aligned}\phi^1 &= \sin f(r) \cos n\theta; \\ \phi^2 &= \sin f(r) \sin n\theta; \\ \phi^3 &= \cos f(r)\end{aligned}$$

$$\psi^{(i)} = \mathcal{N}^{(i)} \begin{pmatrix} v_1(r) e^{il\theta} \\ iv_2(r) e^{i(l+n)\theta} \\ u_1(r) e^{i(l+1)\theta} \\ iu_2(r) e^{i(l+n+1)\theta} \end{pmatrix}$$

$$\mathcal{H} = \begin{pmatrix} g \cos f + m & ge^{-in\theta} \sin f & -e^{-i\theta} \left(\partial_r - \frac{i\partial_\theta}{r} \right) & 0 \\ ge^{in\theta} \sin f & -g \cos f + m & 0 & -e^{-i\theta} \left(\partial_r - \frac{i\partial_\theta}{r} \right) \\ e^{i\theta} \left(\partial_r + \frac{i\partial_\theta}{r} \right) & 0 & -g \cos f - m & -ge^{-in\theta} \sin f \\ 0 & e^{i\theta} \left(\partial_r + \frac{i\partial_\theta}{r} \right) & -ge^{in\theta} \sin f & g \cos f - m \end{pmatrix}$$

● Generalized angular momentum:

$$J_k = -i\nabla_k + \frac{\gamma_k}{2} \otimes \mathbb{I} + \mathbb{I} \otimes \frac{\tau_k}{2}$$

$$J_3 = -i\frac{\partial}{\partial\theta} + \frac{\hat{\gamma}_3}{2} + n\frac{\tau_3}{2}$$

$$[\mathcal{H}, J_3] = 0, \quad J_3\psi = \kappa\psi; \quad \kappa = \frac{1}{2}(1 + n + 2l)$$

Ground state: $\kappa = 0$

$$\rightarrow l = -\frac{1+n}{2}, \quad l = 0, \quad n = -1$$

● Asymptotic expansion:

$$r \rightarrow \infty$$

$$\vec{\phi} \approx \vec{\phi}_\infty + \delta\vec{\phi} \rightarrow \begin{cases} (-i\hat{\gamma}_k\partial_k + g\tau_3 + m)\psi = 0, \\ (\Delta - \kappa_0)\delta\vec{\phi} = 0. \end{cases}$$

Pair of orthogonal scalar dipoles

$$\delta\vec{\phi} \sim K_n(\sqrt{\kappa_0}r) \begin{pmatrix} \cos(n\theta - \chi) \\ \sin(n\theta - \chi) \\ 0 \end{pmatrix}$$

$$\begin{cases} (\Delta - 4(g \pm m)^2)u_{1,2} = \varepsilon^2 u_{1,2} \\ (\Delta - 4(g \pm m)^2)v_{1,2} = \varepsilon^2 v_{1,2} \end{cases}$$

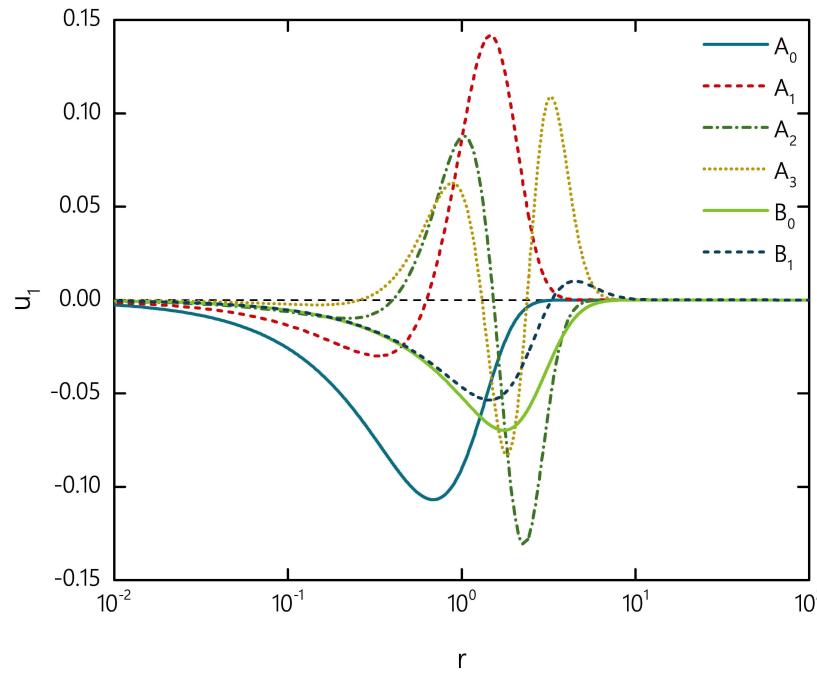
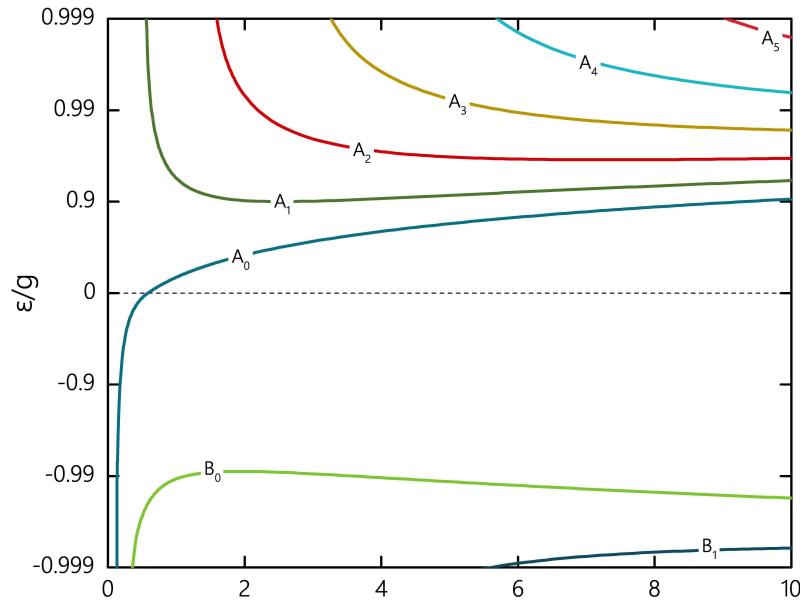


$$v_1 \sim e^{il(\varphi-\chi)} K_l(\sqrt{4(g+m)^2 - \varepsilon^2} r)$$

$$v_2 \sim e^{i(l+n)(\varphi-\chi)} K_{l+n}(\sqrt{4(g-m)^2 - \varepsilon^2} r)$$

Asymptotic fermionic field: a pair of orthogonal 2^l -poles,
together with a pair of collinear 2^{l+n} -poles

