The Yang-Mills Integral Equations

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Topological Solitons, Nonpertubative Gauge Dynamics
and Confinement 2
INFN - Pisa and Department of Physics, University of Pisa
18-20 July 2019

The Main Idea

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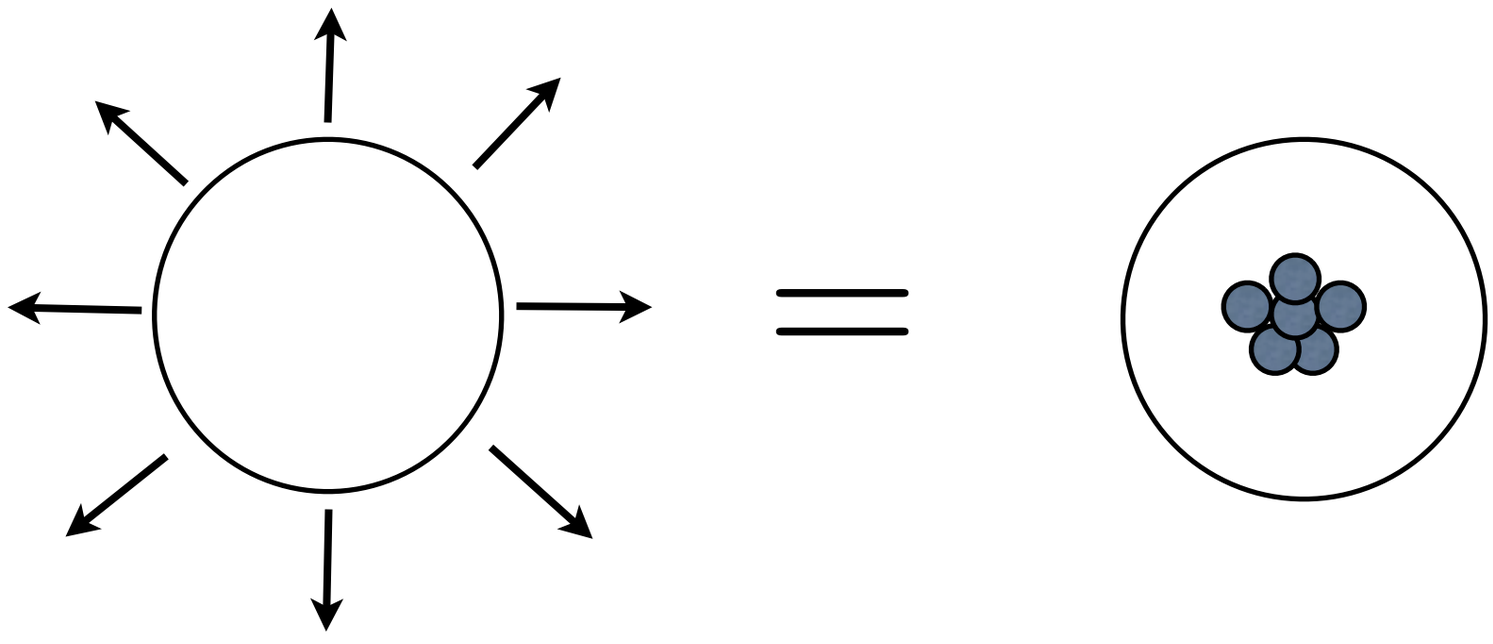
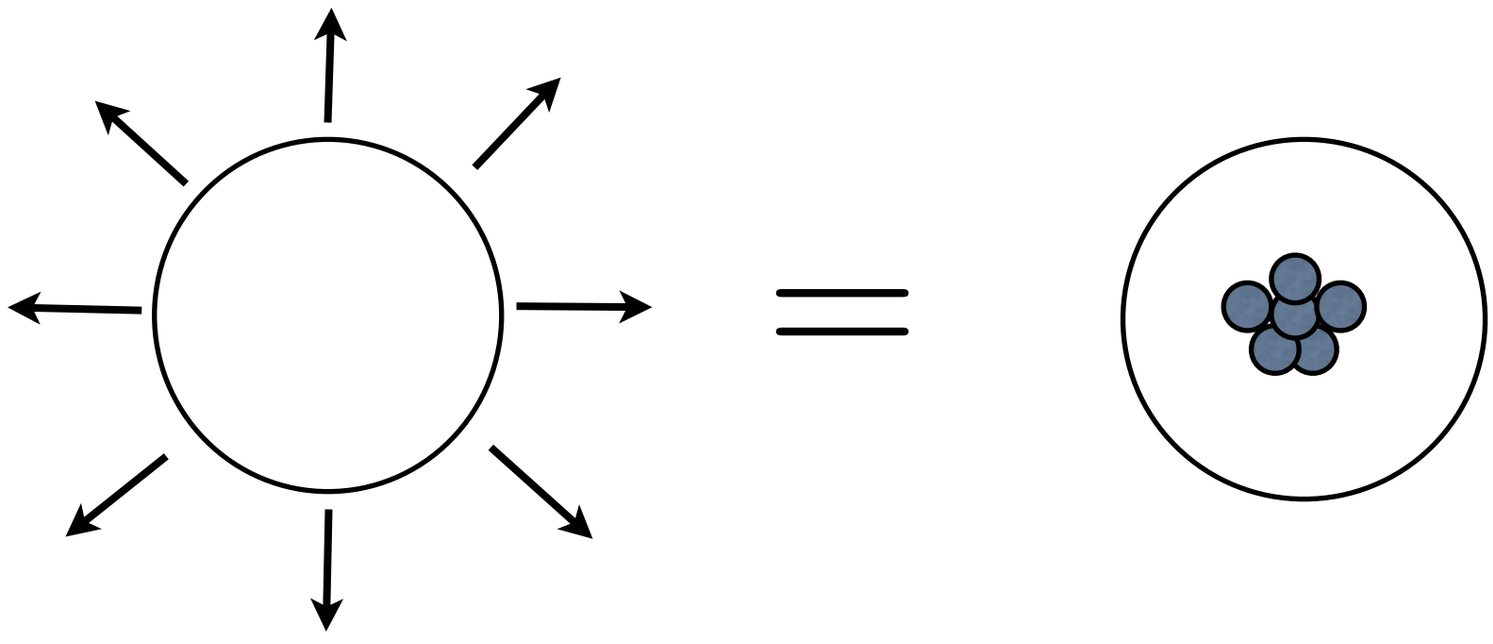


Flux Charge

The Main Idea



Flux = Charge

- It is a conservation law (non-Noether)

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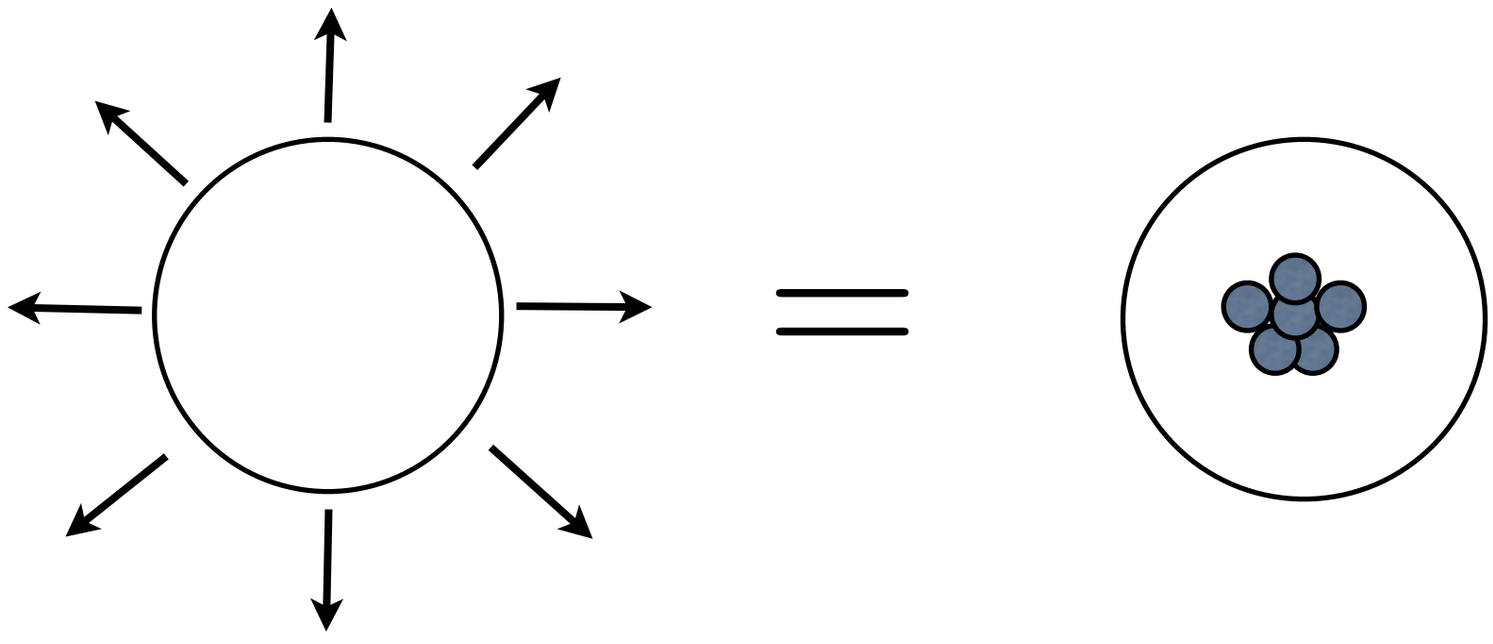

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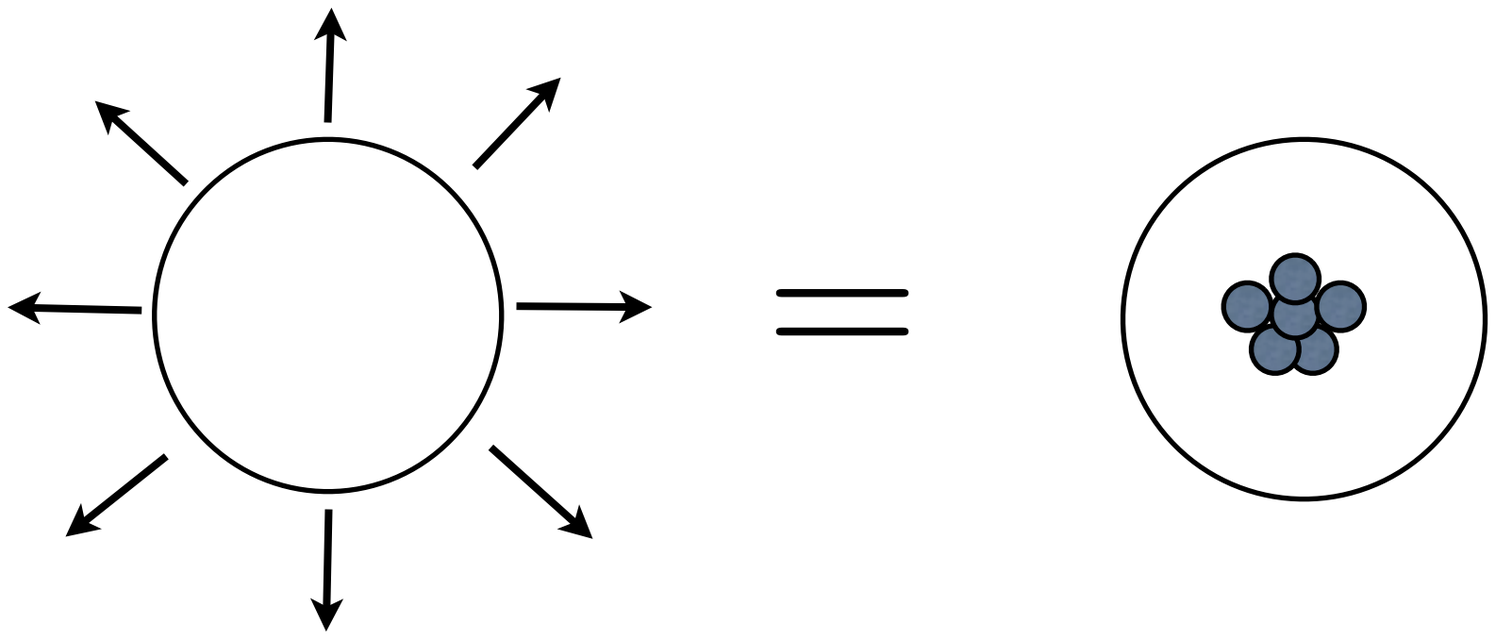

