Localized nonlinear gravitational waves - "geons" in asymptotically anti de Sitter space-times

in collaboration with

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“Particle”-like - localized - solutions

– finite energy, spatially localized, size ($\sim L$) for times $T \gg L/c$
– nonlinearity is essential
– may be the size of particles, stars or galaxies

In many cases there are no time-independent configurations
– but there are solutions oscillating in time

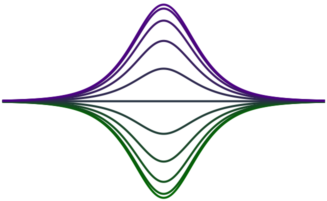
- Real scalar fields in Minkowski space-time:
- oscillons (pulsons) exist in dims. $D = 1, 2, 3, 4$
- Real scalar coupled to Einstein gravity: oscillaton
- Complex scalar with static metric: boson star
- Gravitational or electromagnetic waves: geon

Spherically symmetric real scalar field, with self-interaction potential $U(\phi)$, in case of d spatial dimensions

$$-\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial r^2}+\frac{d-1}{r}\frac{\partial\phi}{\partial r}=U'(\phi)$$

Exactly time-periodic, localized, finite energy solution only exist for $d=1$ and $U(\phi)=1-\cos\phi$

sine-Gordon breather



There are "almost-breather" solutions, weakly emitting energy by scalar field radiation, having a slowly changing frequency

Discovery of pulsons

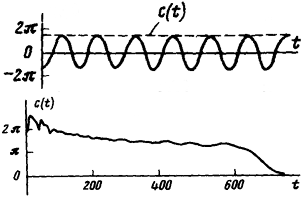
Potential: sine-Gordon or $U(\phi) = \frac{1}{4}(\phi^2 - 1)^2$

– spherically symmetry

Numerical solutions in $d = 3$ spatial dimensions

Bogolyubskii and Makhan'kov, *JETP Letters*, **25,** 107 (1977)

Evolution of the scalar field at the center:



Sudden decay after a few thousands oscillations

– no such decay for $d = 1$ and $d = 2$

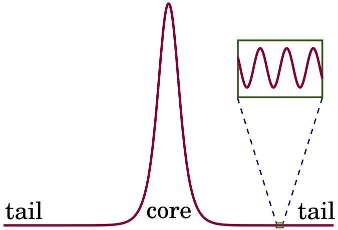
Pulsons were renamed oscillons by Marcelo Gleiser in 1995

Seidel and Suen (1991): numerical observation of spherically symmetric, localized, oscillating solutions for a self-gravitating, real scalar field coupled to gravity – oscillaton

– no sudden decay observed numerically for oscillatons

– slow radiation of energy $\rightarrow$ slowly changing frequency

– lifetime is "infinite"



general structure of oscillons/pulsons and oscillatons

the tail is a very small amplitude outgoing wave

If the central amplitude is ε, then the tail amplitude is proportional to $\exp\left(-\frac{1}{\varepsilon}\right) \longrightarrow$ radiation rate decreases in time

1+D dimensional anti-de Sitter space-time

AdS_{1+D} is the maximally symmetric Lorentzian manifold ($O(2, D)$) with constant negative scalar curvature. Its line element in Schwarzschild area coordinates:

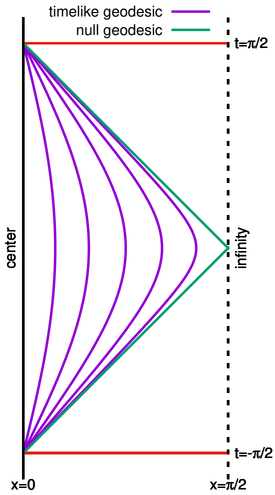
$$ds^2 = -(1 + k^2 r^2)dt^2 + \frac{dr^2}{1 + k^2 r^2} + r^2 d\Omega^2_{D-1}.$$

This metric satisfies Einstein's equations with negative cosmological constant $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, where $\Lambda = -\frac{1}{2}D(D-1)k^2$. An observer at a fixed r undergoes constant outwards acceleration:

$$a = \frac{k^2 r}{\sqrt{1 + k^2 r^2}} \xrightarrow[r\to\infty]{} k$$

AdS background corresponds to an effective attractive force

$D = 3$ AdS spacetime in compactified coordinates



Global spatially compactified coordinates

$$ds^2 = \frac{L^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x \, d\Omega^2\right)$$

where $L^2 = -3/\Lambda$

– each point corresponds to a 2-sphere with radius $L \tan x$

– metric is static in these coordinates

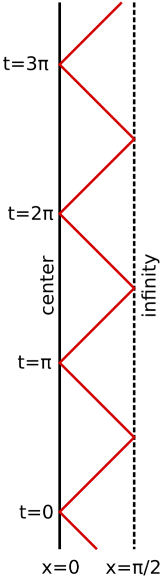
– center is at $x = 0$, infinity at $x = \frac{\pi}{2}$