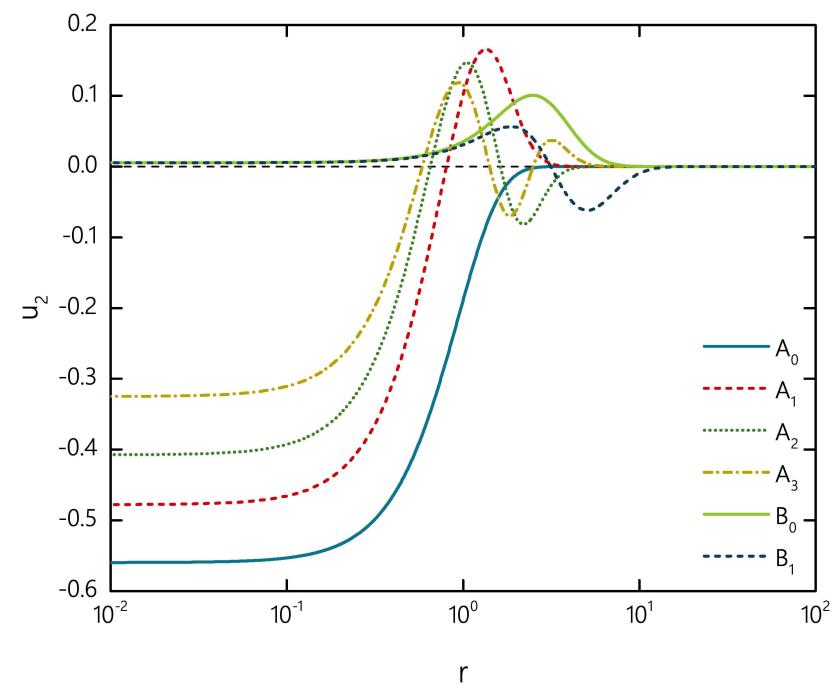
Fermions coupled to a baby Skyrmion



Massless localized fermions:

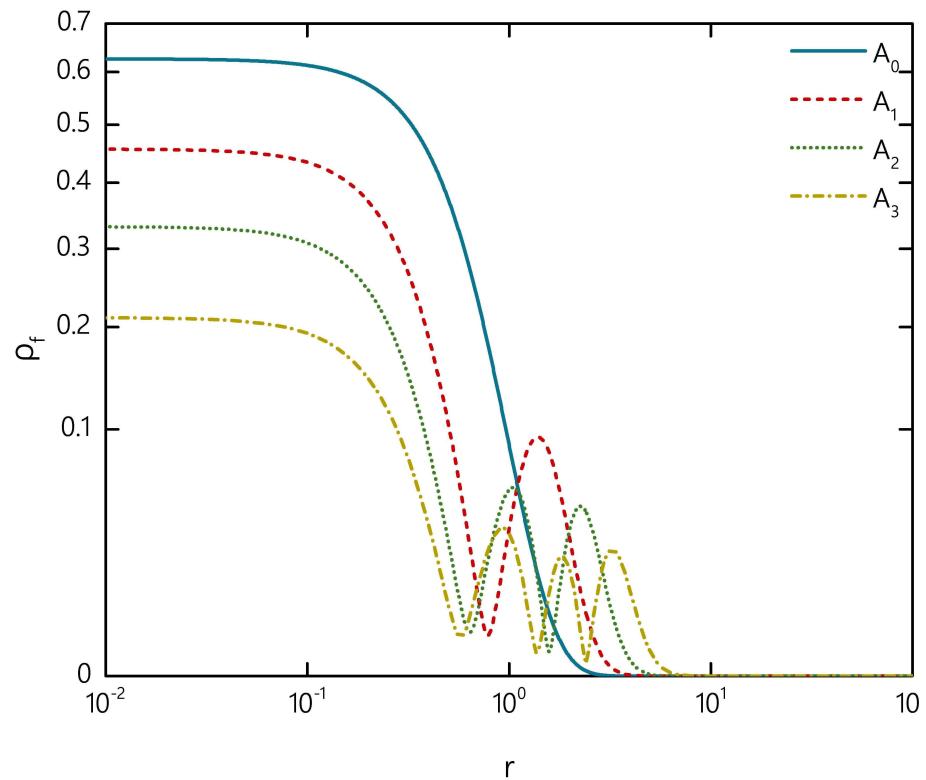
$$m = 0, \quad u_2 = v_1, \quad u_1 = -v_2$$

- ➊ Solutions found numerically by the shooting method;
- ➋ Spectral flow is in agreement with the index theorem
- ➌ There are two types of the modes, the solutions are characterized by the number of nodes