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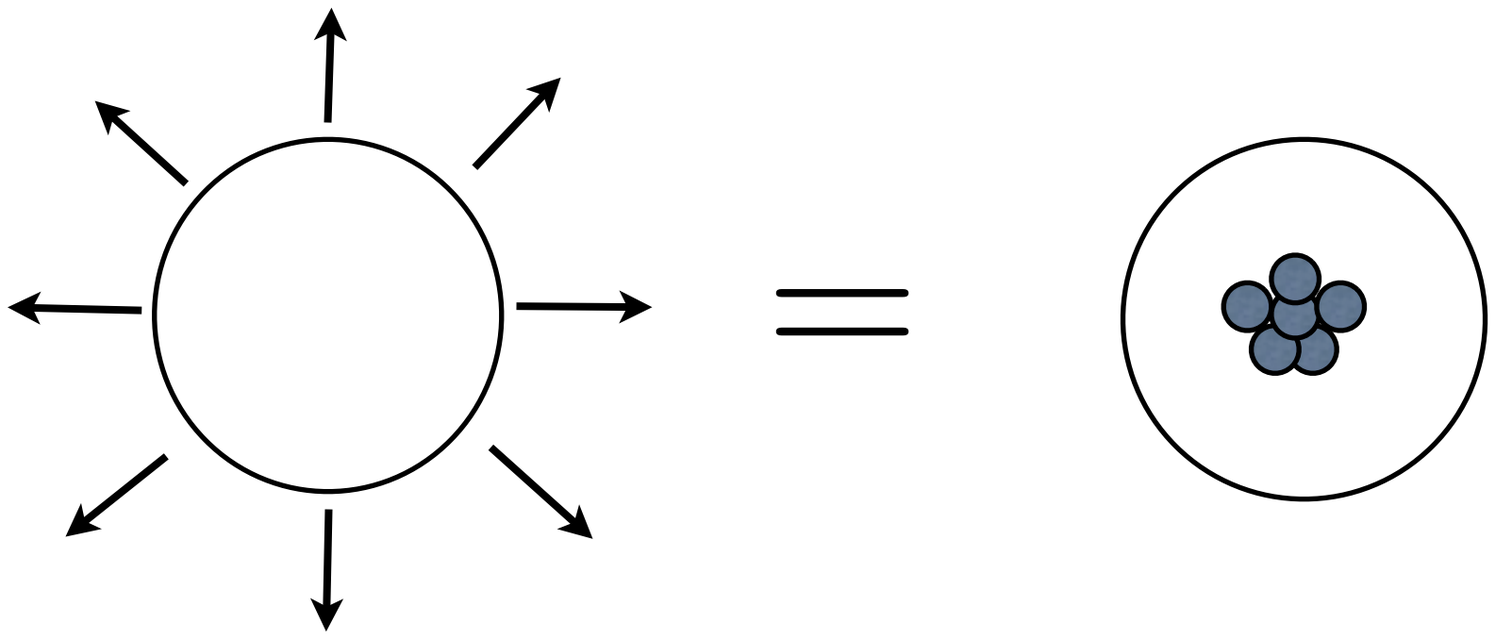

Flux = Charge

- It is a conservation law (non-Noether)
- It underlies gauge theories
- It underlies integrable theories
- Use it to construct the integral eqs. for Yang-Mills, its conserved charges and hidden symmetries in loop space

The $(1+1)$-dimensional case

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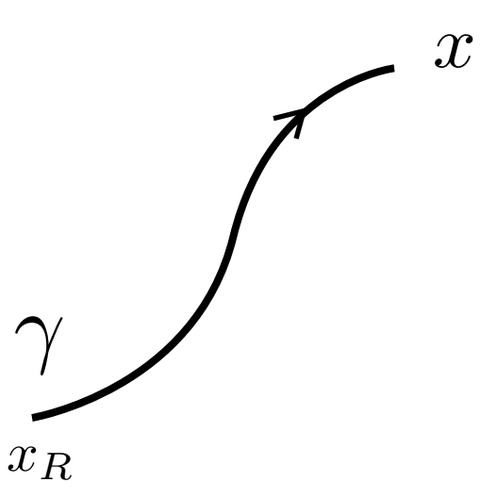

Let $g(x)$ and A_μ be functionals of the fields of the theory and impose on any curve γ the integral equation

$$g\,(x)\;g^{-1}\,(x_R) = P_1\, e^{-\int_\gamma d\sigma\, A_\mu\, \frac{d\,x^\mu}{d\sigma}}$$

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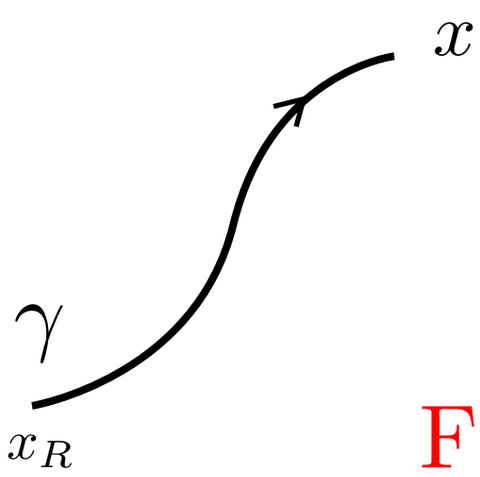

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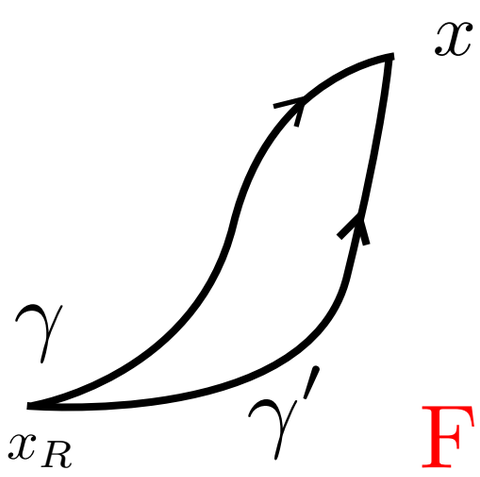

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path independent

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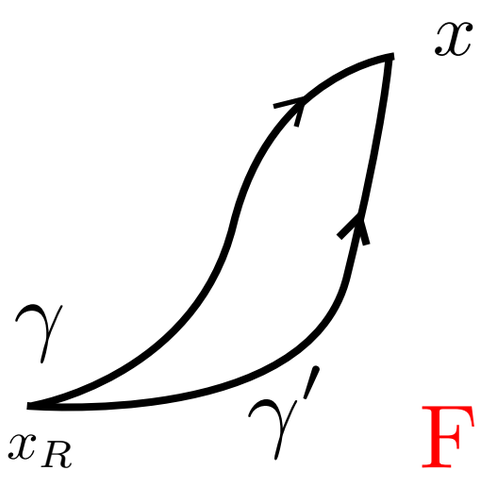

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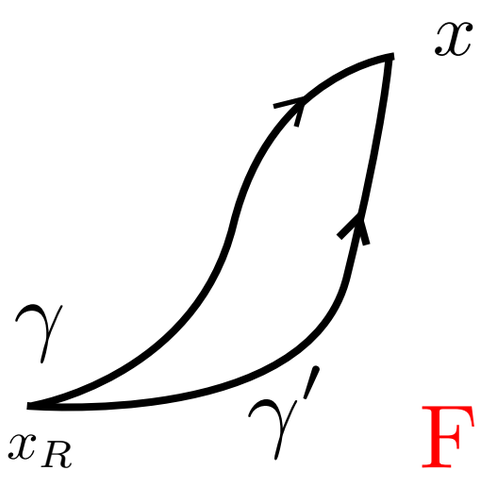

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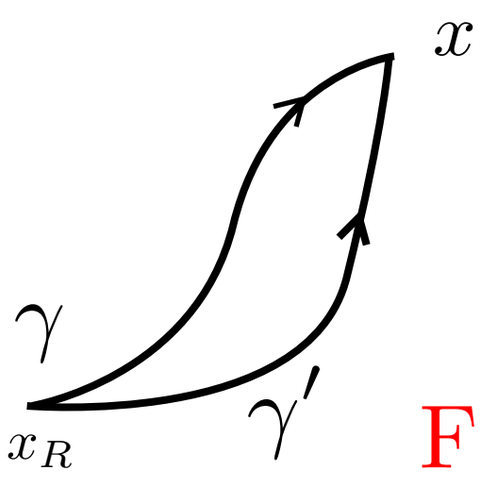

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Lax-Zakharov-Shabat equation

That is what integrable theories are about

The sine-Gordon model

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$$A_0 = \frac{i}{4}\begin{pmatrix} \frac{\partial\theta}{\partial x^1} & \frac{\omega}{c}\left(e^{i\,\theta/2} + \frac{1}{\lambda}\,e^{-i\,\theta/2}\right) \\ \frac{\omega}{c}\left(e^{i\,\theta/2} + \lambda\,e^{-i\,\theta/2}\right) & -\frac{\partial\theta}{\partial x^1} \end{pmatrix}$$

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Note that λ is arbitrary

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hidden symmetries

Loop Algebra $\quad \left[T_i^m\,,\,T_j^n\right] = i\,\varepsilon_{ijk}\,T_k^{m+n} \qquad (T_i^n = \lambda^n\,T_i)$

Kac-Moody Alg. $\quad \left[T_i^m\,,\,T_j^n\right] = i\,\varepsilon_{ijk}\,T_k^{m+n} + C\,m\,\delta_{m+n,0}\,\delta_{i,j}$

Path independency

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$F_{\mu\nu} = 0$ means that $W = P\, e^{-\int_\gamma d\sigma\, A_\mu \frac{d\, x^\mu}{d\, \sigma}}$ is path independent

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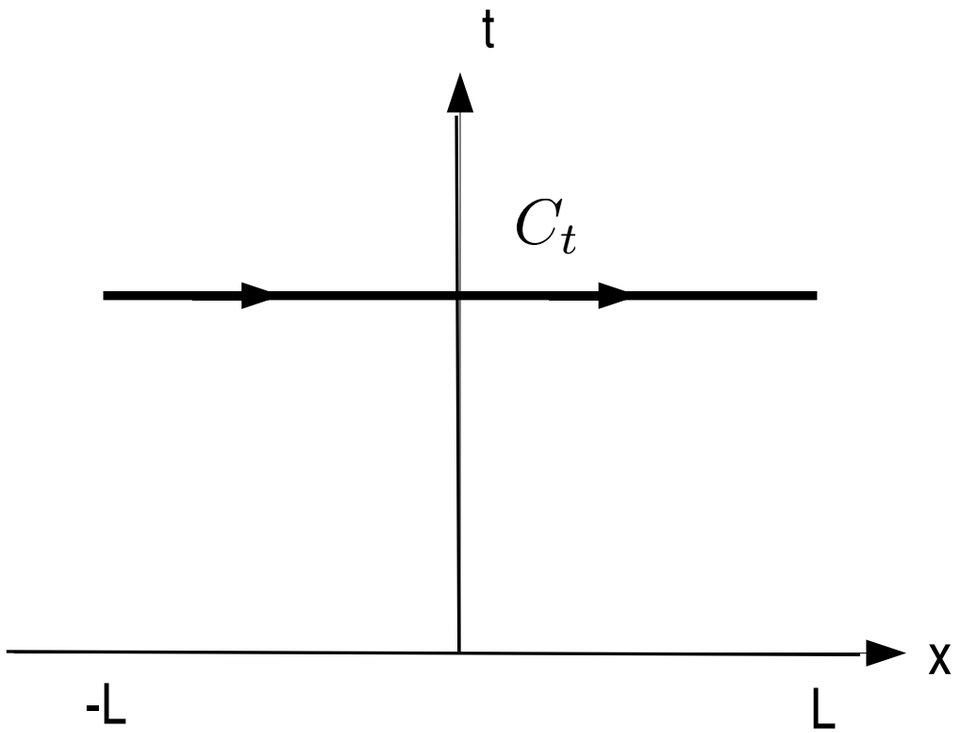

