Skyrmions coupled to omega mesons

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Topological Solitons, Nonperturbative Gauge Dynamics and Confinement 2
Pisa, July 2019

The Skyrme–ω model

- Skyrme field $\varphi : \mathcal{M} \to S^3 \subset \mathbb{R}^4$

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) = (\sigma, \pi_1, \pi_2, \pi_3)$$

Baryon current $B^\mu = \frac{1}{12\pi^2}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}\varphi_a\partial_\nu\varphi_b\partial_\rho\varphi_c\partial_\sigma\varphi_d$

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$$\mathcal{L} = \frac{1}{8} \partial_\mu \varphi \cdot \partial^\mu \varphi - V(\varphi) - \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{1}{2} \omega_\mu \omega^\mu + g \omega_\mu B^\mu$$

- $V(\varphi) = \frac{M^2}{4}(1 - \varphi_0), \qquad \omega_{\mu\nu} = \mathrm{d}\omega(\partial_\mu, \partial_\nu)$
- Coupling constant $g > 0$. Upper bound from $\omega \to \pi\pi\pi$ process

Previous work

- 1984, Adkins-Nappi: $B = 1$ hedgehog, rigid body quantization, fit to N, Δ masses. $g = 96.7$
- 2003, Park-Rho-Vento: included ρ, computed skyrme **crystal** using truncated Fourier series
- 2009, Sutcliffe: $B = 1, 2, 3, 4$ imposing rational map “ansatz” for φ and truncated expansion in spherical harmonics for ω. Set energy scale by putting $F_\pi = F_\pi^{exp}$. Chose g so that $m_4 = m_\alpha^{exp}$. $g = 34.7$
- [2009, Foster-Sutcliffe: $(2+1)$-dimensional version $B = 1, 2, 3, 4$ full numerics using “a heat flow method”]
- Why is this a hard problem?

Some notation

- $\mathcal{M}^{1,3}$ spacetime
  - Hodge isomorphism $\star$
  - Wave operator $\Box = \partial_\mu \partial^\mu$
- M^3 space
  - Hodge isomorphism $*$
  - Laplacian $\Delta = -\partial_i \partial_i$
- $\Omega =$ volume form on S^3, normalized s.t. $\int_{S^3} \Omega = 1$

The field equations

$$
\begin{aligned}
\frac{1}{4}P\Box\varphi + (\mathrm{grad}\, V)\circ\varphi + g\star(\mathrm{d}\omega\wedge\Xi_\varphi) &= 0 \\
-\star\mathrm{d}\star\mathrm{d}\omega + \omega + gB &= 0
\end{aligned}
$$

- $P : u \mapsto u - (\varphi\cdot u)\varphi$
- $B = B_\mu dx^\mu = \star\varphi^*\Omega$
- Ξ_φ is a two-form on $\mathcal{M}$ valued in $\varphi^{-1}TS^3$

$$h(Z, \Xi_\varphi(X, Y)) = \Omega(Z, \mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))$$

$$(\Xi_\varphi)^b_{\mu\nu} = \frac{1}{12\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{abcd}\varphi_a\partial^\rho\varphi_c\partial^\sigma\varphi_d$$

The **static** field equations

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- $\partial_t\varphi = 0 \Rightarrow B = B_0 dt \Rightarrow \omega = f dt$

The **static** field equations

$$\begin{aligned} \frac{1}{4} P \Delta \varphi + (\mathrm{grad}\, V) \circ \varphi + g * (\mathrm{d}f \wedge \Xi_\varphi) &= 0 \\ (\Delta + 1) f &= -g * \varphi^* \Omega \end{aligned}$$

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- $\partial_t\varphi = 0 \Rightarrow B = B_0 dt \Rightarrow \omega = f dt$
- This is **not** the condition for minimizing static **energy**:

$$E(\varphi, f) = \int_M \left(\frac{1}{8}|\mathrm{d}\varphi|^2 + V(\varphi) + \frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}f^2\right)$$

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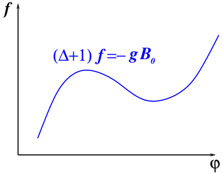
- $\partial_t\varphi = 0 \Rightarrow B = B_0 dt \Rightarrow \omega = f dt$
- Coincides with Euler-Lagrange equation for the **constrained** variational problem

$$
\begin{aligned}
E(\varphi, f) &= \int_M \left(\frac{1}{8}|\mathrm{d}\varphi|^2 + V(\varphi) + \frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}f^2\right) \\
(\Delta + 1)f &= -g * \varphi^*\Omega \qquad (*)
\end{aligned}
$$

- Want to minimize $E(\varphi, f)$ subject to the constraint (*).

