Self-dual (BPS) impurities (background fields)

A new tool for studying of soliton dynamics

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contents

- self-dual (BPS) models in (1+1) dim
  - **trivial moduli space $\mathcal{M}$**

- **self-dual (BPS) preserving impurity** in (1+1) dim
  - **spectral structure depends on a position on the moduli space $\mathcal{M}$**
    - no static force between solitons
    - static multi-soliton solutions
    - soliton-mode interactions

- applications
  - interaction of kinks: soliton-mode interaction $\rightarrow$ spectral walls
  - kink-antikink annihilation $\rightarrow$ zero static force limit
  - beyond geodesic flow approximation

  - SD impurities in higher dim, cond-mat....

self-dual (BPS) models in (1+1) dim

an example and definition

- the canonical scalar model $\phi(x,t)$ in 1+1 dimensions

  - **static energy**

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\phi_x^2 + U(\phi) \right]$$

  U two vacuum potential: $U(\phi_{\pm}^v) = 0$ and $\phi_+^v > \phi_-^v$

  - **topological charge**

$$Q = \frac{\phi(+\infty) - \phi(-\infty)}{\phi_+^v - \phi_-^v} \in \mathbb{Z}$$

  from the conserved topological current: $\partial_\mu j^\mu = 0$, $j^\mu = (\phi_+^v - \phi_-^v)^{-1}\epsilon^{\mu\nu}\partial_\nu\phi$

  - **topological solitons** = kinks

  finite energy solutions with nonzero topological charge

  - **EL**

$$\delta E = 0 \quad \Rightarrow \quad \phi_{xx} - U_\phi = 0$$

- the canonical scalar model $\phi(x,t)$ in 1+1 dimensions - **self-dual sector**
  - **topological bound**

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dx \left(\frac{1}{\sqrt{2}} \phi_x \pm \sqrt{U} \right)^2 \mp \sqrt{2} \int_{-\infty}^{\infty} dx \sqrt{U} \phi_x \\ &\geq \sqrt{2} \left| \int_{-\infty}^{\infty} dx \sqrt{U} \phi_x \right| = \sqrt{2} \left| \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi \sqrt{U} \right| \\ &= \sqrt{2} (\phi_+^v - \phi_-^v) \left\langle \sqrt{U} \right\rangle |Q| \end{aligned}$$

where

$$\left\langle \sqrt{U} \right\rangle = \frac{1}{\phi_+^v - \phi_-^v} \int_{\phi_-^v}^{\phi_+^v} d\phi \sqrt{U} \equiv \frac{1}{\phi_+^v - \phi_-^v} (F(\phi_+^v) - F(\phi_-^v))$$

  - **Bogomolny eq.** - saturation of the bound

$$\frac{1}{\sqrt{2}} \phi_x \pm \sqrt{U} = 0$$

  - **BOG $\Rightarrow$ EL**

$$0 = \partial_x \left(\frac{1}{\sqrt{2}} \phi_x \pm \sqrt{U} \right) = \frac{1}{\sqrt{2}} \phi_{xx} \pm \frac{1}{2\sqrt{U}} U_\phi \phi_x = \frac{1}{\sqrt{2}} \left(\phi_{xx} - U_\phi \right)$$

  - **BOG = zero pressure eq.**

  - **SD** sols = lower order sols, saturate the bound $\Rightarrow$ top. stability
    energy fixed by asymptotic

BPS-ness = a generic feature in (1+1) dim

- trivial moduli space (1 dim target space)
  energy equivalent solutions **only** in one soliton sector
  flow on the moduli = translation
  no spectral structure flow on the moduli

kink-(anti)kink scattering = completely non-BPS process

- no moduli space description
- very complicated dynamics
  static force between solitons
  interactions between a soliton and internal modes
  radiation

  mixed up.... $\rightarrow$ $K\bar{K}$ scattering in ϕ^4 not explained

the BPS property useless in (1+1) dim

(half of) **self-duality preserving defects**

- destroying the BPS-ness

  - **breaking translational invariance** $\rightarrow$ **impurity**

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\phi_x^2 + U(\phi) + \phi\sigma(x) \right]$$

$\sigma(x)$ spatially localized defect

  - **EL**

$$\phi_{xx} - U_\phi - \sigma(x) = 0$$

  - **zero pressure configuration** $\not\Rightarrow$ **EL**

$$T^{11} = \frac{1}{2}\phi_x^2 - U(\phi) - \phi\sigma(x) \Rightarrow \phi_x \left(\phi_{xx} - U_\phi - \sigma(x)\right) = \phi\sigma_x$$

the static equation of motion $\Rightarrow \phi\sigma_x = 0$

impurity breaks the self-duality of the static e.o.m.