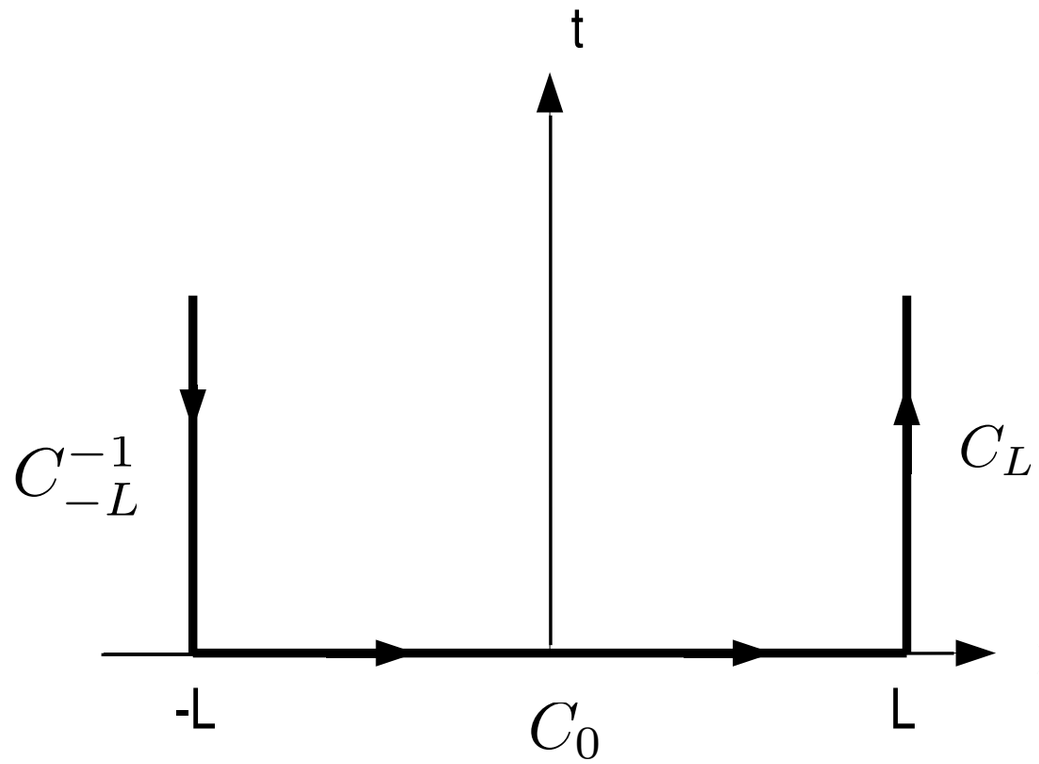

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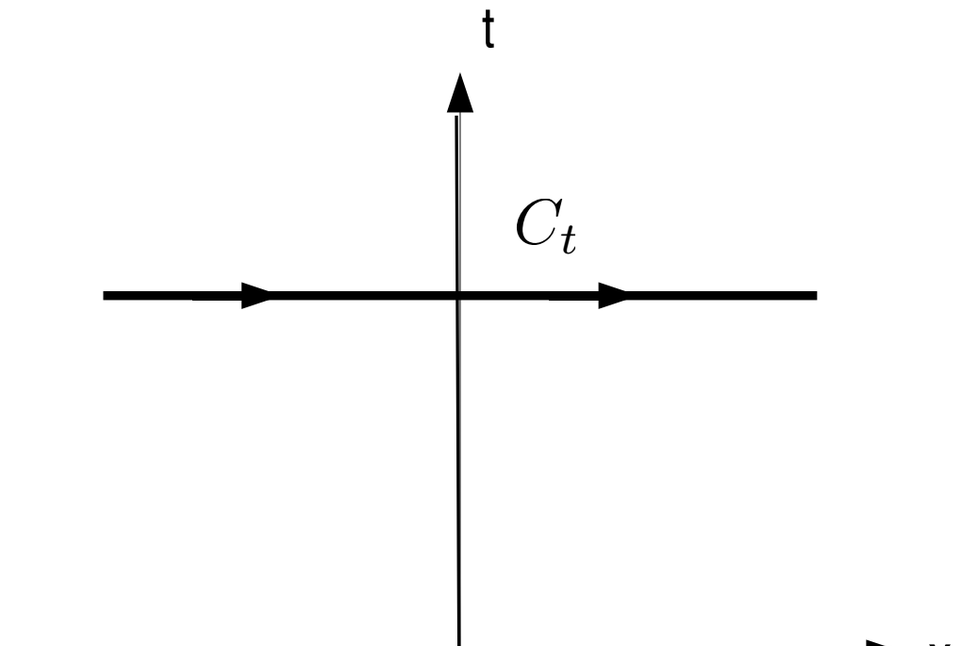

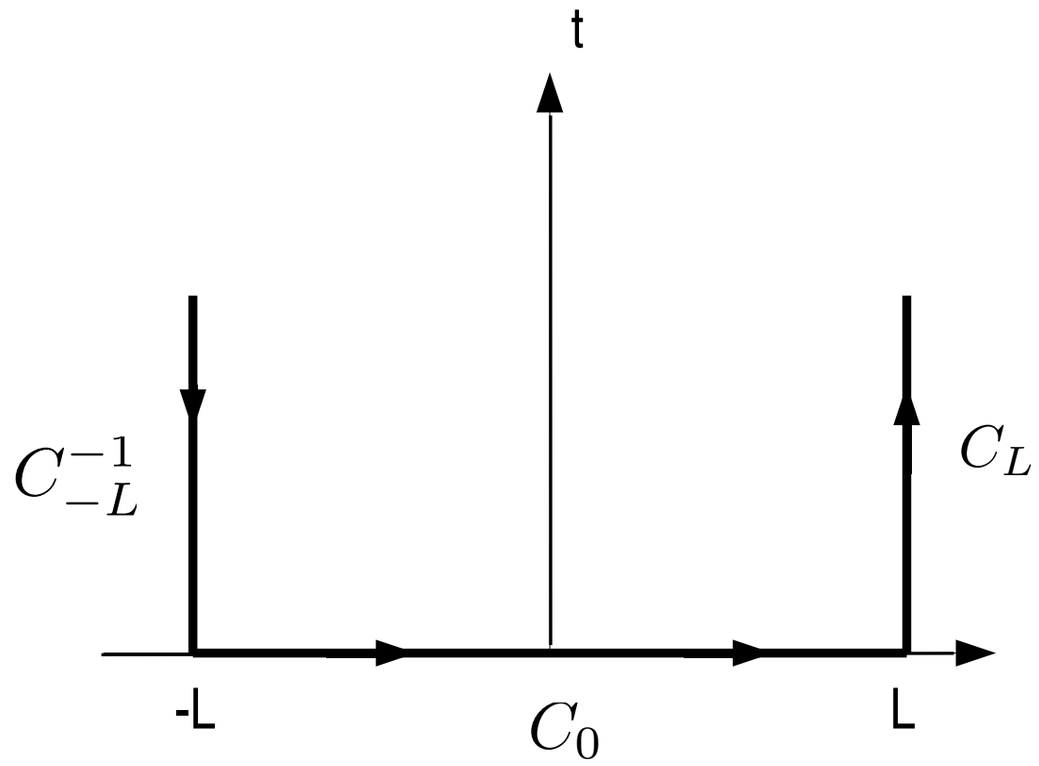

Boundary Condition $A_t(-L, t) = A_t(L, t)$

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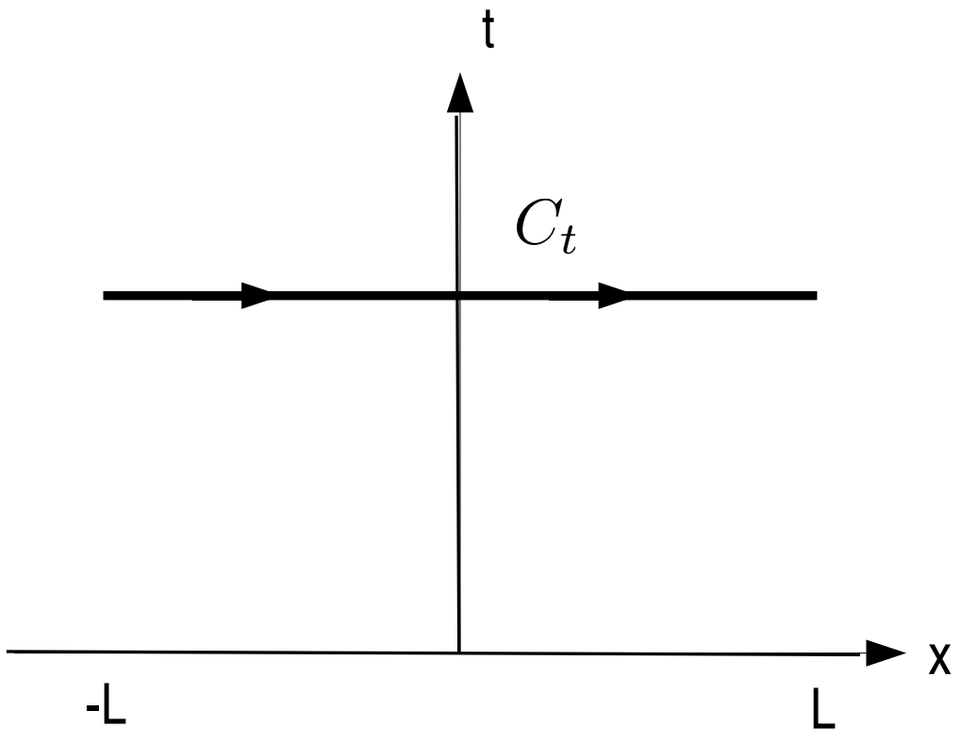

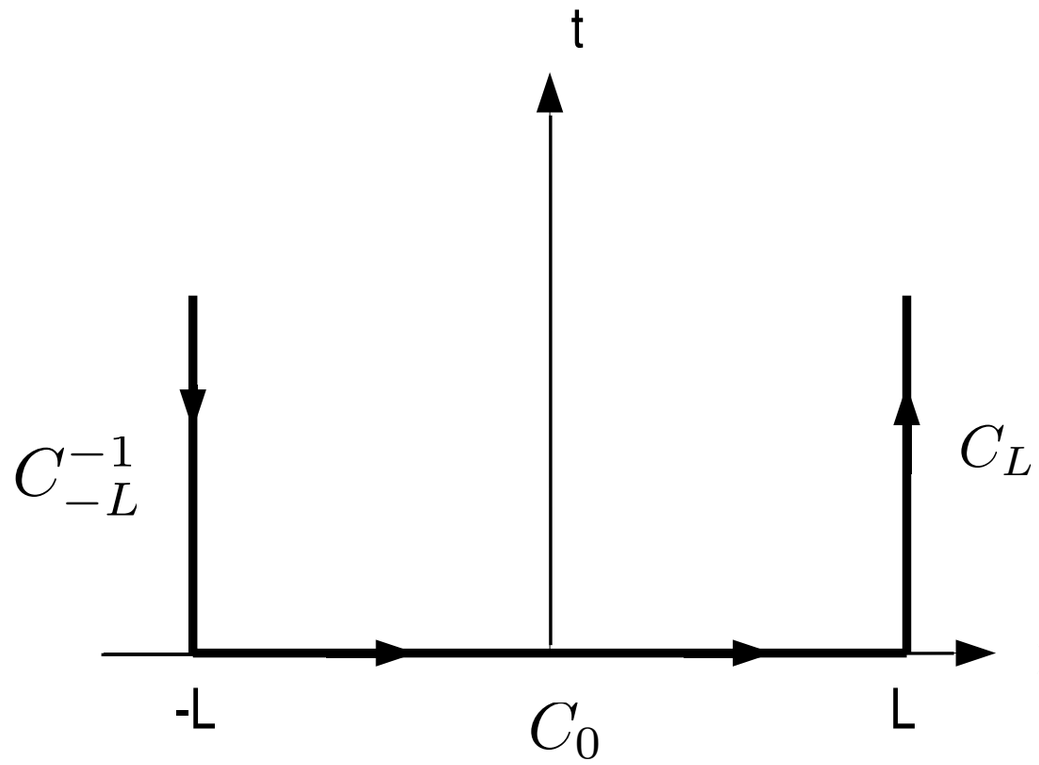

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iso-spectral evolution $\quad W(C_t) = U\, W(C_0)\, U^{-1}$