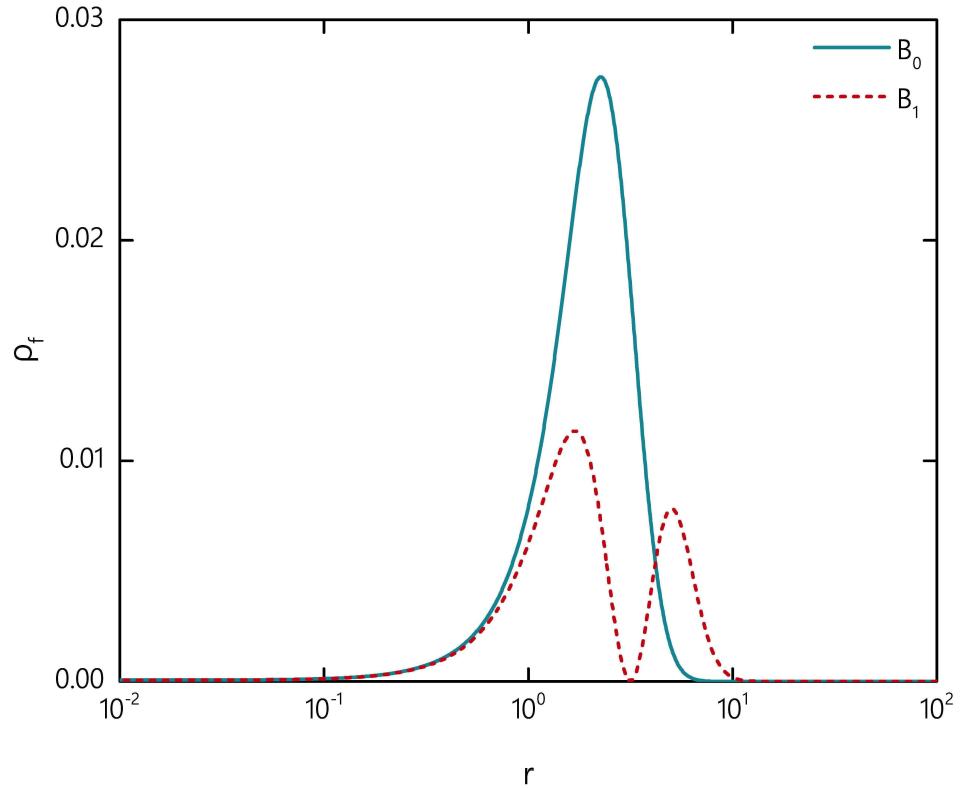


Fermionic density

$$m = 0, \quad g = 1, \quad \kappa_0 = 0.1$$

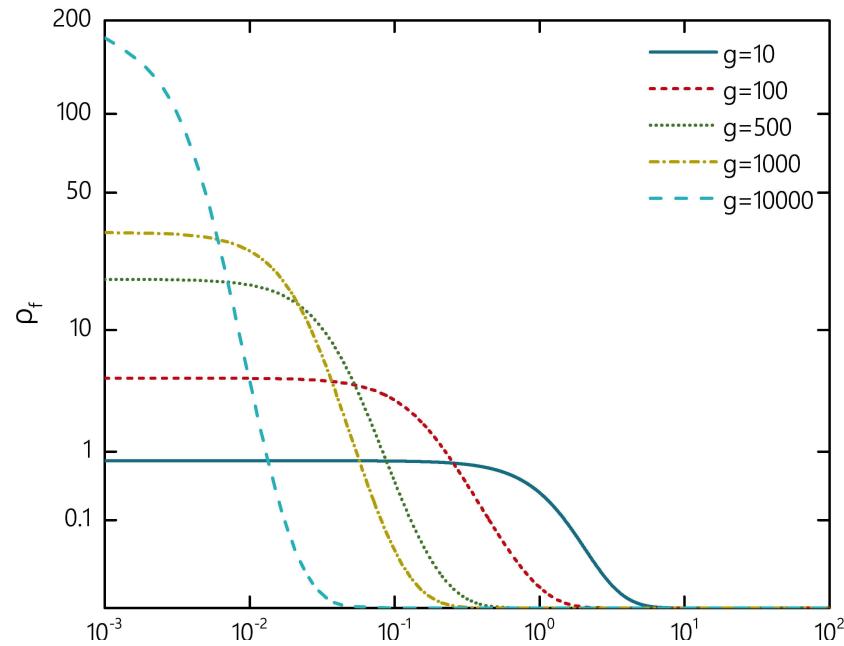


A-modes

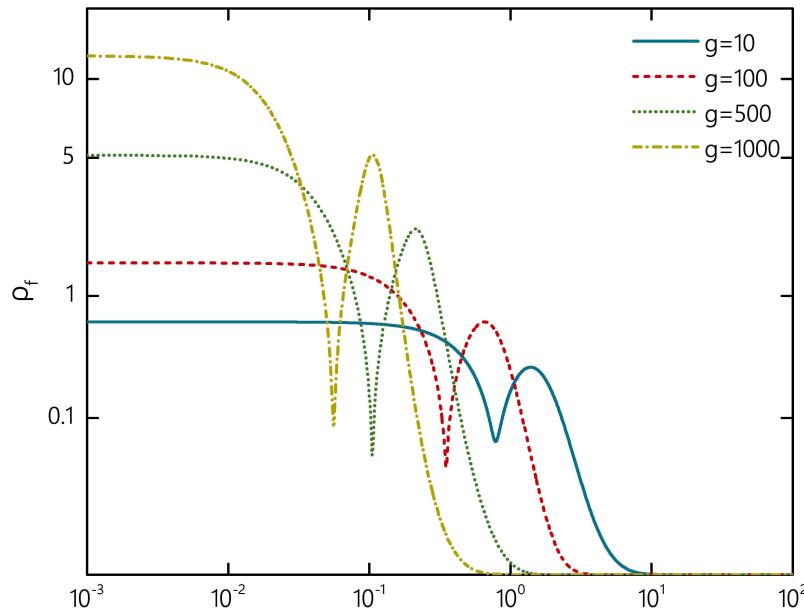
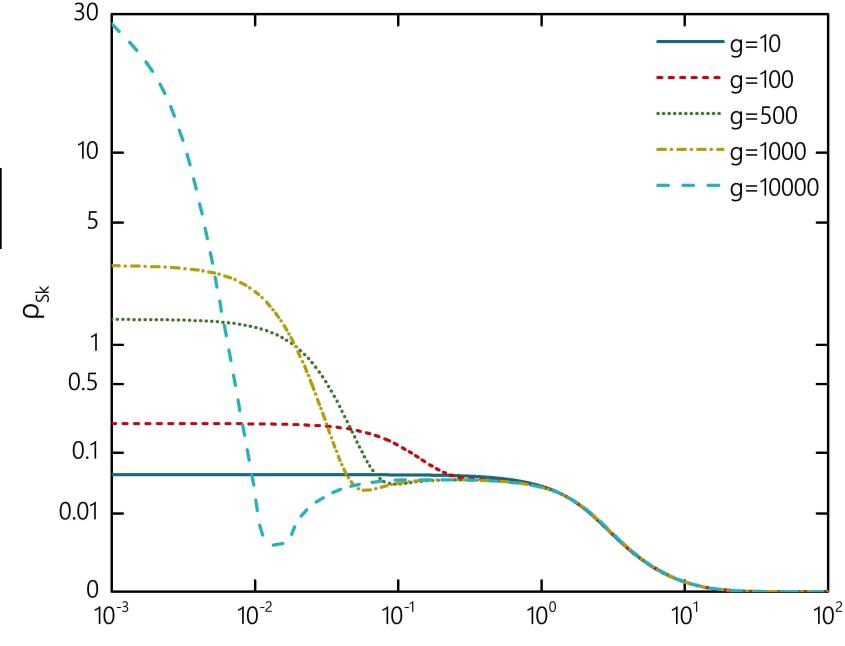