– range of time coordinate: $-\infty < t < \infty$

– radial outwards acceleration of constant x observers is $\frac{\sin x}{L}$

– timelike geodesics meet again at a point

Instability of anti-de Sitter spacetime



A light ray can travel to infinity and back in a finite time

This is related to the (conjectured) instability of AdS
– a wave packet can bounce back many times to the center, it becomes more and more concentrated, and in the end it collapses to a black hole
– smaller amplitude $\longrightarrow$ more bounces needed
– demonstrated numerically by Bizoń and Rostworowski in 2011 for a spherically symmetric massless scalar field coupled to gravity

The assumption of reflective boundary conditions is essential
– energy cannot disperse at infinity
– $\Lambda < 0$ is like a bounding box

Klein-Gordon breather on AdS background

Consider 1st (minimally coupled) KG equation on AdS bckg:

$$\nabla^{\mu}\nabla_{\mu}\phi = m^2\phi, \quad m^2 \geq 0$$

and assume spherical symmetry:

$$-\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial^2\phi}{\partial x^2} + 2\frac{D-1}{\sin(2x)}\frac{\partial\phi}{\partial x} = \frac{m^2}{k^2\cos^2 x}\phi$$

This eq. admits breathers: $\phi = p(x)\cos(\omega\tau/k)$ with $p(x)$ regular

$\phi(t,x)$ localized in space and time-periodic!

in asymptotically flat (or deSitter) $D > 1$ space-times $\nexists$ breathers!

exceptions $\rightarrow$ "V"-shaped potentials, signum-Gordon (H. Arodź et al.)

for $\phi = p(x)\cos(\omega\tau/k)$ the AdS KG eq. becomes:

$$\frac{d^2p}{dx^2} + 2\frac{D-1}{\sin(2x)}\,\frac{dp}{dx} = \frac{m^2}{k^2\cos^2 x}p - \frac{\omega^2}{k^2}p$$

generic solution for $p(x)$ is singular either at $x = 0$ or $x = \pi/2$

If the frequency takes on special values:

$$\omega = \omega_+ = (\lambda_+ + 2n)k \quad \lambda_\pm = \frac{1}{2}\left(D \pm \sqrt{D^2 + 4\frac{m^2}{k^2}}\right)$$

for $n \geq 0$ integer, then $p(x)$ is globally regular and the breathers:

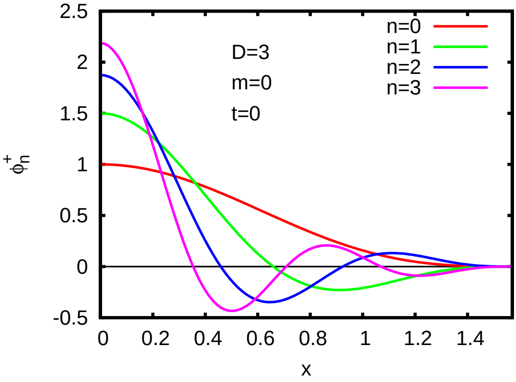
$$\phi_n^+ = \cos\left(\frac{\omega_+}{k}\tau\right)(\cos x)^{\lambda_+} P_n^{(D/2-1,\lambda_+-D/2)}(\cos(2x))$$

$P_n \rightarrow$ Jacobi polynomial

Simplest explicit solutions:

$$\phi_0^{\pm} = \cos\left(\frac{\omega_\pm}{k}\tau\right)(\cos x)^{\lambda_\pm}$$

$$\phi_1^{\pm} = \cos\left(\frac{\omega_\pm}{k}\tau\right)(\cos x)^{\lambda_\pm}\left[\frac{D}{2} - (\lambda_\pm + 1)\sin^2 x\right]$$



Periodic solutions for $m = 0$ in $3 + 1$ spacetime dimensions at time $t = 0$

Solutions for $m > 0$ are similar, but more compact

Numerical simulations $\longrightarrow$ all these solutions are stable

Scalar breathers in asymptotically AdS spacetime

Einstein's equations coupled to a massless real scalar field:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad , \quad T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}$$

together with the wave equation

$$\nabla^{\mu}\nabla_{\mu}\phi = 0$$

look for spherically symmetric solutions:

$$\mathrm{d}s^2 = \frac{L^2}{\cos^2 x}\left(-Ae^{-2\delta}\mathrm{d}t^{\,2} + \frac{1}{A}\mathrm{d}x^2 + \sin^2 x\,\mathrm{d}\Omega^2_{D-1}\right)$$

where $A = A(t,x)$ and $\delta = \delta(t,x)$; $L^2 = -D(D-1)/2/\Lambda$

– anti-de Sitter space-time corresponds to $A = 1$ and $\delta = 0$

Small-amplitude expansion

The scalar field and the metric functions are expanded in powers of a small parameter ε

$$\phi = \sum_{n=0}^{\infty} \phi^{(2n+1)} \varepsilon^{2n+1} \,, \quad A = 1 + \sum_{n=1}^{\infty} A^{(2n)} \varepsilon^{2n} \,, \quad \delta = \sum_{n=1}^{\infty} \delta^{(2n)} \varepsilon^{2n}$$