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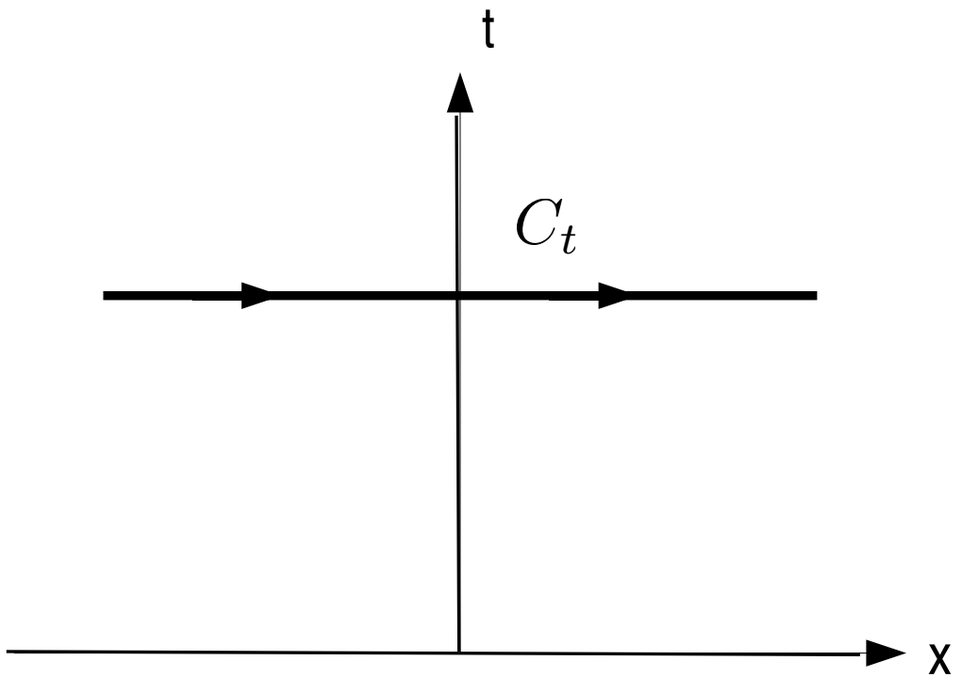

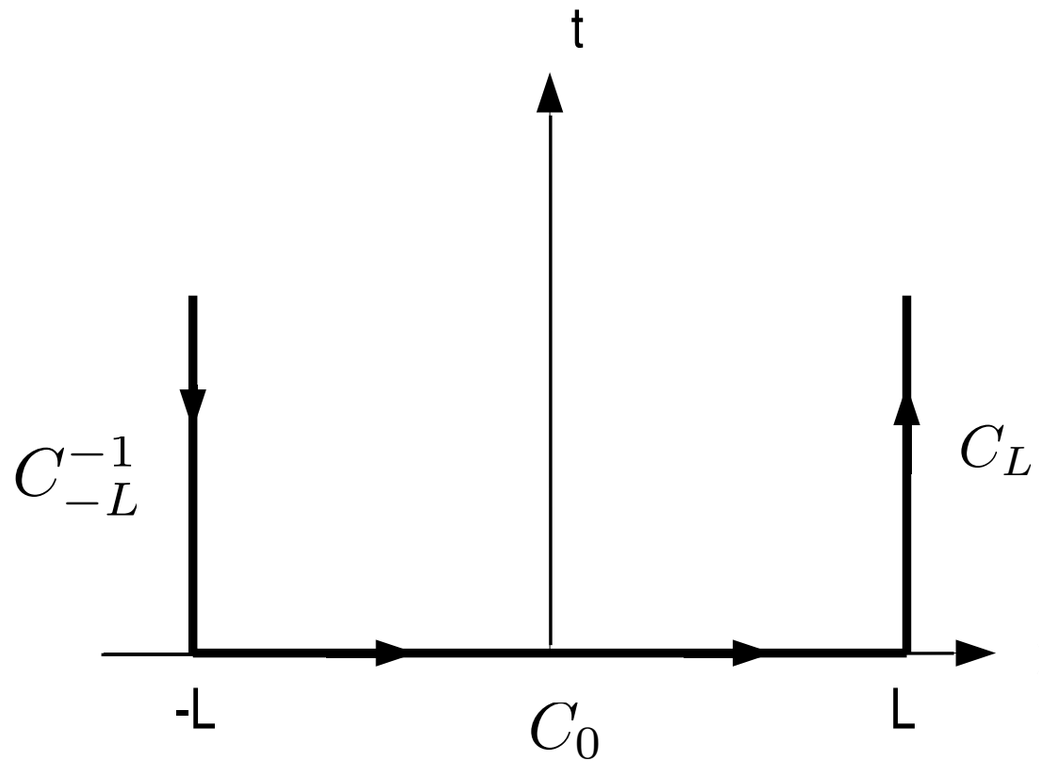

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Eigenvalues of W are conserved in time

power series in λ: infinite number of conserved quantities

The $(2+1)$-dimensional case

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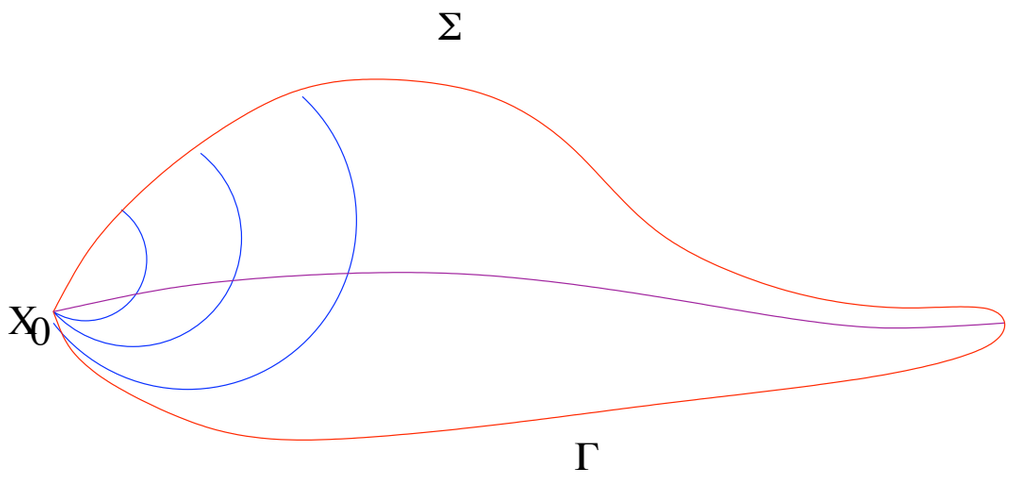
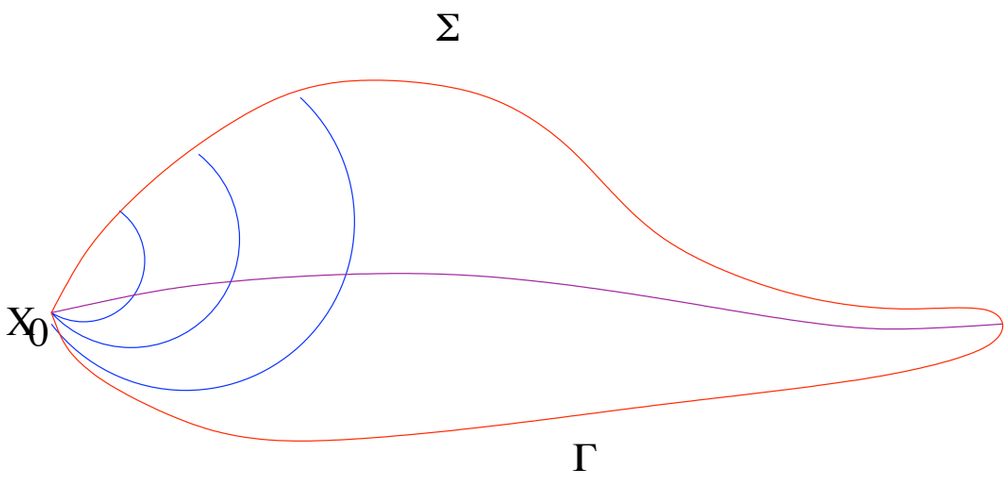
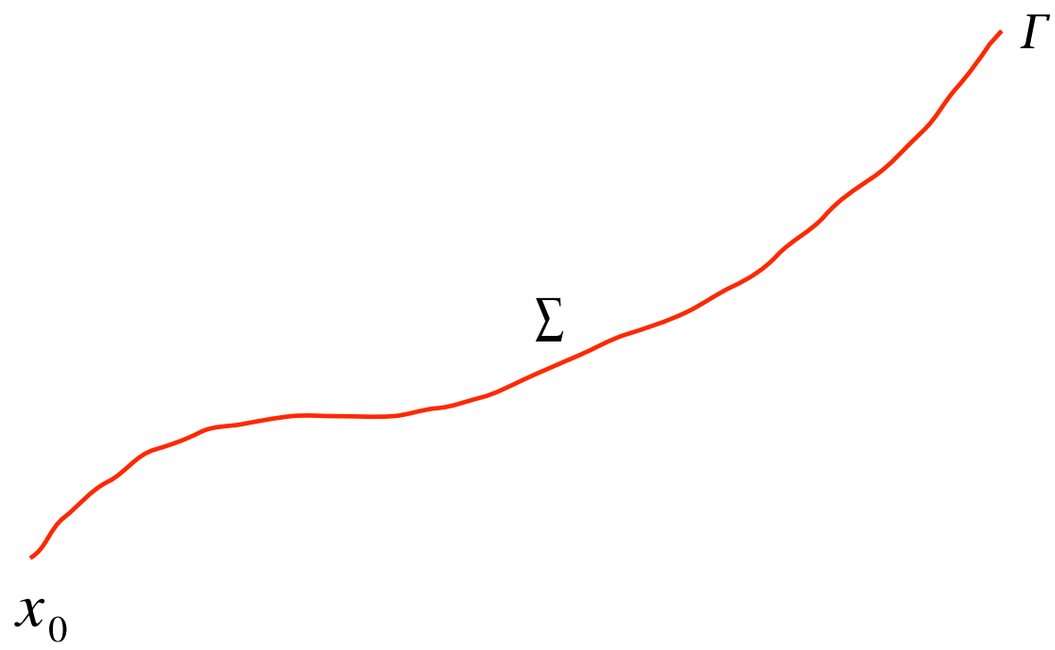


space-time surface

The $(2+1)$-dimensional case





space-time surface $\longleftrightarrow$ path in loop space

Loop Space: $\mathcal{L}^{(1)} = \{ f : S^1 \to M \mid \text{north pole} \to x_0 \}$

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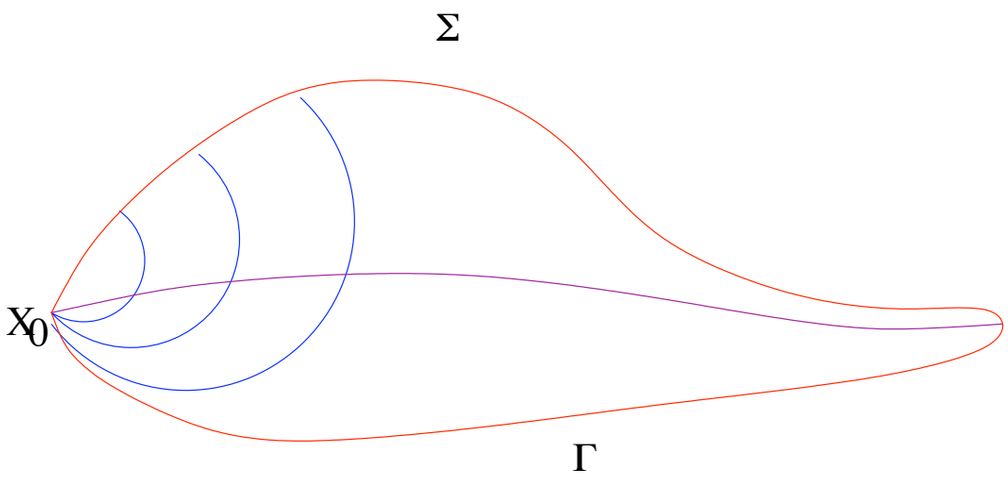