no moduli space

- restoring the BPS-ness with an impurity - **self-dual defects**

Adam, Wereszczynski PRD (18); Adam, Romanczukiewicz, Wereszczynski JHEP (19)

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\phi_x^2 + U(\phi) + 2\sigma W(\phi) + \sqrt{2}\sigma\phi_x \right] + \int_{-\infty}^{\infty} dx \sigma^2$$

where $U = W^2$

- **impurity terms**

$\sigma(x) W \longrightarrow$ typical impurity coupling

$\sigma(x)\phi_x \longrightarrow$ local version of the topological term

$\sigma\phi_x \neq \partial(\sigma\phi)$ as $\sigma = \sigma(x)$

$\sigma^2(x) \longrightarrow$ sets the energy zero, non-dynamical term

- **unrestricted impurity**

$\sigma(x) \longrightarrow$ any spatial distribution

- **unrestricted potential**

- **even more possibilities** Adam, Queiruga, Wereszczynski JHEP (19)

$\rightarrow$ see below: solvable self-dual impurities

- **topological bound**

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} dx \left(\frac{1}{\sqrt{2}} \phi_x + (\sigma + W) \right)^2 - \sqrt{2} \int_{-\infty}^{\infty} dx \phi_x W \\
&\geq -\int_{-\infty}^{\infty} dx \phi_x \sqrt{2} W = -Q \int_{\phi_-^v}^{\phi_+^v} d\phi \sqrt{2} W
\end{aligned}
$$

- **Bogomolny eq.** only one!

$$
\frac{1}{\sqrt{2}} \phi_x + W + \sigma = 0
$$

saturates the bound

equivalent to the zero pressure condition

- **EL eq.** $W = \sqrt{U}$

$$
\phi_{xx} - U_\phi - \sigma \frac{U_\phi}{\sqrt{U}} + \sqrt{2} \sigma_x = 0
$$

BOG eq. $\Rightarrow$ EL eq.

$$
\frac{1}{\sqrt{2}} \phi_{xx} + \sigma_x + \frac{1}{2\sqrt{U}} U_\phi \phi_x = 0 \quad \Rightarrow \quad \phi_{xx} + \sqrt{2} \sigma_x - \frac{U_\phi}{\sqrt{U}} (\sigma + \sqrt{U}) = 0
$$

- **BPS and non-BPS soliton asymmetry**

non-BPS solitons kink if $W = \sqrt{U}$

solves full EL eq.

non-BPS solution

interacts with the impurity: attraction or repulsion

BPS solitons antikink if $W = \sqrt{U}$

$\rightarrow$ restoration of self-duality in $Q = -1$ sector

solves the BOG eq.

static: does not interact with the impurity

any position of the BPS antikink w.r.t. the impurity - energetically equivalent

lumps

$\rightarrow$ restoration of self-duality in $Q = 0$ sector

lump - antikink binding energy $E_B = 0$

- generalized translation

T : BPS $\ni \phi \to \phi_T \in$ BPS, $\quad E[\phi] = E[\phi_T]$

no impurity $\to \phi_T = \phi(x + x_0)$

$\to$ trivial action on the lumps

$\to$ zero mode although the translation inv. broken

- moduli space $\mathcal{M}$

position of the BPS antikink w.r.t. the impurity

move on $\mathcal{M}$ generated by the generalized translation T

spectral structure depends on a position on $\mathcal{M}$

leading order dynamics of BPS solitons **geodesic motion on $\mathcal{M}$**

$\to$ **DW that do not get stuck on impurities**

beyond leading order dynamics of BPS solitons

$\to$ **mode - BPS soliton interaction**

examples, applications and results

spectral wall

role of bound modes in the soliton dynamics

- **spectral wall**

  Adam, Oles, Romanczukiewicz, Wereszczynski PRL (19)

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\phi_x^2 + W^2(\phi) + 2\sigma W(\phi) + \sqrt{2}\sigma\phi_x \right] + \int_{-\infty}^{\infty} dx\sigma^2$$

pre-potential $W = (1-\phi^2)/\sqrt{2} \longrightarrow$ ϕ^4 theory
kink-form preserving impurity $\sigma = \frac{\alpha}{\cosh^2 x}$