Numerical method: arrested Newton flow

- Configuration space $\mathscr{C}$: submanifold of $C^\infty_B(M, S^3) \times C^\infty(M)$ on which (*) holds



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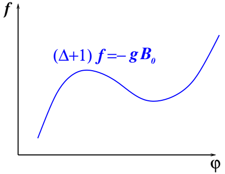

- Want to minimize $E : \mathscr{C} \to \mathbb{R}$

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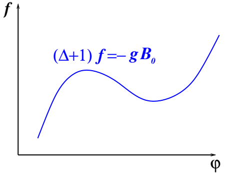

- Want to minimize $E : \mathscr{C} \to \mathbb{R}$
- Start at some $X(0) \in \mathscr{C}$ with $\dot{X}(0) = 0$, solve Newton's equation for motion in potential E

$$\ddot{X} = -\operatorname{grad} E$$

If $E(t) > E(t - \delta t)$ set $\dot{X}(t) = 0$ and restart the flow.

- Much faster than gradient flow.

Numerical method: arrested Newton flow

- $\mathscr{C}$ is a **graph** over $C^\infty_B(\mathsf{M}, S^3)$

$$P\varphi_{tt} = -\frac{1}{4}P\Delta\varphi - (\mathrm{grad}\, V)\circ\varphi - g * (\mathrm{d}f \wedge \Xi_\varphi)$$

Numerical method: arrested Newton flow

- $\mathscr{C}$ is a **graph** over $C_B^\infty(\mathsf{M}, S^3)$

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- Discretize space, 4th order Runge-Kutta for time stepping
- After each time step must solve $(\Delta + 1)f = -gB_0 f$

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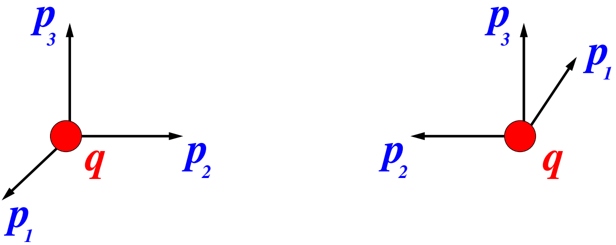
- EL equation for

$$Q(f) = \int_{\mathsf{M}} \left(\frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}f^2 + gB_0 f\right)$$

  Minimize $Q(f)$ via conjugate gradient method
- Iterative: start with good f, typically converges in 0–3 cycles

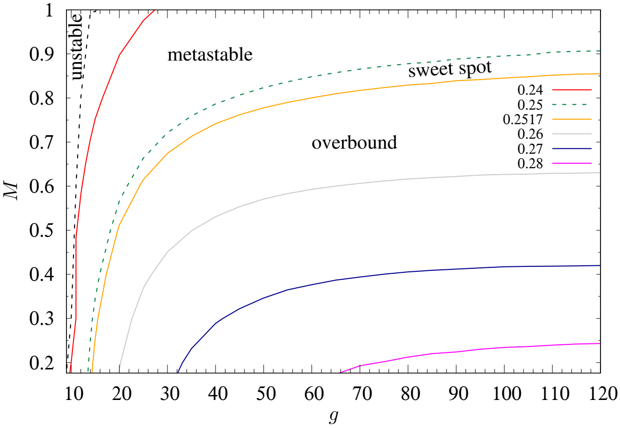
Possibility of low binding energies



$$\mathcal{L}_{linearized} = \frac{1}{8}\partial_\mu \pi \cdot \partial^\mu \pi - \frac{M^2}{8} - \frac{1}{2}\partial_\mu f \partial^\mu f + \frac{1}{2} f^2 + \rho \cdot \pi - \rho_0 f$$

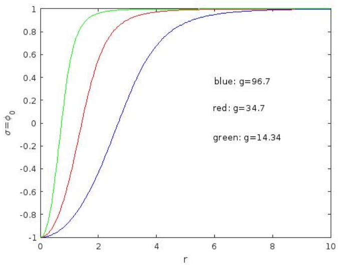
- $M < 1$ scalar dipole interaction dominates at long range
- Scalar monopole interaction is **repulsive**
- Expect BE to vanish as $M \to 1$ (skyrmions unbound for $M > 1$)

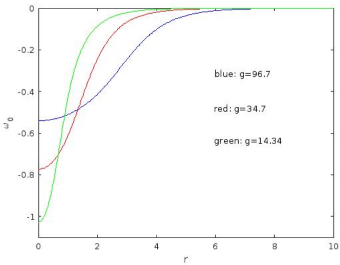
m_1 / m_4 (classical)



- $m_1 / m_4 = 0.2517$, $M = 0.176 \Rightarrow g = 14.34$

$B = 1$ skyrmion





- Length unit: $\hbar c / m_\omega$

Calibration: $g = 14.34$

- Charge radius of proton

$$r_E = \left(\int_{\mathbb{R}^3} r^2 \{ I_3^{normalized} + \frac{1}{2} B_0 \} \right)^{1/2} \equiv 0.875 fm$$

sets $m_\omega = 461.4 MeV$ ($m_\omega^{exp} = 782.7 MeV$)

