Composite Non-Abelian BPS Strings, Grassmannian and Flag Manifolds

Edwin Ireson

William I. Fine Theoretical Physics Institute,
University of Minnesota

Topological Solitons, Non-Perturbative Gauge Dynamics and Confinement 2019

Based on arXiv:1905.09946 (M.Shifman, E.I.) and arXiv:1907.XXXXX (E.I.)

Elementary Non-Abelian String

4D $\mathcal{N} = 2$ SQCD, $U(N)$ gauge group, N flavours Φ_k^A

$$\frac{1}{4g^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + |D\Phi|^2 + \frac{1}{2}\mathrm{Tr}\left(\Phi^\dagger T^a \Phi\right) + \frac{1}{8}\left(\mathrm{Tr}\left(\Phi^\dagger\Phi\right) - N\xi\right)^2 \quad (1)$$

SSB vacuum locks colour and flavour to $U(N)_{\text{diag}}$. Without breaking $\mathbb{Z}_N$ center symmetry can only make ANO BPS strings, where every colour experiences winding, has tension $T = 2\pi N\xi$. Lower tension objects exist!

Break $\mathbb{Z}_N$ by winding only one colour i.e. charging an object with exactly one unit of flux of one colour, BPS object of tension $T = 2\pi\xi$.

Leftover $U(N)_{\text{diag}}$ transformation endows vortex string with $\mathbb{CP}(N-1)$ degree of freedom

$\mathbb{CP}(N-1)$ String: Main Ingredient

$\mathbb{CP}(N-1)$ string has one unit of magnetic flux in one colour: N choices

$$\Phi = U \left(\begin{array}{c|c} \mathbb{1}_{N-1} & 0 \\ \hline 0 & e^{i\theta} \end{array} \right) U^\dagger \tag{2}$$

This endows it with internal degree of freedom living in this space. Low-energy effective action of string dynamics produces NLSM

$$\frac{U(N)}{U(N-1)\times U(1)} = \mathbb{CP}(N-1) \tag{3}$$

Strings are BPS so exert no long-range forces, can imagine creating composite object where multiple colours have many units of flux.

Question

Can we create an NA string with more than a single unit of flux? How does this affect the gauge symmetry breaking? What kind of manifold for the moduli?

Composite strings

Multiple colours can have winding, and more than one unit.

$$\Phi = U \left(\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{2i\theta} \end{array}\right) U^\dagger \tag{4}$$

$\frac{6!}{1!2!3!} = 60$ equivalent ways of writing Ansatz. Left over colour invariance

$$U = \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{pmatrix} \tag{5}$$

which can act on the string. DoFs inside U exist in

$$\frac{U(6)}{U(1)U(2)U(3)} \tag{6}$$

The Flag Family

Generically obtain gauge degrees of freedom living in

$$\mathcal{F}_{\{N_\alpha\}} = \frac{U(N)}{U(N_0) \times \cdots \times U(N_p)}, \quad \sum_{\alpha=0}^{p} N_\alpha = N \tag{7}$$

with two special cases: $\mathbb{CP}(N)$ and the Grassmannian manifold

$$\mathbb{CP}(N-1) = \frac{U(N)}{U(1)U(N-1)}, \quad \mathcal{G}_{M,L} = \frac{U(N)}{U(M)U(L)} \quad (M+L=N) \tag{8}$$

They are manifolds of dimension

$$\left|\mathcal{F}_{\{N_\alpha\}}\right| = N^2 - \sum_{\alpha=0}^{p} N_\alpha^2 = 2\sum_{\alpha>\beta} N_\alpha N_\beta \tag{9}$$

What is the geometrical interpretation of these spaces?

The Flag family

All these spaces are complex projective-type spaces. $\mathbb{CP}(N-1)$ is the space of all lines through the origin of $\mathbb{C}^N$, Grassmannian is set of all M-dimensional hyperplanes in $\mathbb{C}^N$

$$L \in \mathbb{CP}(N-1): \{0\} \subset V_1 \subset \mathbb{C}^N, \quad G \in \mathcal{G}_{M,L}: \{0\} \subset V_M \subset \mathbb{C}^N \tag{10}$$

Generically consider flags, progressive inclusions of spaces

$$F \in \mathcal{F}_{\{N_\alpha\}}: \{0\} \subset V_{N_1} \subset V_{N_1+N_2} \cdots \subset V_{N_1+\cdots+N_p} \subset \mathbb{C}^N \tag{11}$$

Describing these spaces algebraically involves picking basis of coordinates on these subspaces:

$$\begin{aligned} &\text{span}\left(X^{(1)}\right) = V_{N_1}, \text{span}\left(X^{(1)}, X^{(2)}\right) = V_{N_1+N_2} \ldots \\ &\text{span}\left(X^{(1)}, \ldots, X^{(p)}, X^{(0)}\right) = \mathbb{C}^N \end{aligned} \tag{12}$$

Parametrising the Gauge Moduli

Each N_α block has winding q_α. Parametrise the string internal moduli: break up U into orthonormal column blocks

$$U = \left(X^{(0)} \middle| \ldots \middle| X^{(p)} \right), \quad X^{(\alpha)}_{Ai} : A = 1 \ldots N,\ i = 1 \ldots N_\alpha \tag{13}$$

$$X^{(\alpha)^\dagger} X^{(\beta)} = \delta_{\alpha\beta} \mathbb{1}_{N_\alpha}, \quad \sum_{\alpha=0}^{p} X^{(\alpha)} X^{(\alpha)^\dagger} = \mathbb{1}_N \tag{14}$$

Alone these conditions are not sufficient for DoF counting to match manifold dimension, we have not quotiented by the group action stabilizer.