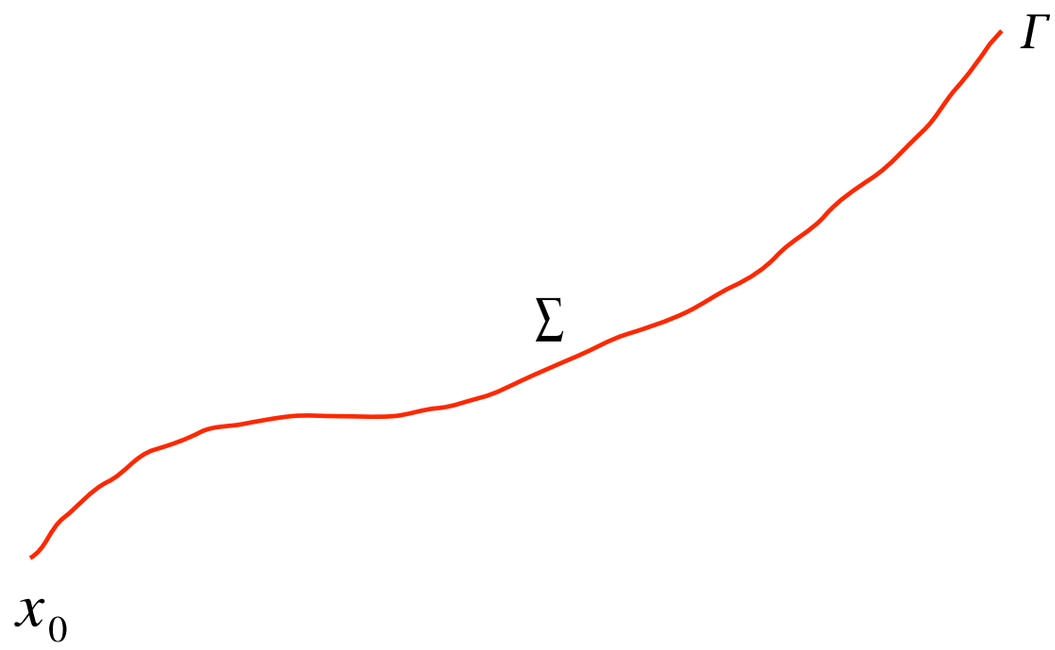

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Introduce a flat connection $\mathcal{A}$ in loop space

$$\mathcal{F} = \delta\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

Construct the charges using path independency!

Not quite that ...

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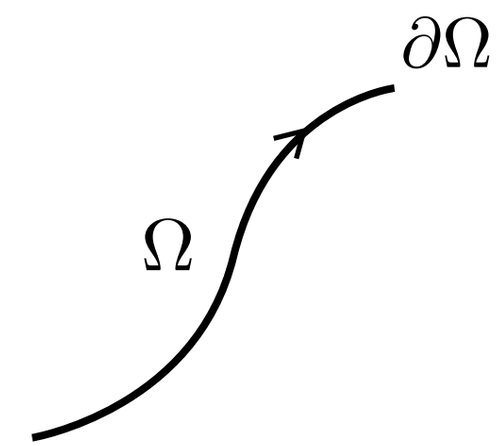
Look for integral equations!!

$$P_{d-1} e^{\int_{\partial\Omega} \mathcal{B}} = P_d e^{\int_{\Omega} \mathcal{A}}$$



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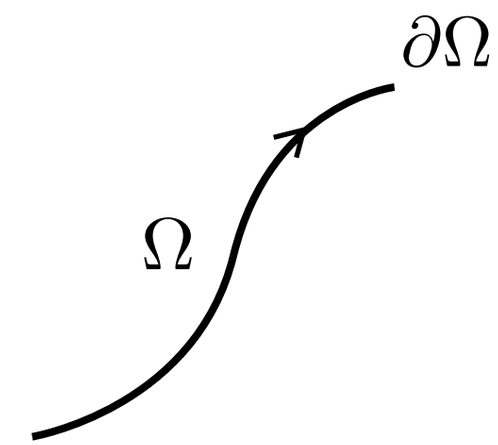

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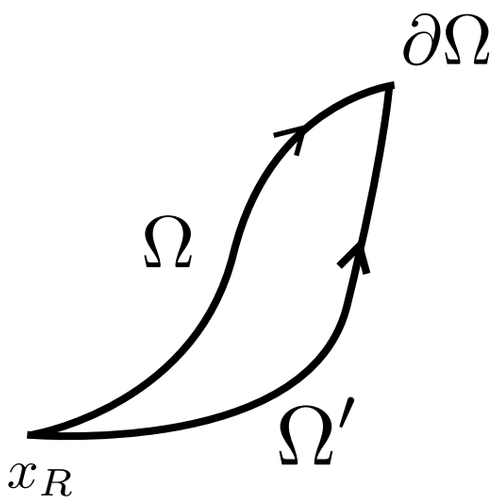

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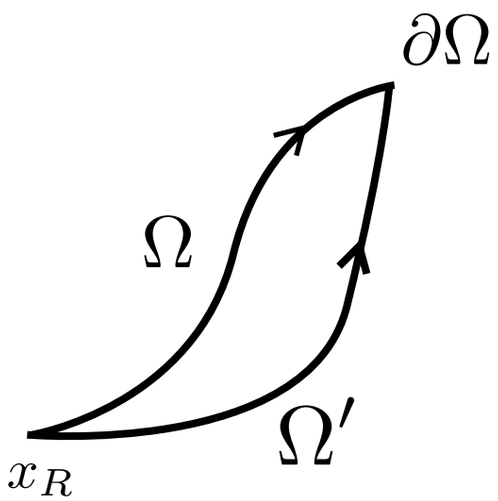

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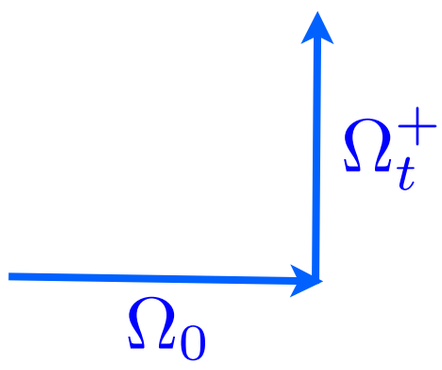

$$P_d e^{\int_{\Omega_t^+} \mathcal{A}} \; P_d e^{\int_{\Omega_0} \mathcal{A}}$$

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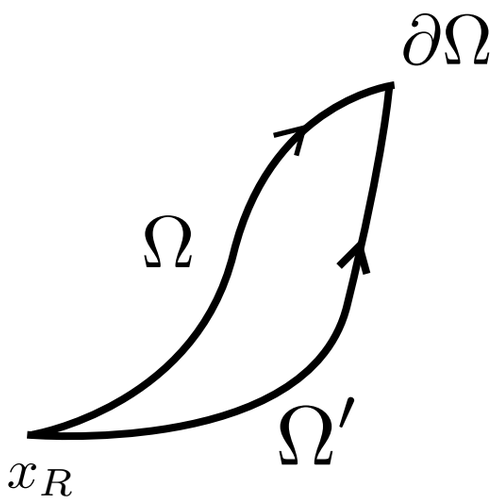

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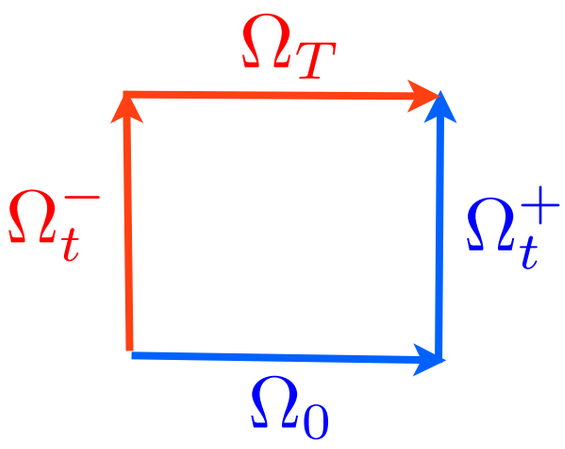

$$P_d e^{\int_{\Omega_T} \mathcal{A}} P_d e^{\int_{\Omega_t^-} \mathcal{A}} = P_d e^{\int_{\Omega_t^+} \mathcal{A}} P_d e^{\int_{\Omega_0} \mathcal{A}}$$

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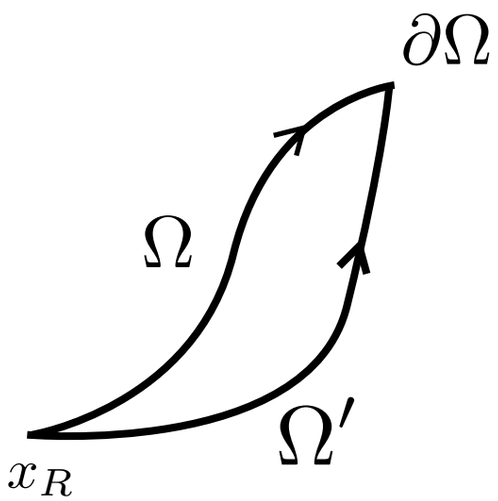

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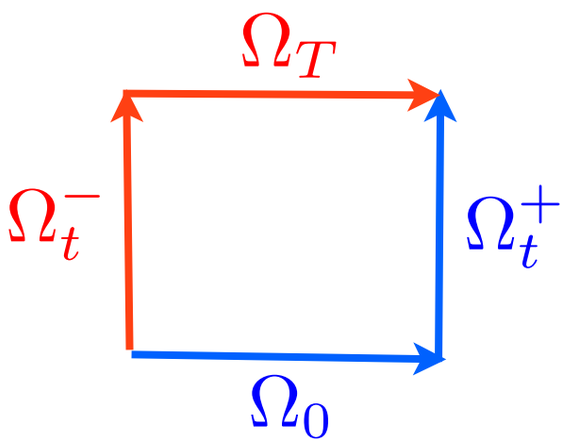
Flux — Charge

Path Independency

$$= P_d e^{\int_{\Omega'} \mathcal{A}}$$

Loop Space: $\mathcal{L}^{(d-1)} = \left\{ f : S^{d-1} \to M \mid \text{north pole} \to x_R \right\}$



$$P_d e^{\int_{\Omega_T} \mathcal{A}} P_d e^{\int_{\Omega_t^-} \mathcal{A}} = P_d e^{\int_{\Omega_t^+} \mathcal{A}} P_d e^{\int_{\Omega_0} \mathcal{A}}$$

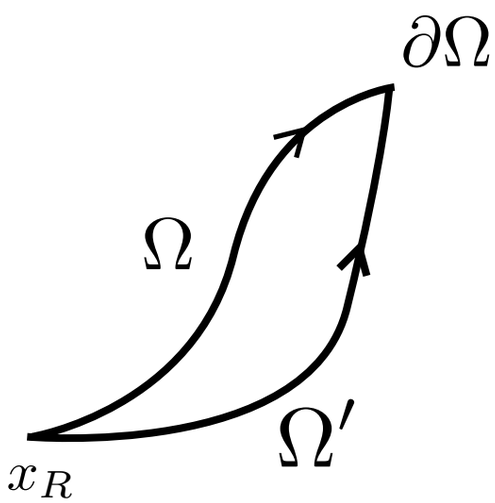
boundary conditions

Not quite that ...