$\rightarrow$ **the energy bound**

$$E \geq -\frac{4}{3}Q$$

$\rightarrow$ **no impurity** $\alpha = 0$

$$\phi_{k,a} = \pm 4\tanh(x - x_0)$$

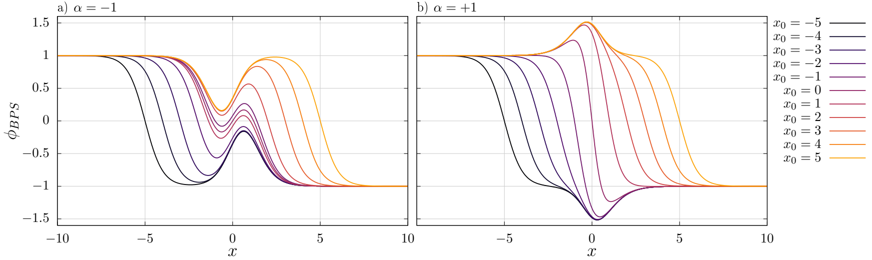
$\rightarrow$ **impurity** structure of static solutions

non-BPS kink $\tanh x$

BPS antikinks

BPS lumps (top. trivial)

lump (Q=0) and BPS anti-kink (Q=-1)

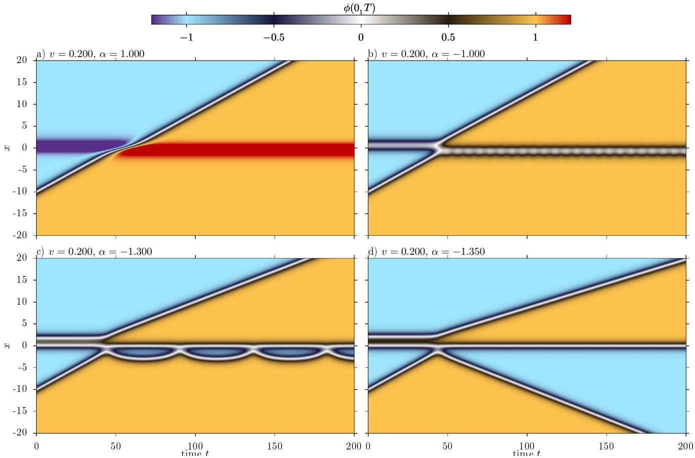


moduli space $\mathcal{M} = \{\phi(x; x_0) : x_0 \in \mathbb{R}, x_0 \sim$ position of topological 0$\} \cong \mathbb{R}$

$$E[\phi(x; x_0)] = 4/3$$

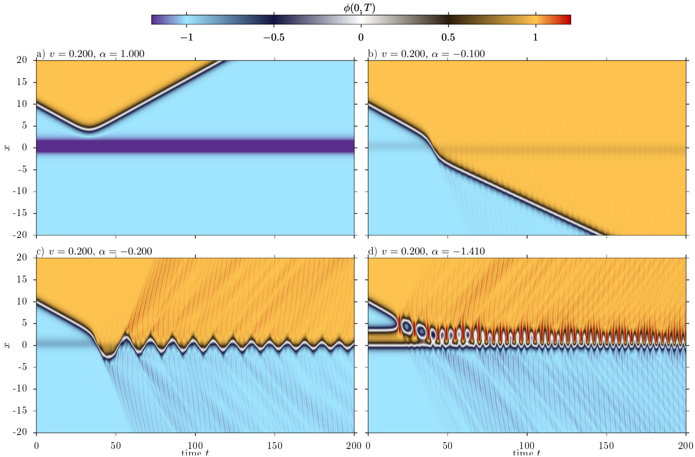
generalized translation $T : x_0 \rightarrow x_0 + a \quad \Rightarrow \quad$ flow on the moduli space

zero order dynamics $\rightarrow$ geodesic flow interaction of the BPS soliton with the impurity



effortless transition of the BPS solitons through the impurity

interaction of the non-soliton with the impurity



non-effortless transition of the BPS solitons through the impurity

lets excite a mode = beyond the geodesic approximation

$\rightarrow$ usual soliton model in (1+1) dim $\rightarrow$ scattering of solitons

force between solitons $\rightarrow$ deformation of shapes, no static solutions

excitation of modes (mutual distance dependent)

radiation

strong mixing $\rightarrow$ no analytical, quantitative understanding...a lot of mess