$\left\{ X^{(1)} \ldots X^{(\alpha)} \right\}$ can be seen as an orthonormal basis for $V_{N1+\cdots+N_\alpha}$. Specifying all of them produces a Flag, but not uniquely: leftover gauge invariance corresponds directly to invariance under change of basis

Worldsheet Action

This over-representation should be seen on the string as worldsheet gauge invariance. Let $X^{(\alpha)}$ vary along the worldsheet and produce low-energy effective action for the string:

$$S = \frac{4\pi}{g^2} \sum_{\alpha > \beta} I_{\alpha\beta} \int dtdz \, \mathrm{Tr} \left(X^{(\beta)^\dagger} \partial^\mu X^{(\alpha)} \partial_\mu X^{(\alpha)^\dagger} X^{(\beta)} \right) \tag{15}$$

where

$$I_{\alpha\beta} = \int drd\theta \, \left(\rho'^2_{\alpha\beta} + \frac{1}{r^2} \left(q_\alpha f_\beta - q_\beta f_\beta \right)^2 \left(1 - \rho_{\alpha\beta} \right)^2 + \frac{\rho^2_{\alpha\beta}}{2} \left(\phi^2_\alpha + \phi^2_\beta \right) + \left(1 - \rho_{\alpha\beta} \right) \left(\phi_\alpha - \phi_\beta \right)^2 \right) \tag{16}$$

Using BPS equations extensively, we can show

$$I_{\alpha\beta} = (q_\alpha - q_\beta), \quad \text{wlog } \alpha > \beta \longrightarrow q_\alpha > q_\beta \tag{17}$$

No DoF for inter-string distance: needed but difficult (Gorsky, Yung)

Gauge Invariance

Let us act with a local symmetry transformation on the fields X:

$$X^{(\alpha)}_{Ai} \longrightarrow X^{(\alpha)}_{Ai} + X^{(\alpha)}_{Aj}\alpha(x)_{ji} + O(\alpha^2) \tag{18}$$

Then, the generic worldsheet element transforms as

$$\mathrm{Tr}\left(X^{(\beta)^\dagger}\partial_\mu X^{(\alpha)}\partial^\mu X^{(\alpha)\dagger}X^{(\beta)}\right) \longrightarrow \mathrm{Tr}\left(X^{(\beta)^\dagger}\partial_\mu X^{(\alpha)}\partial^\mu X^{(\alpha)\dagger}X^{(\beta)}\right. \tag{19}$$
$$\left.+X^{(\beta)^\dagger}X^{(\alpha)}\partial_\mu\alpha(x)X^{(\alpha)^\dagger}\partial^\mu X^{(\beta)} + X^{(\beta)^\dagger}\partial_\mu X^{(\alpha)}\partial^\mu\alpha^\dagger(x)X^{(\alpha)\dagger}X^{(\beta)}\right) + \dots$$

Thanks to PI measure orthonormality relations the α dependent terms vanish identically. This proves that we have at least the gauge invariance that we require:

$$U(N_1)\times U(N_2)\times \dots U(N_p) \tag{20}$$

But can we have more?

Block Merger

Structure of the $l_{\alpha\beta} = (q_\alpha - q_\beta)$ terms means we could actually have more symmetry. If $q_\alpha = q_\beta$ then $X^{(\alpha)}$, $X^{(\beta)}$ can merge

$$Y = \left(X^{(\alpha)} \middle| \, X^{(\beta)} \right) \tag{21}$$

$U(N_\alpha + N_\beta)$ acts on Y, which can be gauged by the argument above. When all flux numbers become equal, we reduce to Grassmannian action

$$S = \sum_{\alpha > \beta} \frac{4\pi}{g^2} \int dtdz \, \mathrm{Tr} \left(X^{(0)\dagger} \partial_i Y \partial_i Y^\dagger X^{(0)} \right) \tag{22}$$

If $q_\alpha \neq q_\beta$ then the term it leads stops them from merging and forbids any accidentally enhanced symmetry.

$$(q_\alpha - q_\beta) \mathrm{Tr} \left| X^{(\beta)\dagger} \partial_i X^{(\alpha)} \right|^2 \tag{23}$$

Gauged Linear Sigma Model

Gauge invariant interactions resulting from integrating out vector fields with no kinematics and PI measure constraints can be added as auxiliary.