Look for integral equations!!

$$P_{d-1} e^{\int_{\partial\Omega} \mathcal{B}} = P_d e^{\int_\Omega \mathcal{A}}$$

Flux — Charge



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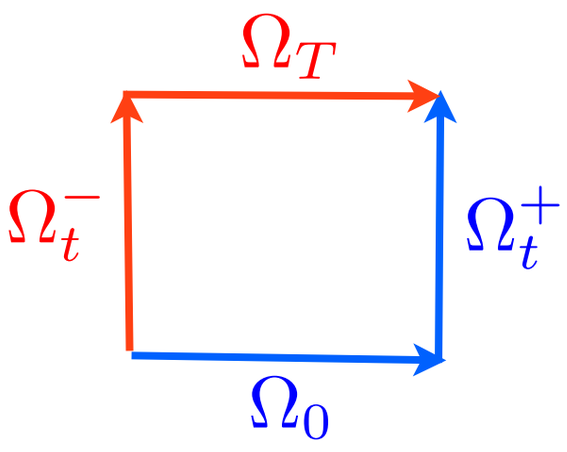


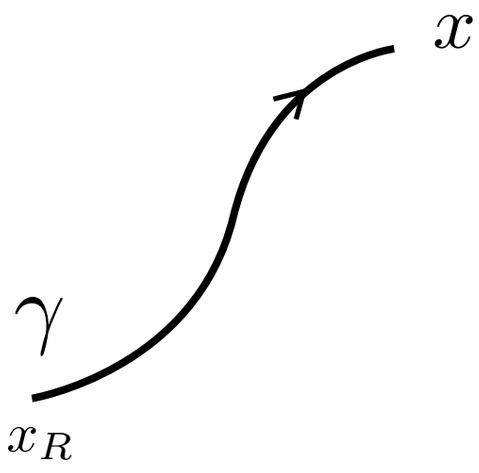
$$P_d e^{\int_{\Omega_T} \mathcal{A}}\, P_d e^{\int_{\Omega_t^-} \mathcal{A}} = P_d e^{\int_{\Omega_t^+} \mathcal{A}}\, P_d e^{\int_{\Omega_0} \mathcal{A}}$$

boundary conditions

iso-spectral evolution $P_d e^{\int_{\Omega_T} \mathcal{A}} = U(T)\, P_d e^{\int_{\Omega_0} \mathcal{A}}\, U(T)^{-1}$

Non-Abelian Stokes Theorem

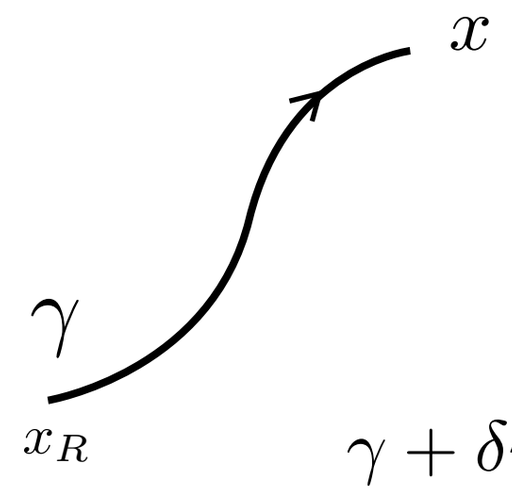
Non-Abelian Stokes Theorem



$$\frac{dW}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma} W = 0$$

$$W(\gamma) = P_1 \, e^{-\int_\gamma A_\mu \frac{dx^\mu}{d\sigma}} \, W_R$$

Non-Abelian Stokes Theorem



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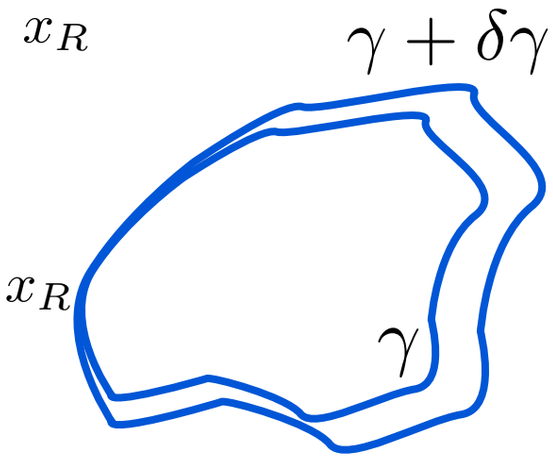

$$W^{-1}(\gamma)\delta W(\gamma) = \int_0^{2\pi} d\sigma \, W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma} \delta x^\nu$$

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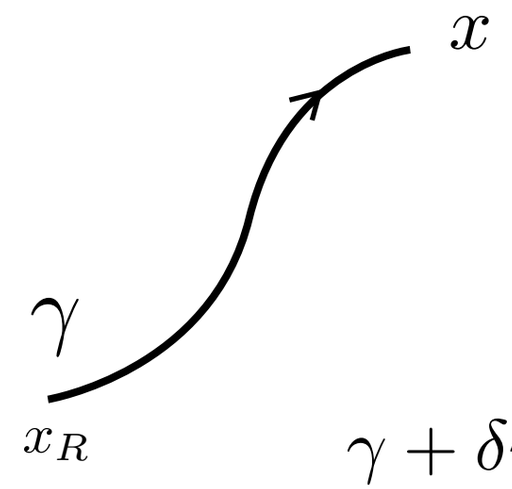

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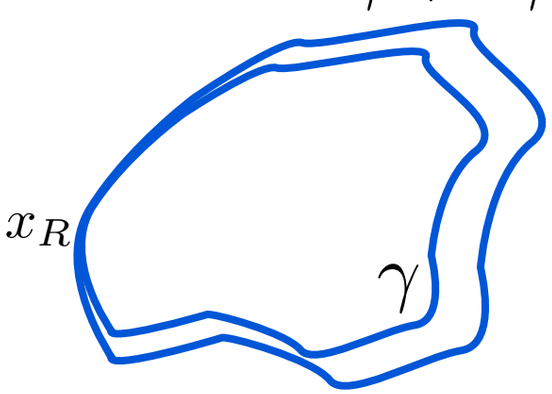

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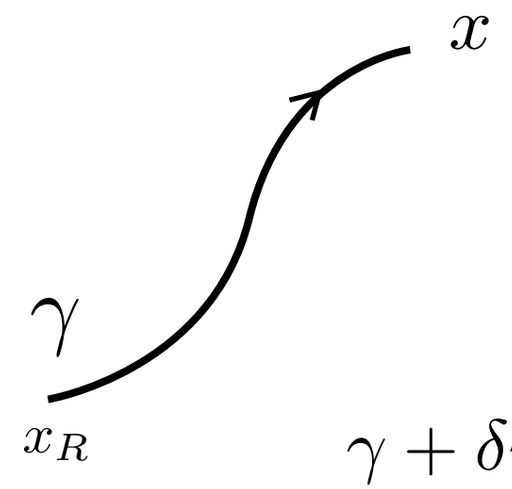
Non-Abelian Stokes Theorem



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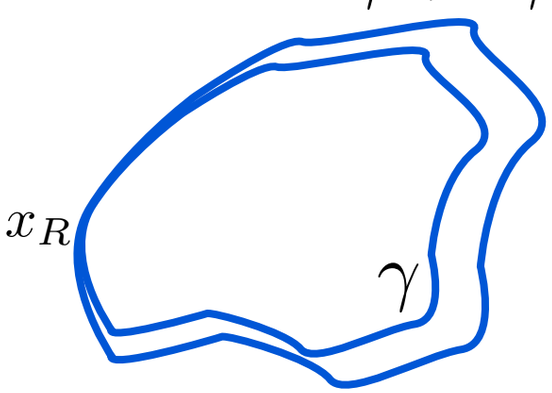

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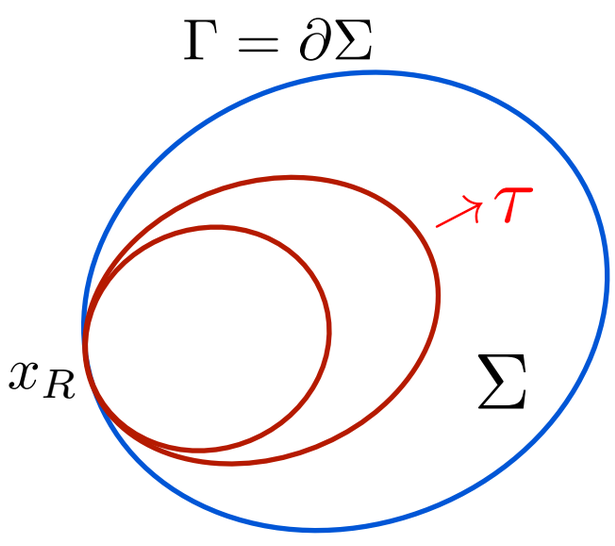
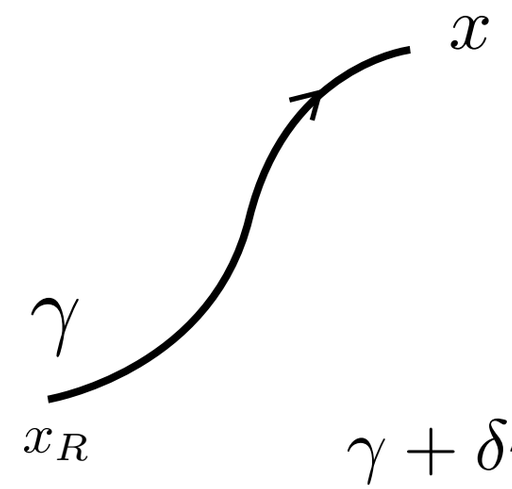


$$W = P_2 \, e^{\int_\Sigma d\sigma d\tau W^{-1} \, F_{\mu\nu} \, W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}}$$

Non-Abelian Stokes Theorem



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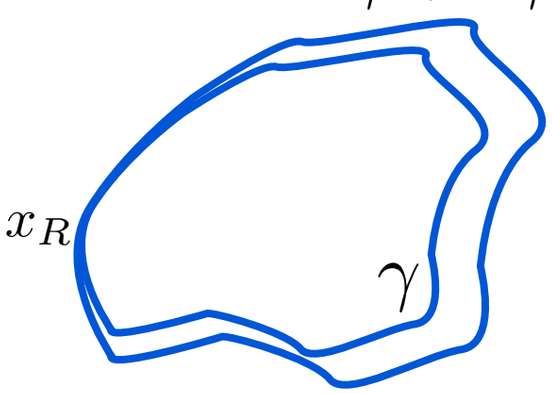

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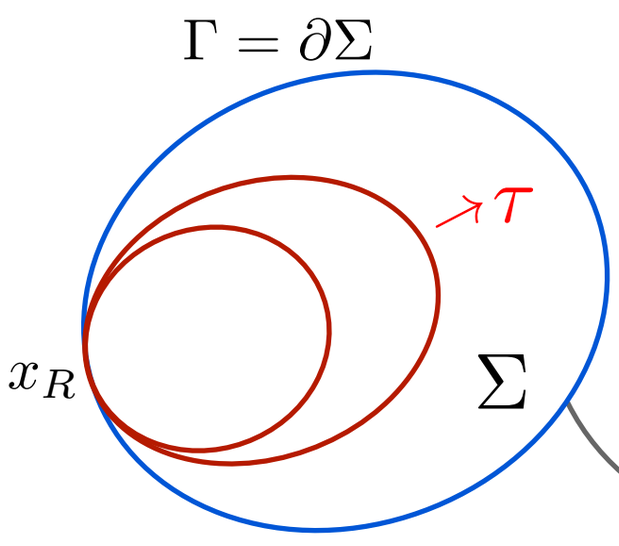
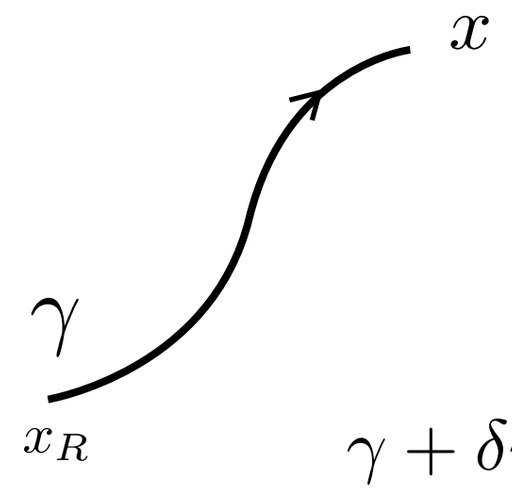


$$W = P_2 \, e^{\int_\Sigma d\sigma d\tau W^{-1} \, F_{\mu\nu} \, W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}}$$

$$W = P_1 \, e^{-\int_\Gamma d\sigma A_\mu \frac{dx^\mu}{d\sigma}}$$

Non-Abelian Stokes Theorem



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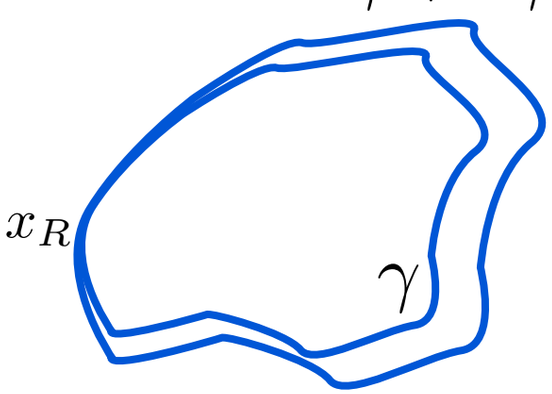