B-modes

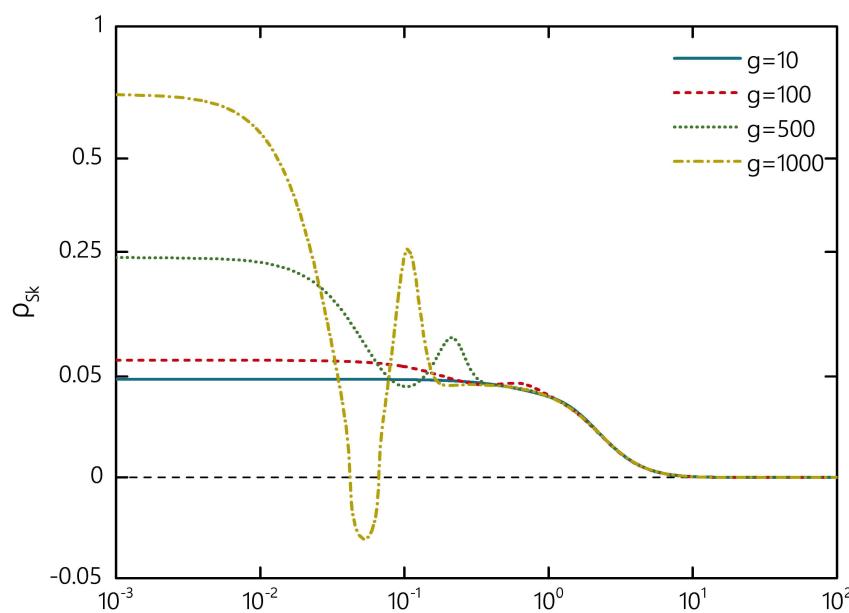
Fermionic and topological densities



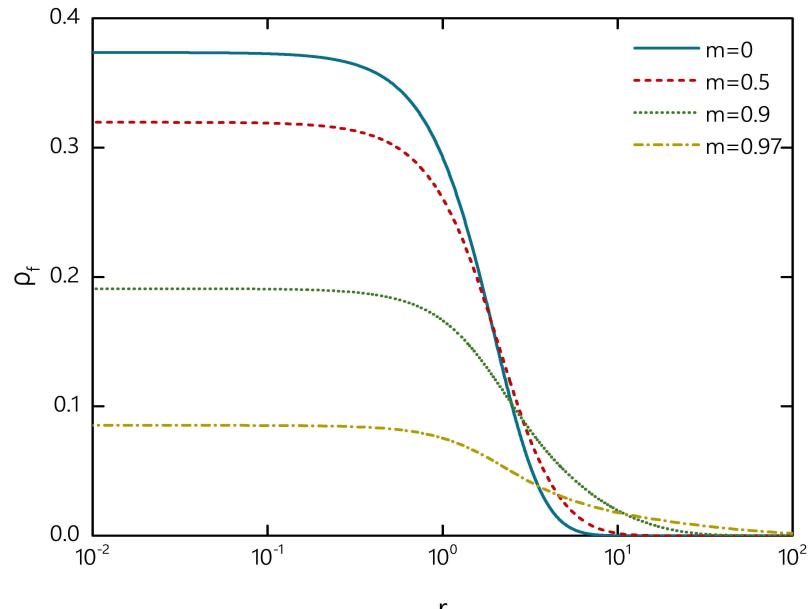
A_0



A_1

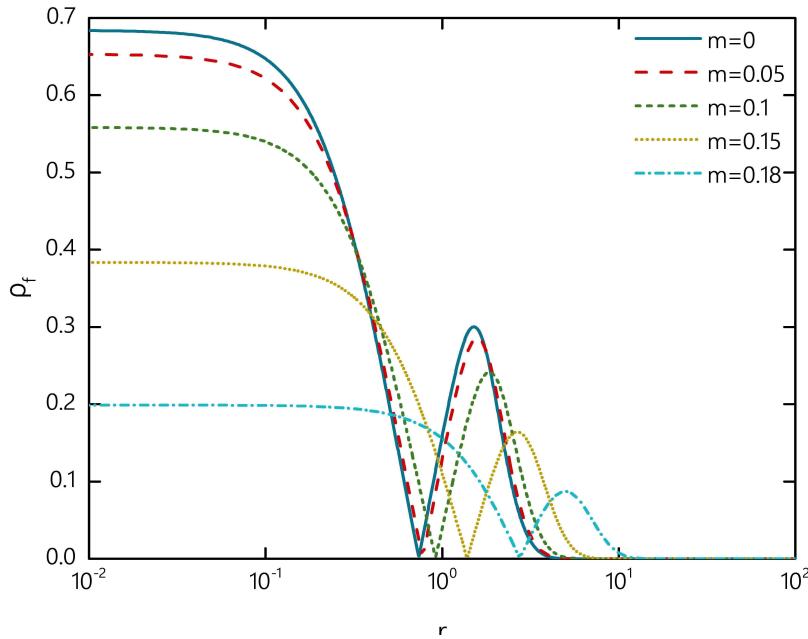


Fermion-Skyrmion system: vanishing potential

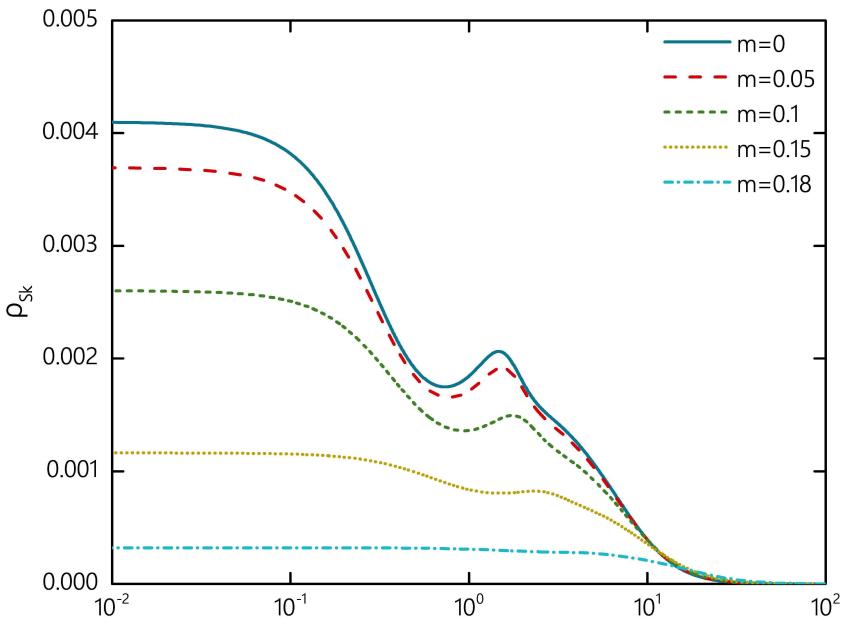
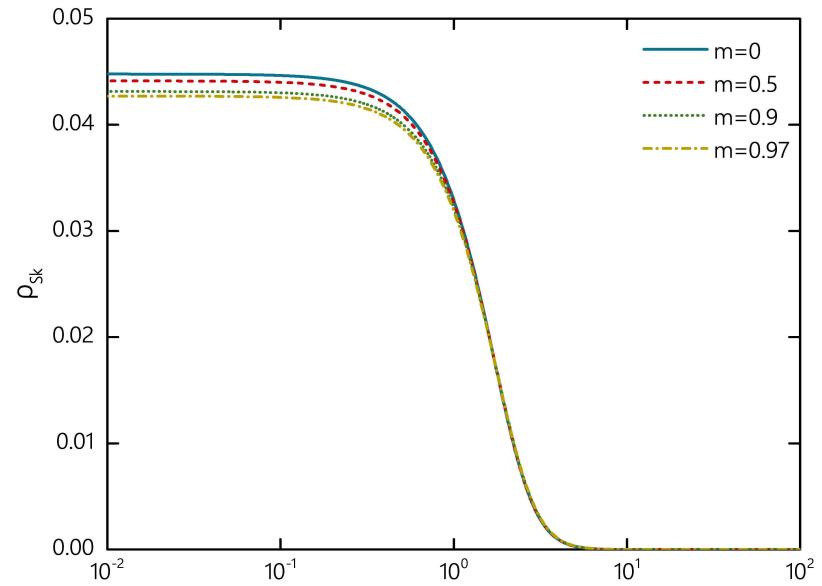