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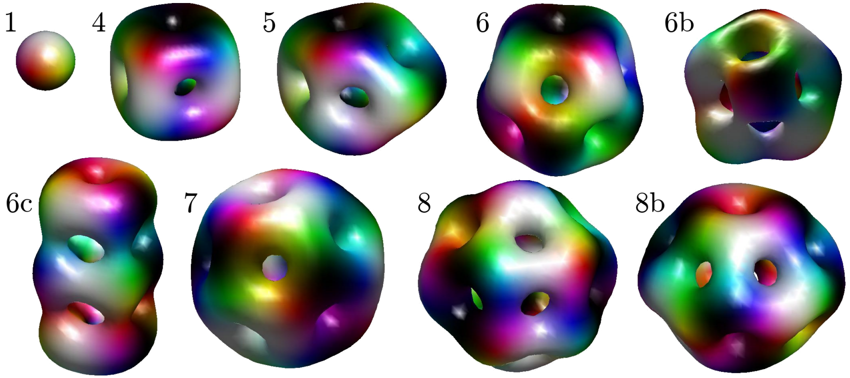
- Energy unit F_π^2 / m_ω
- Choose F_π such that $m_1 = m_N^{exp} = 938.0 MeV$. Then $m_4 \equiv m_\alpha^{exp}$.

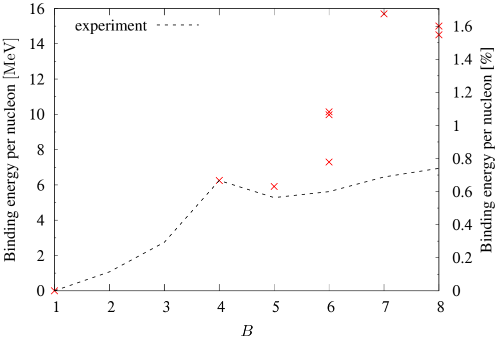
$$F_\pi = 144.1 MeV \qquad (F_\pi^{exp} = 130 MeV)$$

- $\omega \to \pi\pi\pi$ bound: $F_\pi g / m_\omega < 23.9$. We have $F_\pi g / m_\omega = 10.8$

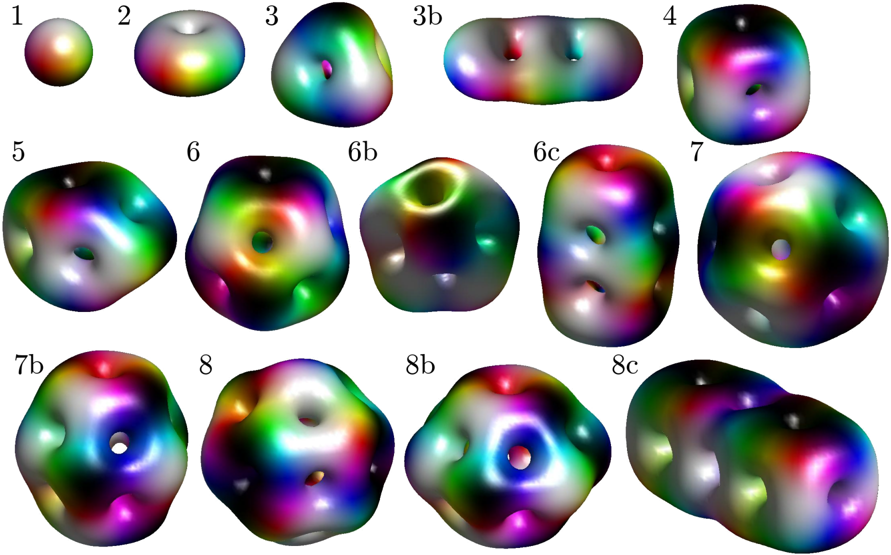
Skyrmions $g = 14.34$



Classical binding energies $g = 14.34$



Skyrmions $g = 34.7$



Concluding remarks

- Unlike conventional Skyrme model, have a genuine coupling constant: changing it really affects the skyrmions
- Small g regime may be interesting: low binding energies
- Numerical problem is tractable using arrested Newton flow/conjugate gradient scheme
- Open problems:
  - Quantization (of course).
  - Topological energy bound? $E(\varphi) \geq const \times B$?
  - **Existence** of energy minimizers (even on compact M)? Much harder than $E_2 + E_0 + E_6$ model
  - Isospin symmetry breaking term

    $$\mathcal{L}' = \star d\omega \wedge d\pi_1 \wedge d\pi_2$$

    can reproduce m_n–m_p mass difference. What does it do to $B \geq 2$ skyrmions?