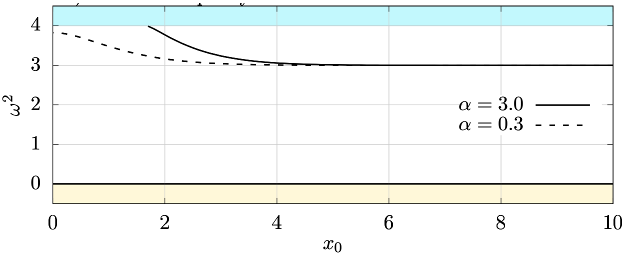
$\rightarrow$ BPS-impurity model

no static force between the BPS soliton and the impurity

spectral structure (vibrational modes) depends on a position on the moduli space

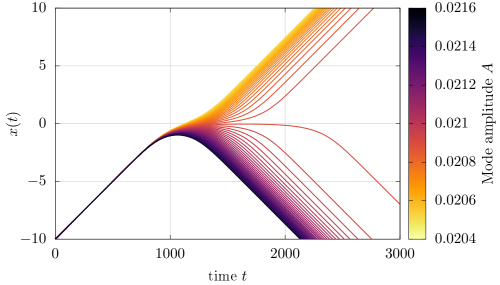
BPS soliton while approaching the impurity changes its spectral structure

clear signature of the role of the modes

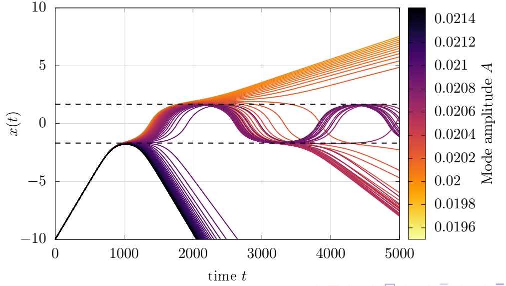


excited BPS $\bar{K}$ colliding on the impurity $\phi = \phi_{BPS}(x, t; x_0) + \eta(x, t; x_0)$

$\alpha = 0.3$



$\alpha = 3$



spectral wall

a **well defined region in space** at which

soliton nontrivially interacts

due to the mode transition to the continuous spectrum

the effect first discovered for the BPS-impurity models but...

$\longrightarrow$ should be present in a beyond geodesic-approx. dynamics of any BPS model

whenever a mode enters the continuous spectrum

Abelian Higgs vortices at critical coupling

$\longrightarrow$ role in kink dynamics in non-BPS processes?

whenever a mode enters the continuous spectrum

ϕ^4, ϕ^6...

solvable self-dual impurity

- **solvable self-dual impurity models**

Adam, Oles, Queiruga Romanczukiewicz, Wereszczynski, JHEP (19) [1905.06080]

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\phi_x^2 + W^2 + W^2\sigma^2 + 2W\sigma^2 + \sqrt{2}\sigma W \phi_x \right]$$

obtained by $\sigma \rightarrow W\sigma$

- **topological bound**

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\phi_x + \sqrt{2}W + \sqrt{2}\sigma W\right)^2 - \sqrt{2}W\phi_x \right] \\ &\geq -\sqrt{2}\int_{-\infty}^{\infty} dx W\phi_x = \sqrt{2}Q\int_{\phi_-}^{\phi_+} W d\phi \end{aligned}$$

- **Bogomolny eq.**

$$\frac{1}{\sqrt{2}}\phi_x + W(1+\sigma) = 0 \;\; \Rightarrow \;\; \frac{1}{\sqrt{2}}\phi_y + W = 0$$

solvable $\frac{dy}{dx} = 1 + \sigma(x) \;\; \Rightarrow \;\; y(x) = x + \int_{-\infty}^{x} \sigma(x')dx' \equiv x + \Delta_\sigma(x)$

- **generalized translation** pure coordinate transformation

trivial translation in $y \rightarrow$ moduli space coordinate

analytically known

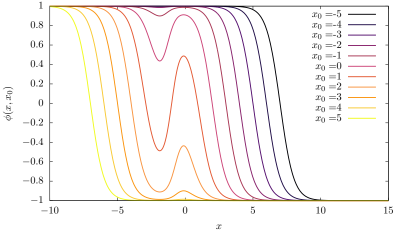
→ BPS solutions $\phi_{BPS}(x) = -\tanh(x + \Delta_\alpha(x) + x_0), \quad \Delta_\alpha(x) = \alpha\,(\tanh x + 1)$
lumps = vacua of no-impurity model: $\phi = \pm 1$