$$g = 1, \\ \kappa_0 = 0$$

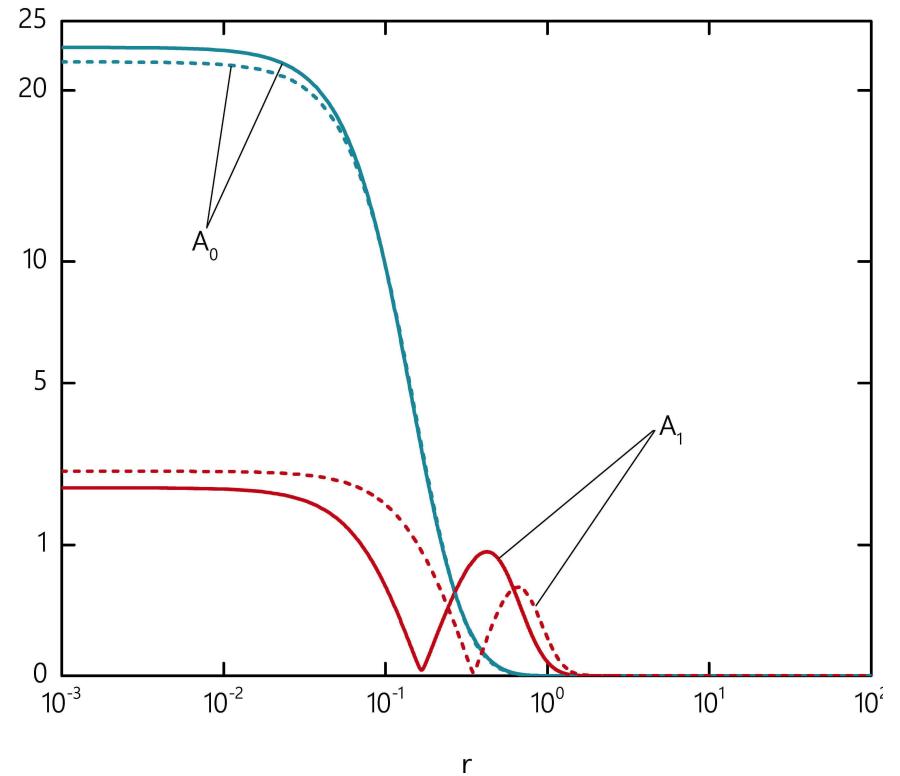
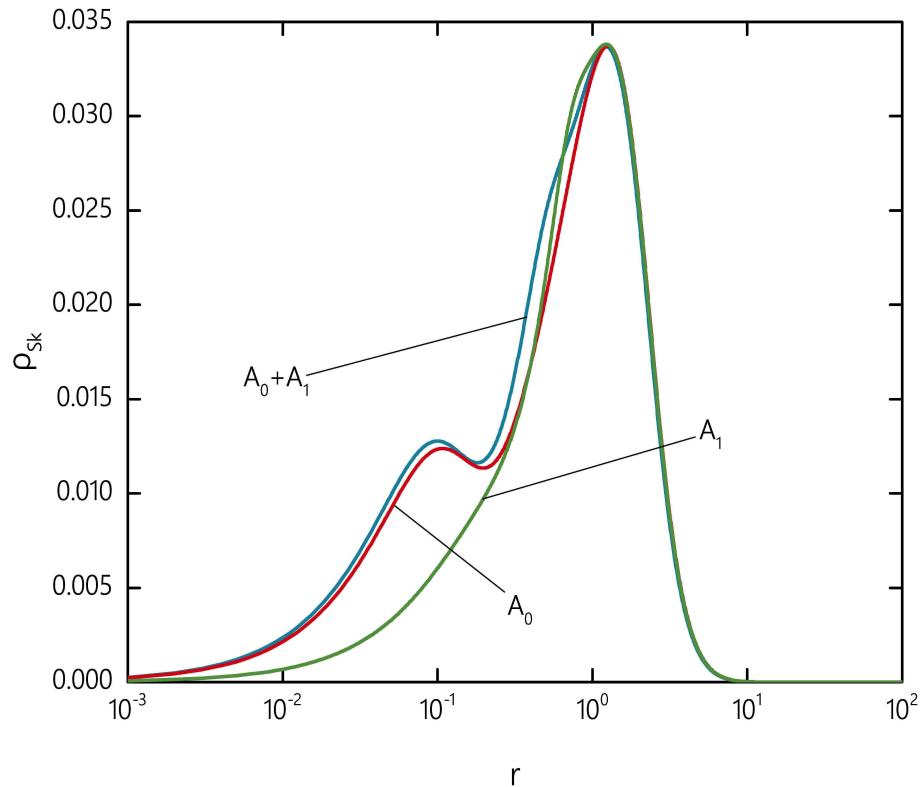
A_0



A_1



Fermion-Skyrmion system: Filling factor 2



Magnetic Skyrmions

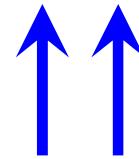
- Heisenberg model:

$$H = \sum_{i < j=1} J_{ij} S_i S_j$$

classical nearest neighbour interaction



$$\frac{J}{2} \int d^2x \partial_n \vec{m} \cdot \partial_n \vec{m}$$



mean field approximation

- Dzyaloshinskii-Moriya interaction:

$$H_{DM} = \sum_{ij} D_{ij} (S_i \times S_j)$$

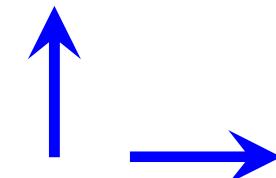


$$D \int d^2x \vec{m} \cdot (\nabla \times \vec{m})$$

- Zeeman interaction:

$$H_{ext} = \vec{B} \cdot \vec{m}$$

Chiral magnetic Skyrmions:



$$E = \int d^2x \left(\frac{J}{2} (\nabla \vec{m})^2 + D \vec{m} \cdot (\nabla \times \vec{m}) - \vec{B} \cdot \vec{m} \right)$$

Magnetic Skyrmions

Field equation: $J\Delta\vec{m} - 2D\nabla \times \vec{m} + \vec{B} = 0$

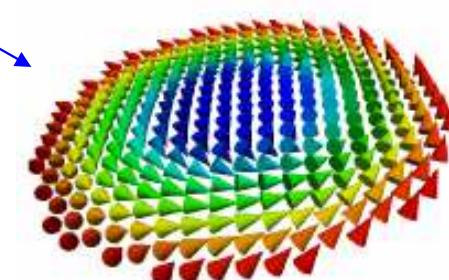
$$\alpha = 2D/J, \quad \mu = |B|/J$$

$$\begin{aligned} m_1 &= \sin f(r) \cos(Q\varphi - \delta); \\ m_2 &= \sin f(r) \sin(Q\varphi - \delta); \\ m_3 &= \cos f(r) \end{aligned}$$

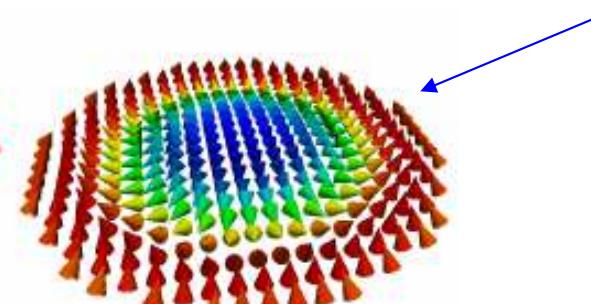
$$E = 2\pi \int r dr \left\{ \frac{1}{2} f'^2 + \frac{Q^2}{2r^2} \sin^2 f - \mu \cos f \right. \\ \left. + \frac{\alpha}{Q-1} \sin(Q\pi) \sin(\delta + Q\pi) \left(f' + \frac{Q}{2r} \sin(2f) \right) \right\}$$