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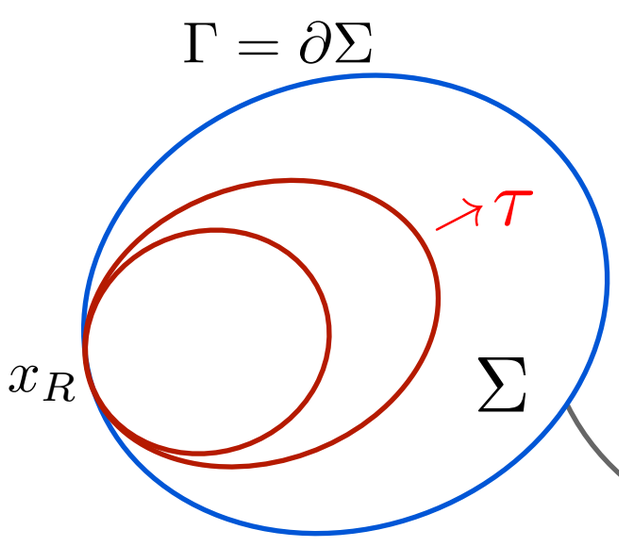

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Generalized Non-Abelian Stokes Theorem

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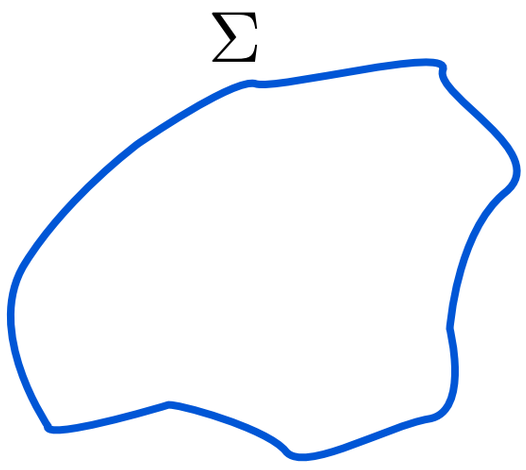
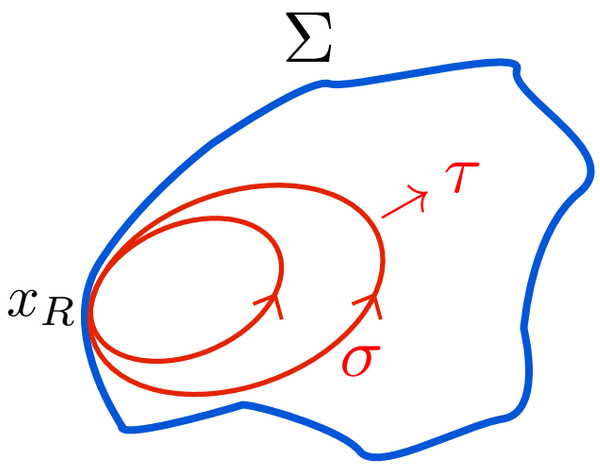


Generalized Non-Abelian Stokes Theorem



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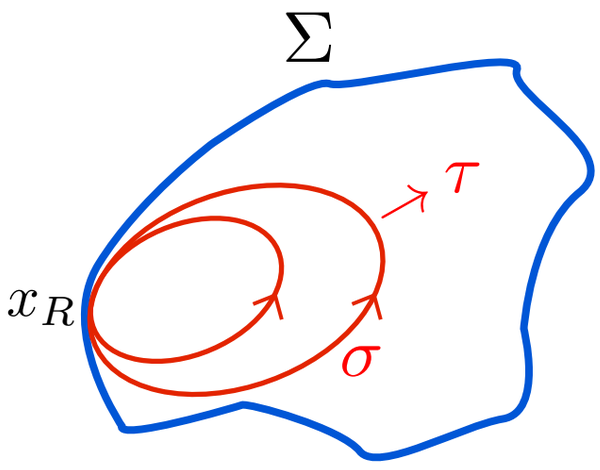
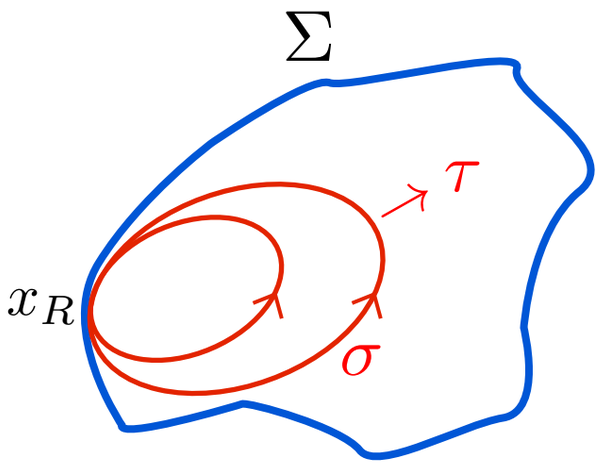


$$\frac{d\,V}{d\,\tau} - V\,T\,(A, B, \tau) = 0$$

$$T\,(B, A, \tau) \equiv \int_0^{2\,\pi} d\sigma\; W^{-1}\,B_{\mu\nu}\,W\,\frac{d\,x^{\mu}}{d\,\sigma}\,\frac{d\,x^{\nu}}{d\,\tau}$$

Generalized Non-Abelian Stokes Theorem



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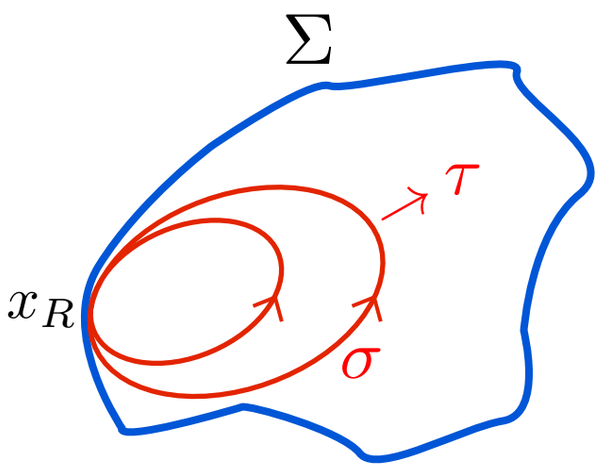
It is a surface ordered integral

$$V(\Sigma) = V_R\; P_2\, e^{\int_\Sigma d\sigma d\tau\; W^{-1}\,B_{\mu\nu}\,W\,\frac{d\,x^{\mu}}{d\,\sigma}\,\frac{d\,x^{\nu}}{d\,\tau}}$$

Generalized Non-Abelian Stokes Theorem



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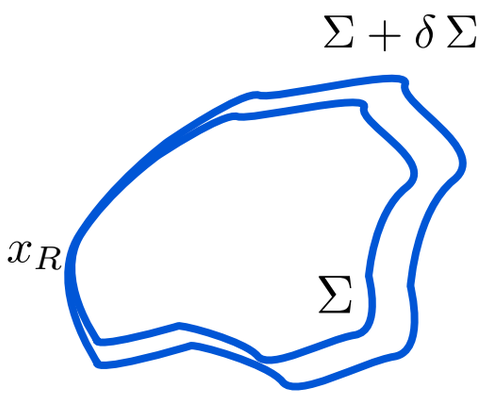
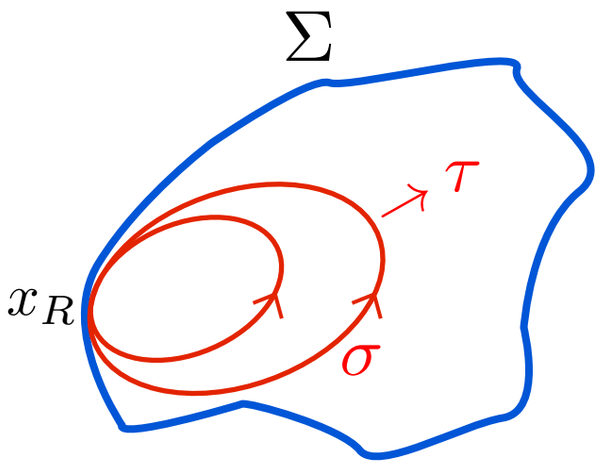
Vary Σ



Generalized Non-Abelian Stokes Theorem



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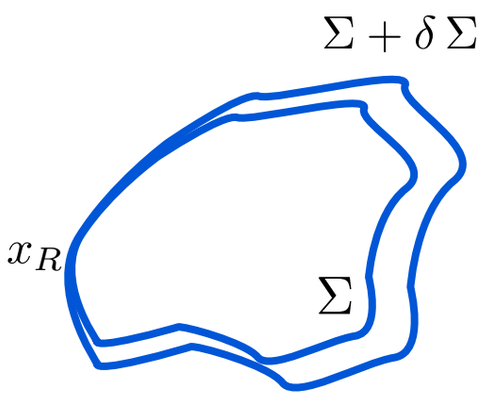

$$\delta V\,V^{-1} \equiv \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\,V\,(\tau)\;\{$$

$$W^{-1}\;\left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right]\,W\frac{d\,x^{\mu}}{d\,\sigma}\;\frac{d\,x^{\nu}}{d\,\tau}\,\delta x^{\lambda}$$

$$-\int_0^{\sigma} d\sigma'\,\left[\,B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\,B^W_{\mu\nu}\,(\sigma)\,\right]\,\frac{dx^{\kappa}}{d\sigma'}\,\frac{dx^{\mu}}{d\sigma}$$

$$\times\left(\frac{d\,x^{\rho}\,(\sigma')}{d\,\tau}\delta x^{\nu}\,(\sigma) - \delta x^{\rho}\,(\sigma')\;\frac{d\,x^{\nu}\,(\sigma)}{d\,\tau}\right)\Bigg\}\,V^{-1}\,(\tau)$$

The generalized non-abelian Stokes Theorem

The generalized non-abelian Stokes Theorem

$$V_R\, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{d\, x^\nu}{d\, \tau}} = P_3 e^{\int_\Omega d\zeta\, \mathcal{K}}\, V_R$$

The generalized non-abelian Stokes Theorem

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$$\mathcal{K} \equiv \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\, V\,(\tau)\, \{$$

$$W^{-1}\, \left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right]\, W \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}\, \frac{d\,x^\lambda}{d\,\zeta}$$

$$- \int_0^\sigma d\sigma'\, \left[\, B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\, B^W_{\mu\nu}\,(\sigma)\,\right] \frac{dx^\kappa}{d\sigma'} \frac{dx^\mu}{d\sigma}$$

$$\times \left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\, \frac{d\,x^\nu\,(\sigma)}{d\,\zeta} - \frac{d\,x^\rho\,(\sigma')}{d\,\zeta}\, \frac{d\,x^\nu\,(\sigma)}{d\,\tau} \right) \Bigg\}\, V^{-1}\,(\tau)$$

The generalized non-abelian Stokes Theorem

$$V_R\, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{d\,x^\nu}{d\,\tau}} = P_3 e^{\int_\Omega d\zeta\, \mathcal{K}}\, V_R$$

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$$W^{-1}\, [D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}]\, W \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}\, \frac{d\,x^\lambda}{d\,\zeta}$$

$$- \int_0^\sigma d\sigma' \left[\, B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\, B^W_{\mu\nu}\,(\sigma)\,\right] \frac{dx^\kappa}{d\sigma'} \frac{dx^\mu}{d\sigma}$$

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O. Alvarez, L. A. Ferreira and J. Sanchez Guillen,
Nucl. Phys. B **529**, 689 (1998) [arXiv:hep-th/9710147].
Int. J. Mod. Phys. A **24**, 1825 (2009) [arXiv:0901.1654 [hep-th]]

The Integral Equations for Yang-Mills

The Integral Equations for Yang-Mills

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

The Integral Equations for Yang-Mills

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu}+\beta \widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

$$\mathcal{J} \equiv \int_0^{2\pi} d\sigma \left\{ ie\beta \widetilde{J}^W_{\mu\nu\lambda} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2 \int_0^\sigma d\sigma' \right.$$
$$\times \left[\left((\alpha-1) F^W_{\kappa\rho} + \beta \widetilde{F}^W_{\kappa\rho} \right)(\sigma'), \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu} \right)(\sigma) \right]$$
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The Integral Equations for Yang-Mills

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Direct consequence of Stokes theorem and Yang-Mills eqs.