$$S = \frac{4\pi}{g^2} \int dtdz \sum_{\alpha} \left(q_\alpha \left| \partial_i X^{(\alpha)} - i X^{(\alpha)} A_i^{(\alpha)} \right|^2 + q_\alpha \mathrm{Tr} D^{(\alpha)} \left(X^{(\alpha)\dagger} X^{(\alpha)} - \mathbb{1} \right) \right.$$
$$\left. + 2 q_\beta \sum_{\beta<\alpha} \left(\mathrm{Tr} D^{(\alpha\beta)} X^{(\alpha)\dagger} X^{(\beta)} + \mathrm{Tr} A_i^{(\alpha\beta)} X^{(\alpha)\dagger} \partial_i X^{(\beta)} + \mathrm{h.c.} \right) \right) \tag{24}$$

This form hints at $\mathcal{N} = (2,2)$ SUSY completion of the theory, which is required since the object is BPS. This can be done for generic Flags.

Then, 1-loop β-function for g^2 in all Flag-type theories is

$$\beta(g^2) = -\frac{N}{4\pi} g^4, \quad \Lambda = M_{\mathrm{UV}} e^{-\frac{4\pi}{N g^2}} \tag{25}$$

which implies a dynamically generated mass gap.

Non-Linear Sigma Model

Introduce coordinates that solve all constraints and fix gauge. $\mathbb{CP}(N-1)$ and Grassmanian yields Fubini-Study metric: for ϕ_{mi} of size $M \times L$

$$\mathcal{L} = \left(\frac{1}{\mathbb{1} + \phi^\dagger \phi}\right)_{ji} \left(\partial\phi^\dagger\right)_{im} \left(\frac{1}{\mathbb{1} + \phi\phi^\dagger}\right)_{mn} \left(\partial\phi\right)_{nj} \tag{26}$$

For general Flag manifold algebra is tricky: with complex $\phi_{\beta\alpha}$ for $\alpha > \beta$ of size $N_\beta \times N_\alpha$, produce block determinants (defined iteratively)

$$\Delta_{0\alpha} = \begin{vmatrix} q_1\phi_{01} & q_2\phi_{02} & \dots & q_\alpha\phi_{0\alpha} \\ \mathbb{1}_{N_1} & (q_2 - q_1)\phi_{12} & \dots & (q_\alpha - q_1)\phi_{1\alpha} \\ 0 & \mathbb{1}_{N_2} & \dots & \dots \\ \dots & 0 & \dots & (q_{\alpha-1} - q_\alpha)\phi_{\alpha-1\ \alpha} \end{vmatrix} \tag{27}$$

from which build orthonormal $X^{(\alpha\geq 0)}$. Block merging seen via Woodbury identities: express larger inverse matrices from products of inverses of blocks.

Kähler Potentials

All Flags (and simpler) are C-Y spaces so should have Kähler, Ricci-flat metric. Fubini-Study for $\mathbb{CP}(N-1)$ and Grassmannian for sure:

$$\mathcal{K} = \mathrm{Tr}\log\left(\mathbb{1} + \phi\phi^\dagger\right) \tag{28}$$

Suggest the following Kähler potential for full Flag

$$\mathcal{K} = \sum_{\alpha=1}^{p}(q_\alpha - q_{\alpha+1})\mathrm{Tr}\log\left(\mathbb{1} + \sum_{\beta=1}^{\alpha}\Delta_{0\beta}\overline{\Delta}_{\beta 0}\right) \tag{29}$$

Dovetails nicely with progressive inclusion nature of Flags, manifestly reproduces block merging, but Δ is not elementary degree of freedom so metric is not easy to read from this.

Field Theory Properties

Strings are BPS: their worldsheet theory is gapped $\mathcal{N} = (2, 2)$ with

$$\mathcal{I}_W = \frac{N!}{N_0! \dots N_p!} \tag{30}$$

discrete SUSY vacua due to LG potential on gauge scalar

$$\prod_{A=1}^{N} \left(\sigma_{ii}^{(\alpha)} - m^A \right) = 0 \quad (\text{or } \Lambda^N) \tag{31}$$

Quantum theory dictated by LG rules & $\mathbb{CP}(N-1)$ vacuum structure: construct vacua and observe low-lying spectrum of kinks between neighbouring vacua thanks to $tt^\star$ equations.

$$\mathcal{G}_{M,L} = \left(\mathbb{CP}(N-1)\right)^M /\!\!/ S_M \tag{32}$$

Conclusions

- Fusing elementary non-Abelian strings produces a rich pattern of gauge symmetry breaking, especially with multiply wound sectors,
- Composite NA strings can be endowed with internal colour degrees of freedom existing in Flag manifold target space, generalising $\mathbb{CP}(N)$,
- Strings are BPS, bearing supersymmetry. This fixes all interaction parameters to be in integer ratios, proportional to differences of flux numbers,
- Multiple presentations for the Sigma model, including Gauged Linear one for usual FT and Non-Linear for geometrical aspects,
- Many properties can now be investigated.

Flag Trivia

Why is a Flag called a Flag?