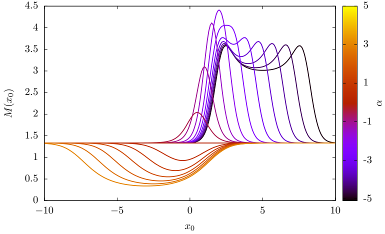
→ moduli space coordinate x_0

→ moduli space metric $\phi(x,t) = \phi_{BPS}(x, x_0(t))$

$$L_{eff} = \frac{1}{2} M(x_0)\dot{x}_0^2, \quad M = \int_{-\infty}^{\infty} dx \left(\frac{d}{dx_0} \phi_{BPS}(x + \Delta_\alpha(x) + x_0(t)) \right)^2$$

→ spectral potential $V(x) = 2(1+\sigma)^2 \partial_\phi (W W_\phi) - \sqrt{2}\sigma_x W_\phi$





but much more... non-localized impurities

kink-antikink annihilation - zero static force limit

non-localized impurities

$$\sigma_j = \frac{j}{2} \tanh x - 1$$

BPS solutions

$$\phi(x; a) = -\frac{\cosh^j x - a}{\cosh^j x + a}, \quad a = -1 + e^{jX}$$

→ **top. trivial sector**

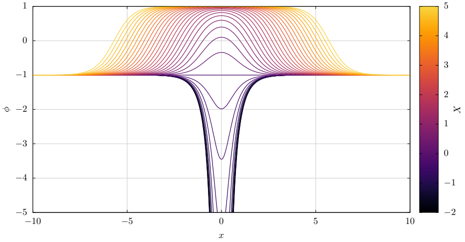
→ a or X moduli coordinate

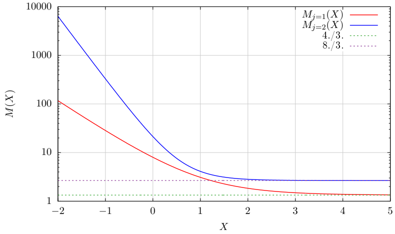
→ $\phi = -1$ lump is a member of the moduli space (has a zero mode)

→ $\phi = 1$ lump is not a member of the moduli space (no a zero mode)

→ moduli space kink-antkink annihilation

lowest order dynamics geodesic flow





beyond geodesic annihilation: spectral walls

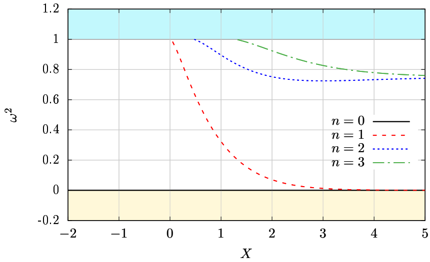
Adam, Oles, Romanczukiewicz, Wereszczynski, 1907.xxxxx

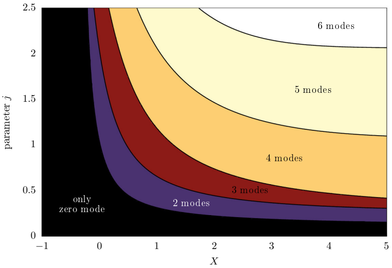
excite a bound mode $j = 1$ and $j = 0.7$

$\rightarrow$ zero mode: symmetric change of the position of the solitons

$\rightarrow$ deep bound mode: asymmetric change of the position of the solitons

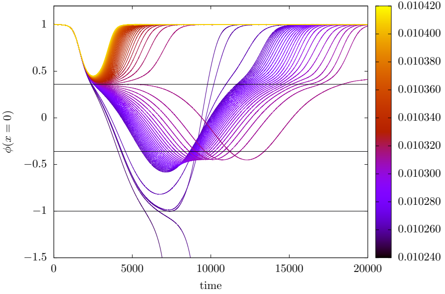
$\rightarrow$ two shape modes of asymptotic K, K^*