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Implies Yang-Mills eqs. in the limit $\Omega \to 0$

Conserved Charges

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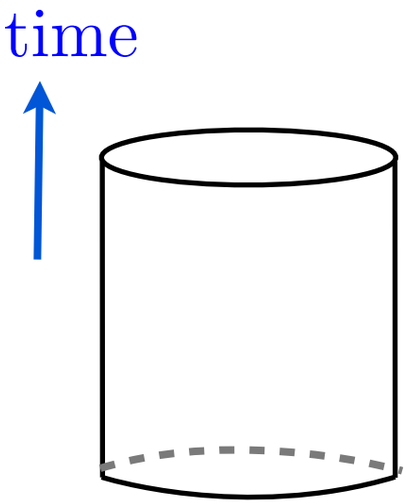

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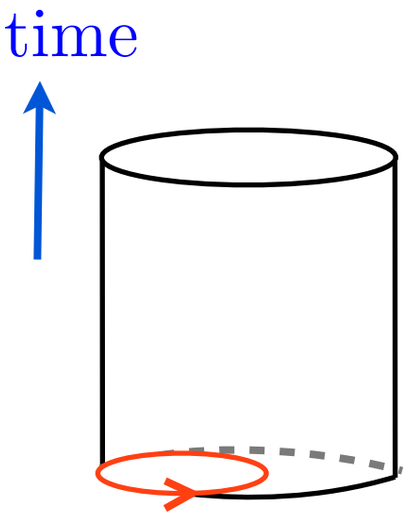

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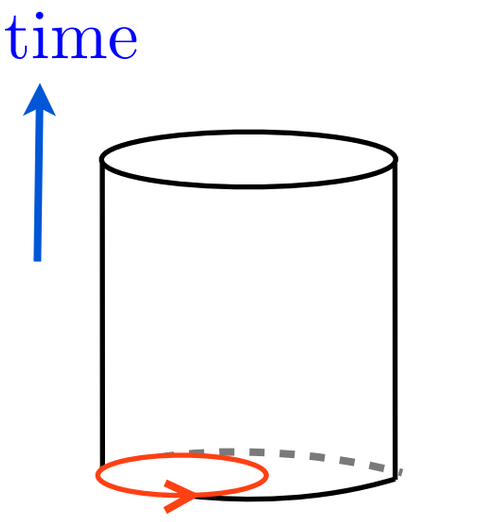

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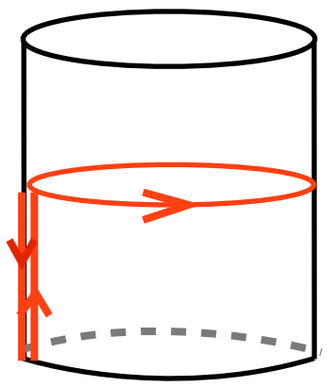

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Conserved Charges

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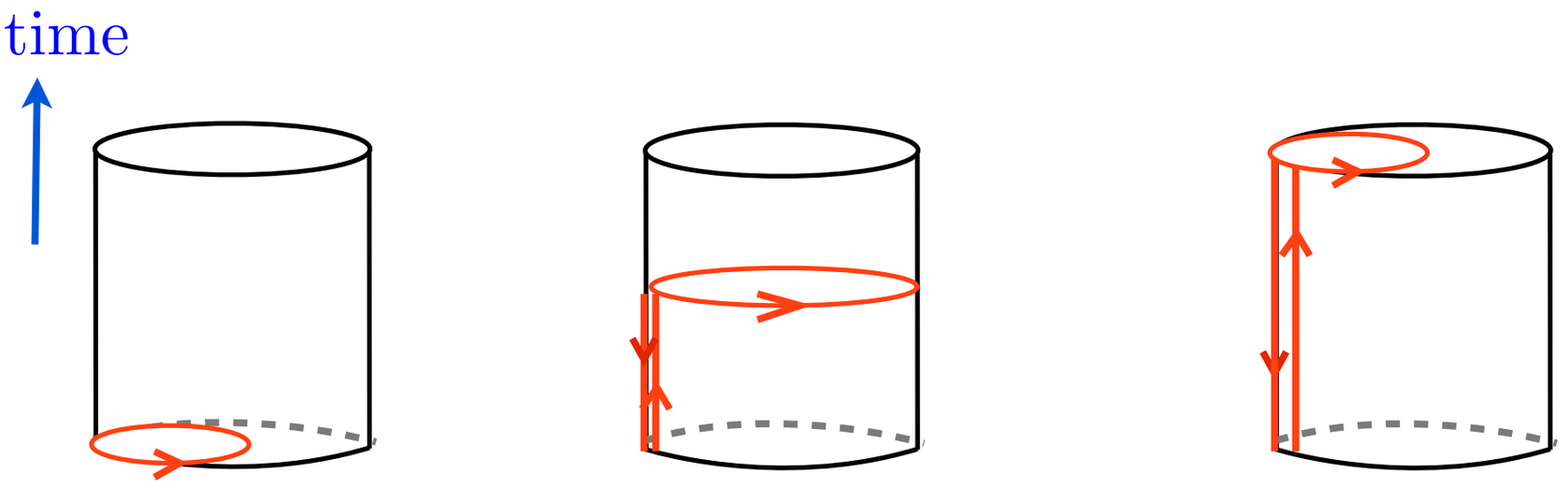

$P_3 e^{\int_{\Omega_0} d\zeta d\tau V \mathcal{J} V^{-1}}$ $P_3 e^{\int_{S^2_\infty \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$ $P_3 e^{\int_{\Omega_t^{-1}} d\zeta d\tau V \mathcal{J} V^{-1}}$

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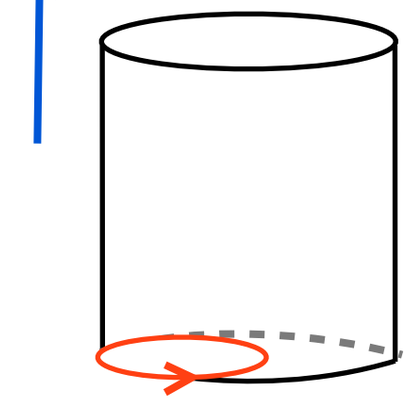
time



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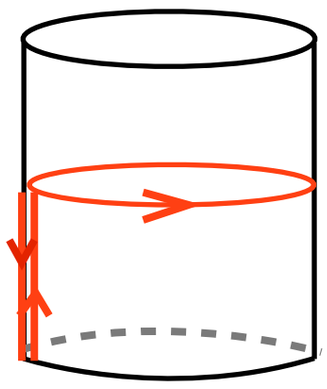

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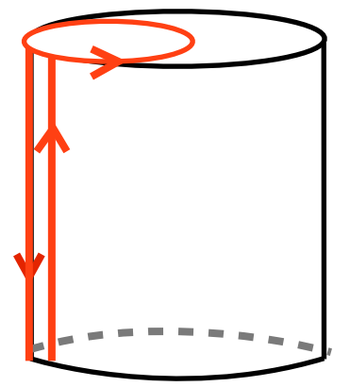

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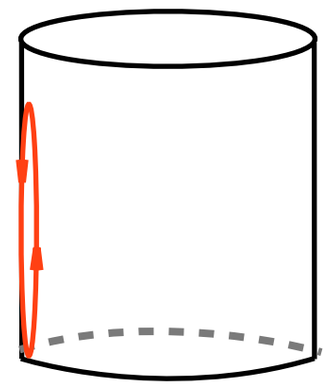

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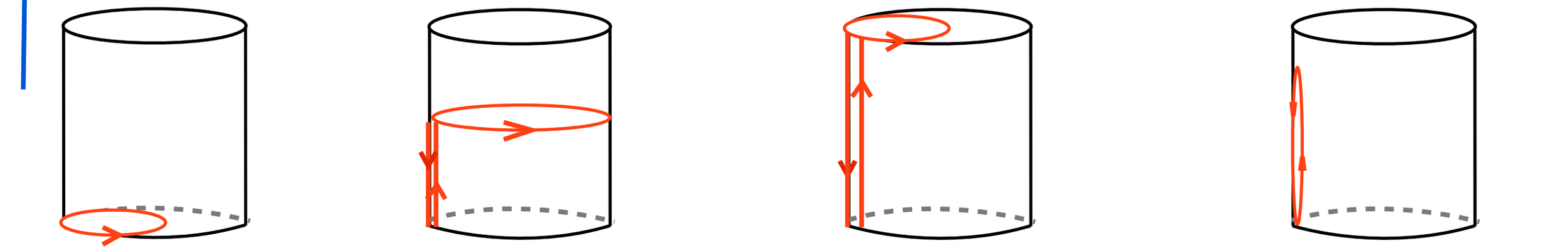

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Iso-spectral evolution:

$$V(\Omega_t) = U(t) \cdot V(\Omega_0) \cdot U^{-1}(t)$$

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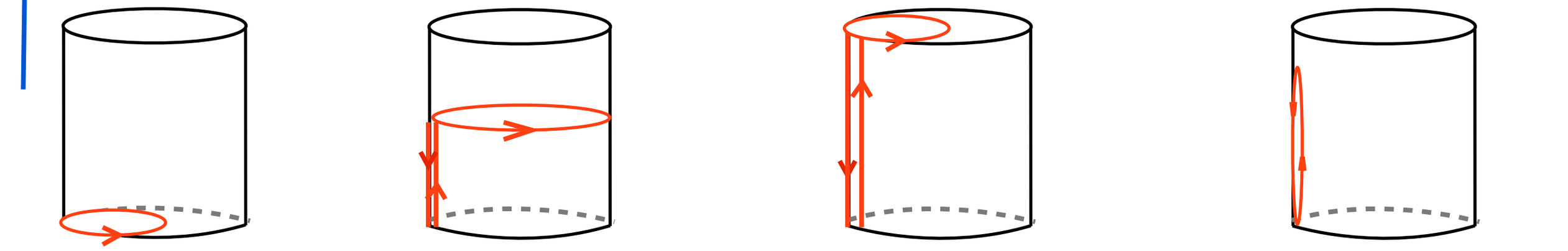

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Iso-spectral evolution: $\boxed{V(\Omega_t) = U(t) \cdot V(\Omega_0) \cdot U^{-1}(t)}$

Eigenvalues of $V(\Omega_t)$ are constant in time

Conserved charges are eigenvalues of the operator

$$V(\Omega_t) = P_2\; e^{ie\int_{\mathcal{S}^{2,(t)}_{\infty}} d\tau\; d\sigma\; \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right) \frac{dx^\mu}{d\sigma}\, \frac{dx^\nu}{d\tau}} = P_3\; e^{\int_{\Omega_t} d\zeta\; d\tau\; V \mathcal{J} V^{-1}}$$

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5) Relevant for the global aspects of Yang-Mills theory

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At spatial infinity the magnetic field has the form:

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In such a case the charge operator becomes:

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• It paves the way to interesting new developments

Thank You

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