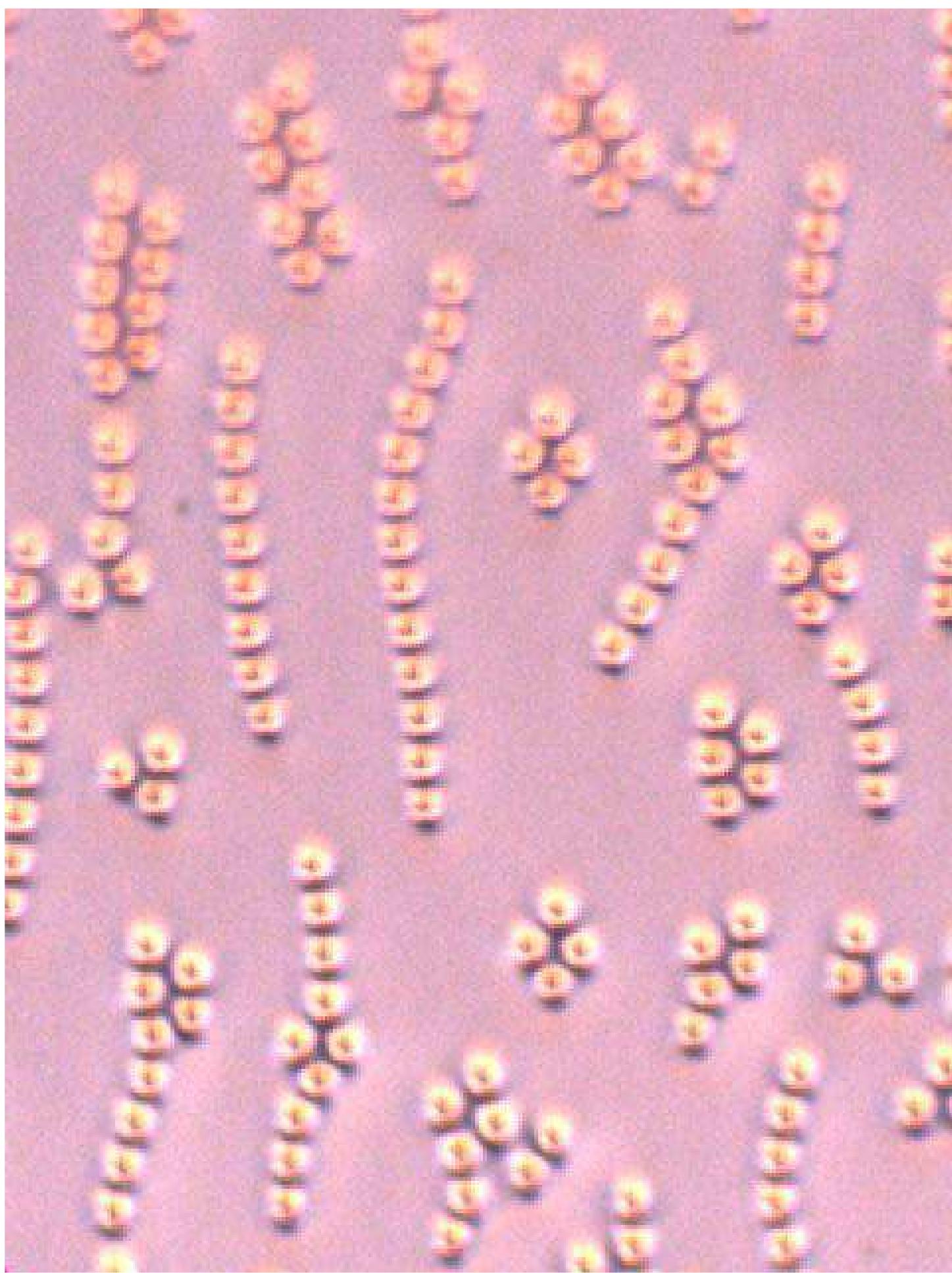
$$\frac{\alpha}{Q-1} \sin(Q\pi) \sin(\delta + Q\pi) = \begin{cases} \alpha \sin \delta & , \quad \text{if } Q = 1 \\ 0 & , \quad \text{if } Q \neq 1. \end{cases}$$

Bloch-type skyrmions: $\delta = \pm \pi/2$



Néel-type skyrmions: $\delta = 0, \pi$





Fermion-magnetic Skyrmi system in 2+1 dim

$$\mathcal{H} = \mathcal{H}_{Sk} + \mathcal{H}_f$$

$$\mathcal{H}_{Sk} = \frac{J}{2} \left(\nabla \vec{\phi} \right)^2 + D\vec{\phi} \cdot \left(\nabla \times \vec{\phi} \right) - \vec{B} \cdot \vec{\phi}$$

$$A_k = \frac{B}{2} (0, -y, x)$$

$$\mathcal{H}_f = \Psi^\dagger \hat{\gamma}^3 \left(-i\hat{\gamma}^k \partial_k + e\hat{\gamma}^k A_k + m + g\vec{\tau} \cdot \vec{\phi} \right) \Psi.$$

Stationary configuration: $\vec{\phi} = \vec{\phi}(r, \theta), \quad \Psi = \psi(r, \theta)e^{-i\varepsilon t}$

Field equations:

Spin-Isospin fermions

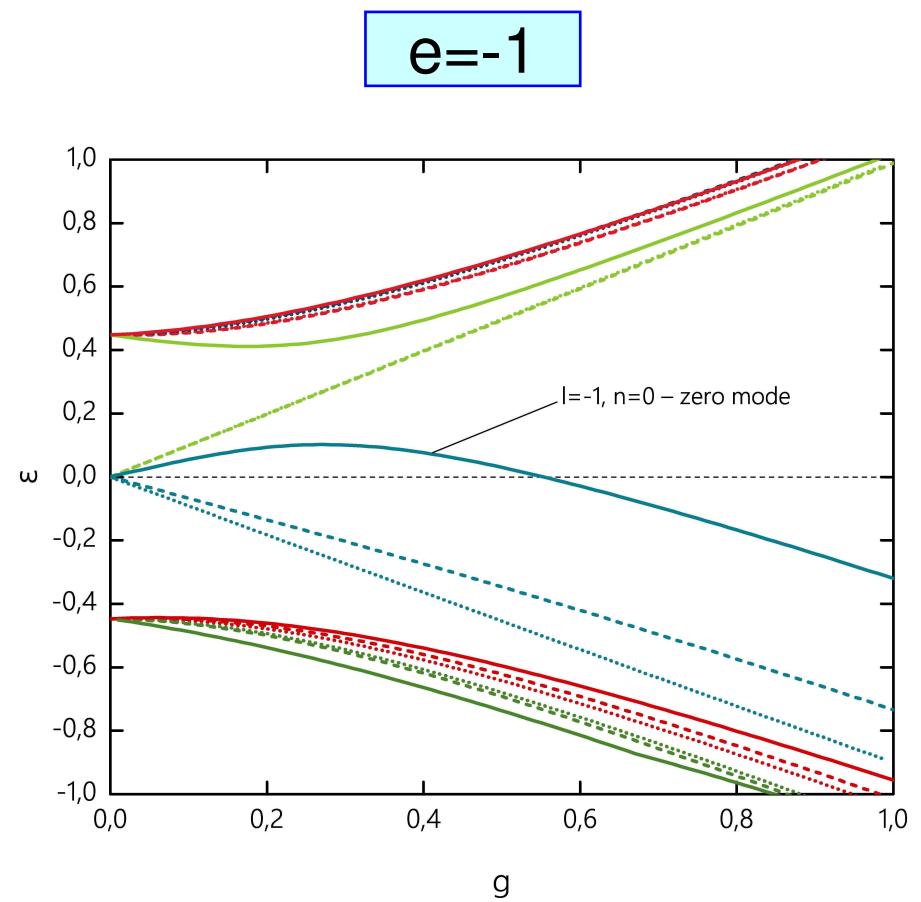
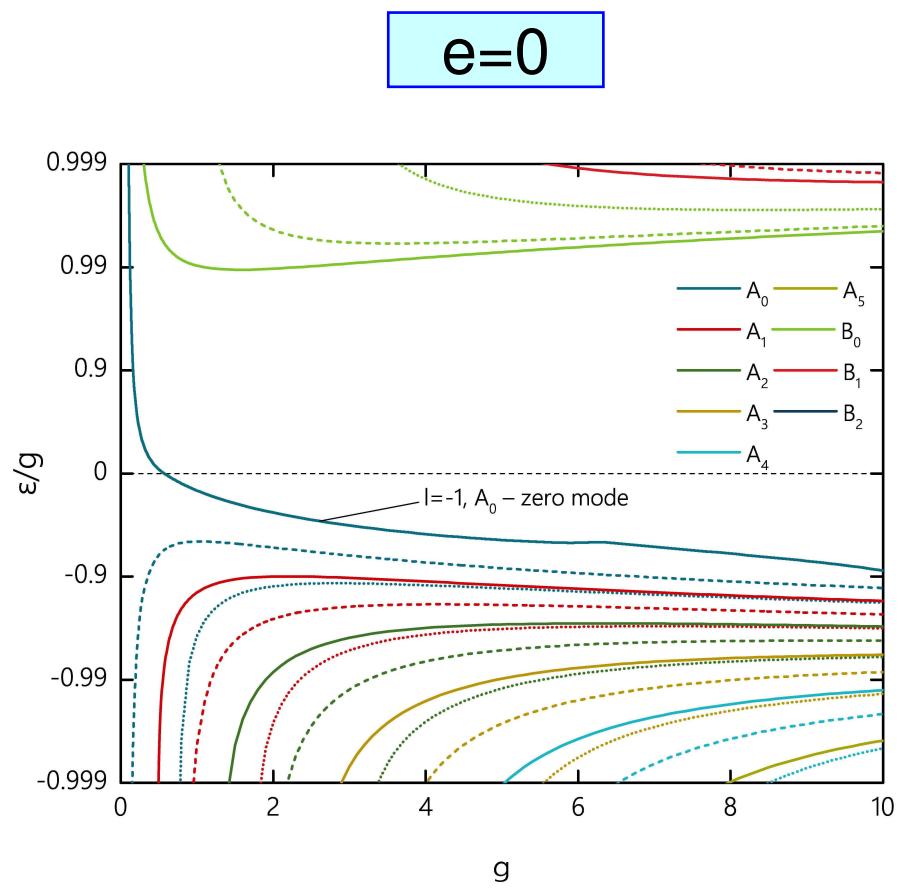
$$\Delta \vec{\phi} - 2\nabla \times \vec{\phi} + \vec{B} - g\psi^\dagger \hat{\gamma}_3 \vec{\tau} \psi = 0$$

$$\hat{\gamma}^3 \left(-i\hat{\gamma}^k \partial_k + e\hat{\gamma}^k A_k + m + g\vec{\tau} \cdot \vec{\phi} \right) \psi = \varepsilon \psi$$

$$\psi^{(i)} = \mathcal{N}^{(i)} \begin{pmatrix} v_1(r)e^{il\varphi} \\ iv_2(r)e^{i(l+n)\varphi} \\ u_1(r)e^{i(l+1)\varphi} \\ iu_2(r)e^{i(l+n+1)\varphi} \end{pmatrix}$$

Generalized angular momentum:

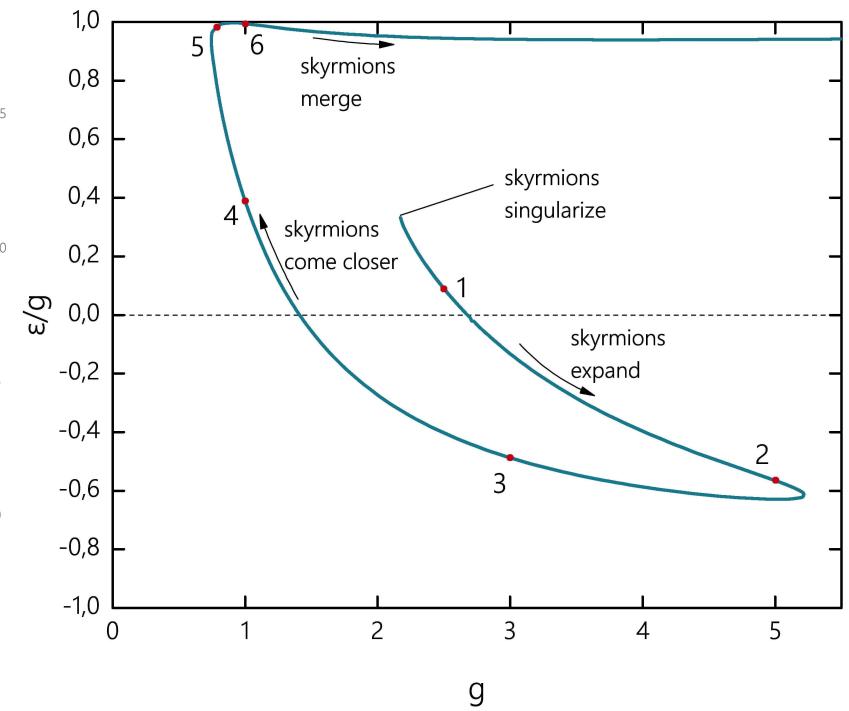
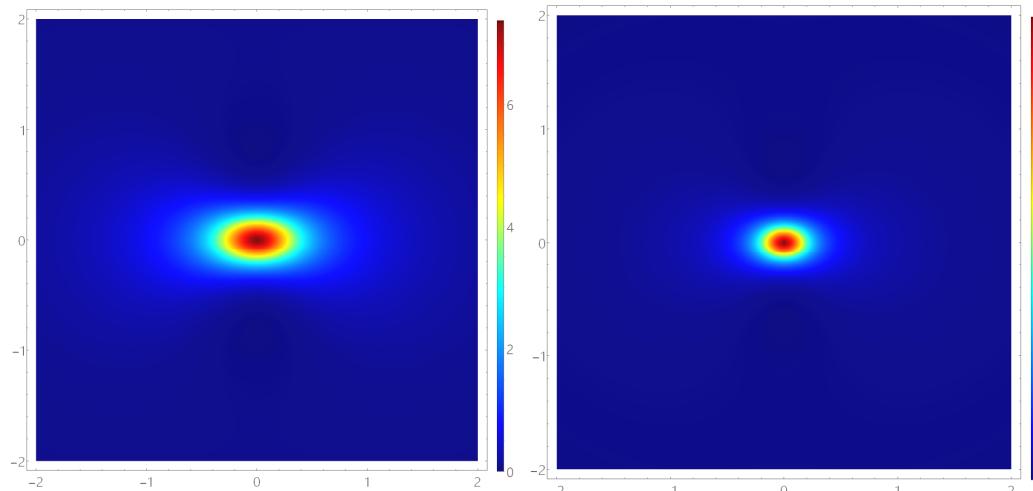
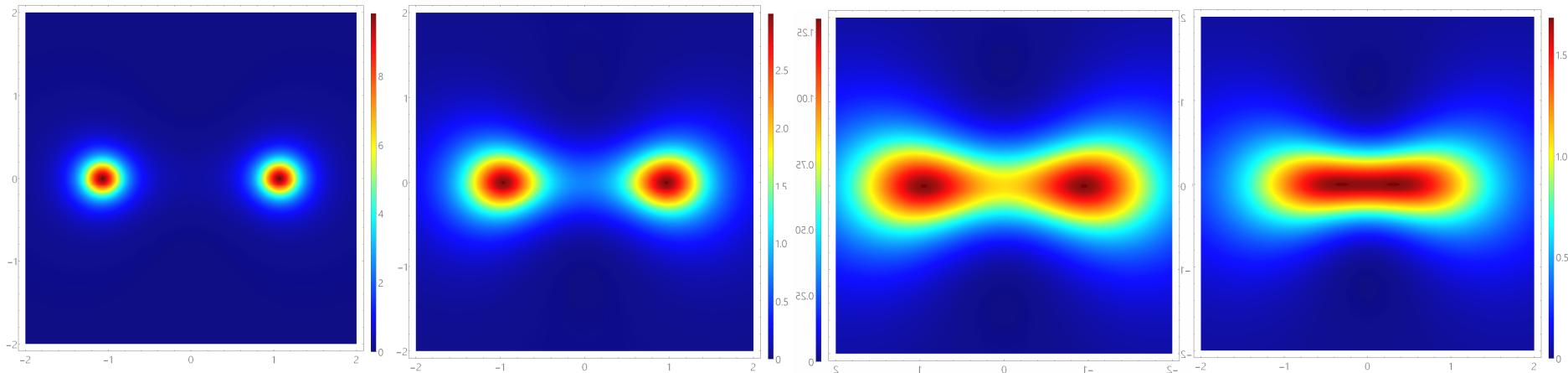
$$J_k = -i\nabla_k + \frac{\gamma_k}{2} \otimes \mathbb{I} + \mathbb{I} \otimes \frac{\tau_k}{2}$$