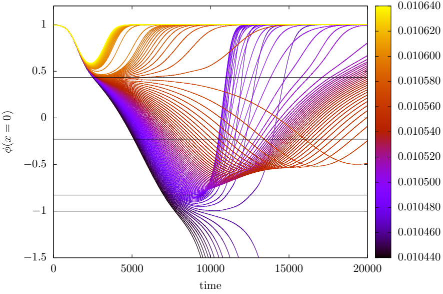




beyond geodesic annihilation: spectral walls

asymmetric superposition of the shape modes

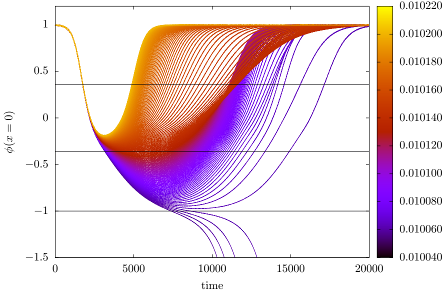




$\rightarrow$ the spectral wall at $\phi_0 = 0.36$
$\rightarrow$ **the vacuum wall** at $\phi_0 = -1$
$\rightarrow$ **bouncing structure possible**

beyond geodesic annihilation: spectral walls

symmetric superposition of the shape modes



→ the spectral wall at $\phi_0 = -0.34$ not too well visible

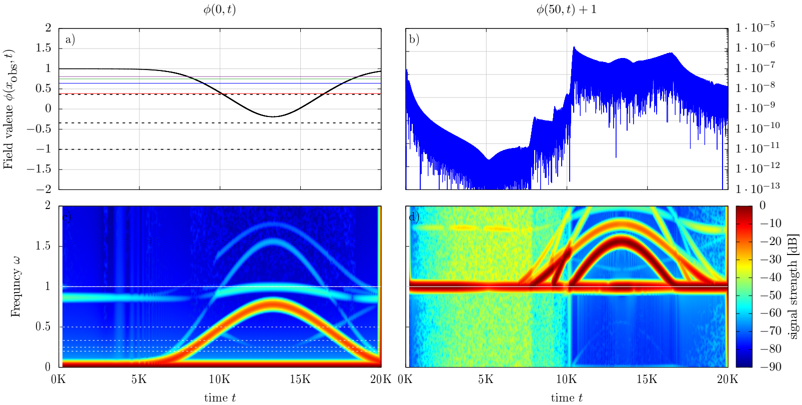
→ **shadow of the vacuum wall**

higher spectral walls and radiation bursts

deepest mode: asymmetric superposition of the translation modes

higher spectral walls $\longrightarrow$ when higher harmonics hit the mass threshold

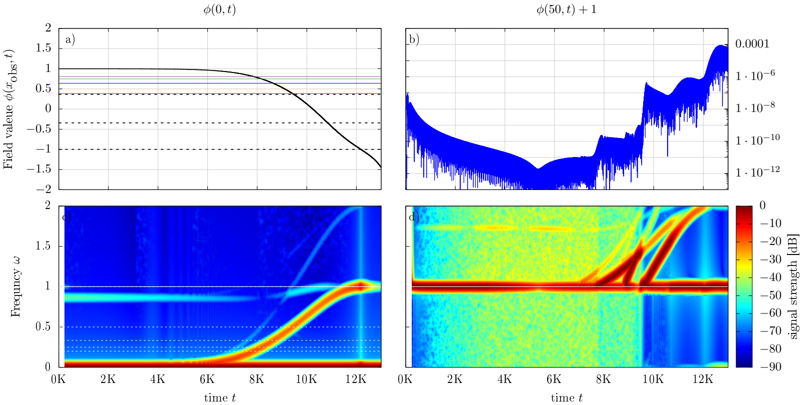
$$n\omega_n = E_{continuum} \quad \Rightarrow \quad \omega_n = \frac{j^2}{n}$$



$\longrightarrow$ higher spectral walls $\longrightarrow$ v. sensitive, before the wall is hit

$\longrightarrow$ **radiation bursts**

higher spectral walls and radiation bursts



$\to$ radiation bursts eight orders of magnitude

summary

self-dual BPS impurity models $\rightarrow$ a new, previously unknown class of models

no restriction on the spatial form of the impurity

can be added for any BPS model

- (1+1) dimensions

  half of BPS solitons of the original theory preserve the BPS property

  no static force between BPS solitons and impurity

  spectral structure depends on a position on the moduli space

  as in higher dim BPS models - AH vortices

  a proper toy-model to study higher dim solitons

  suitable for study of the soliton-mode interactions

  the spectral wall phenomenon

  kinks annihilation (scattering) only via modes and radiation

  a new tool for switching off the static force

  make the SD impurity dynamical Manton, Oles, Wereszczynski, in progress

- applications: condense matter

- (d+1) dimensions

  known, lots of structures....