g=0: Landau levels

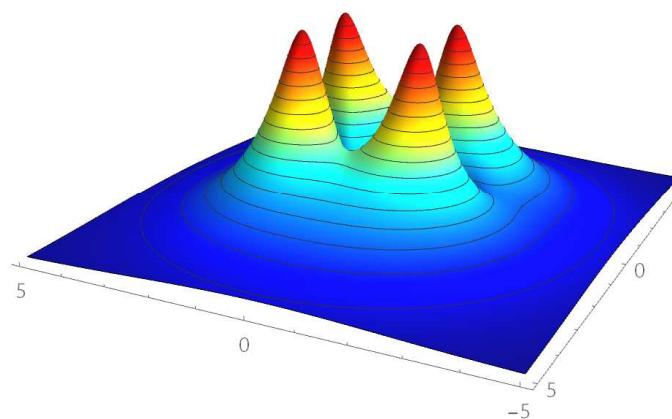
$$\varepsilon_k^2 = m^2 + B(|e|(2k+1) \pm e)$$

Chiral Fermions bounded by a fermionic mode

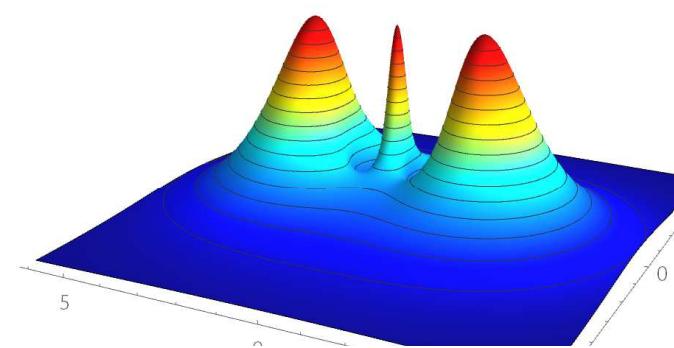


Chiral Fermions bounded by a fermionic mode

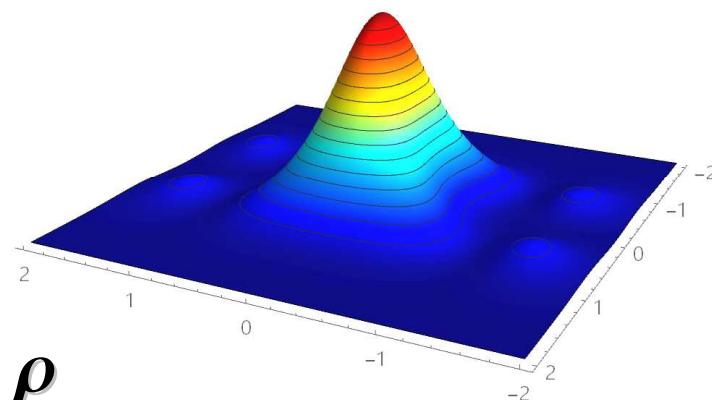
$Q=4$



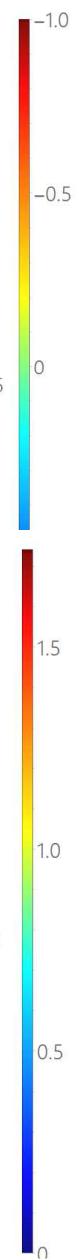
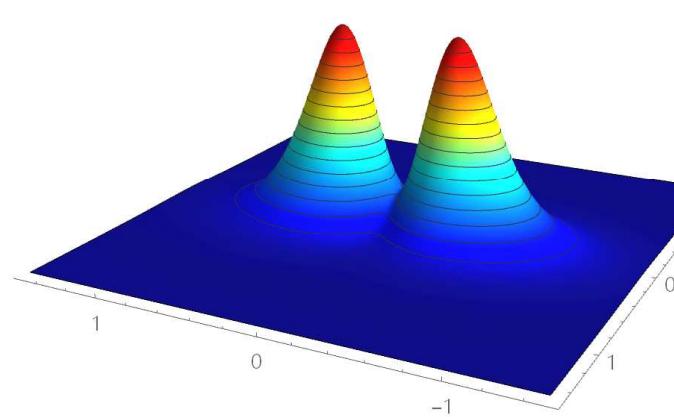
$Q=3$



q_{top}



ρ



Summary

- Backreaction of the localized fermions may strongly affect the form of the solitons
- Localization of the fermions produces additional channels of interaction between the solitons
- Fermionic exchange interaction may bound solitons with repulsive scalar interaction, e.g. Skyrmions with DM interaction, or with holomorphic potential
- Fermion-Skyrmion system in 3+1 dim?
- Other topological solitons with localized fermions?
- Dynamics of the solitons with localized fermionic modes?
- Unconventional superconductivity in magnetic materials?

Thank you!