To first order in ε: the metric is AdS, $\phi^{(1)}(x,t)$ is given as:

$$\phi^{(1)}(x,t) = p_m(x) \cos(\omega_m t) \quad , \quad m \geq 0 \text{ integer}$$

$p_m(x)$ can be given with Jacobi polynomials, and the allowed frequencies are

$$\omega_m = d + 2m \,,$$

in leading order $\rightarrow \phi(x,t) = \phi^{(1)}(x,t)$ is spatially localized, time-periodic.

in higher orders in the ε expansion

$$\omega = \omega^{(0)}(1 + \sum_{j=1}^{\infty} \omega^{(j)} \varepsilon^{j}) \,, \quad \omega^{(0)} = \omega_n \tag{1}$$

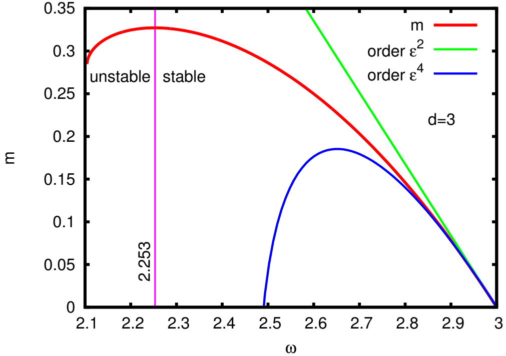
There is a one-parameter family of solutions emerging from each p_n linearized mode

The mass of the "breathers" in the perturbative small amplitude expansion

$$m = \frac{\pi}{32}\varepsilon^2 + m_4(D)\varepsilon^4 + \ldots \,, \; m_4(D=3) \approx -1.05316 \,. \tag{2}$$

ε determines the value of the amplitude of the field at the center at $t = 0$

comparing the numerical results to the perturbative (small-ε) expansion:



AdS breather becomes unstable when the total mass starts to decrease with increasing central density

Perturbative expansion is in excellent agreement with the numerical results up to $\varepsilon \approx 1$.

AdS gravitational breathers – geons

Localized time-periodic vacuum solutions with regular center and no horizon for $\Lambda < 0$

– typical size given by the length-scale $L = \sqrt{-\frac{3}{\Lambda}}$

There are no spherically symmetric vacuum geon solutions

Small-amplitude expansion: consider a one-parameter family of solutions depending on a parameter ε, and expand the metric as

$$g_{\mu\nu} = \sum_{k=0}^{\infty} \varepsilon^k g_{\mu\nu}^{(k)}$$

$g_{\mu\nu}^{(0)}$ is the AdS metric

$$\mathrm{d}s_{(0)}^2 = \frac{L^2}{\cos^2 x}\left[-\mathrm{d}t^2 + \mathrm{d}x^2 + \sin^2 x(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]$$

$g_{\mu\nu}^{(0)}$ has components that diverge as $\left(\frac{\pi}{2} - x\right)^{-2}$ at infinity

We require that for $k \geq 1$ all $g_{\mu\nu}^{(k)}$ diverge at most as $\left(\frac{\pi}{2} - x\right)^{-1}$

$\longrightarrow g_{\mu\nu}$ is asymptotically AdS

We use real spherical harmonics $\mathbb{S}_{lm}$

- defined for $l \geq 0$ and $-l \leq m \leq l$ integers
- ϕ dependence is $\cos(m\phi)$ for $m \geq 0$, and $\sin(|m|\phi)$ for $m < 0$

Spherical harmonic decomposition

Tensors can be decomposed into

- scalar-type part (polar, even parity)
- vector-type part (axial, odd parity)
- tensor-type part – only for $d \geq 4$ space dimensions

Vector spherical harmonics $\mathbb{V}_{(lm)i}$ for $d = 3$ has the components

$$\mathbb{V}_{(lm)\theta} = \frac{1}{\sqrt{l(l+1)}} \frac{1}{\sin\theta} \frac{\partial \mathbb{S}_{lm}}{\partial \phi} \,, \quad \mathbb{V}_{(lm)\phi} = \frac{-1}{\sqrt{l(l+1)}} \sin\theta \frac{\partial \mathbb{S}_{lm}}{\partial \theta}$$

Perturbations for each l, m can be considered separately
– they are only coupled by lower order terms in the ε expansion

For each choice of l and m, and at each k order in the ε expansion

– scalar-type perturbations are described by the function $\Phi_{lm}^{(k,s)}$

– vector-type perturbations are described by the function $\Phi_{lm}^{(k,v)}$

All these scalars satisfy the equations (dropping the indices)

$$\boxed{-\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial^2\Phi}{\partial x^2}-\frac{l(l+1)}{\sin^2 x}\Phi=\frac{\bar{\Phi}}{\sin^2 x}}$$

where $\bar{\Phi}$ are known functions of t, x, already determined at lower than k order in ε

Boundary conditions: metric perturbation be asymptotically AdS:

- for vector-type perturbations $\lim_{x\to\frac{\pi}{2}}\Phi=0$
- for scalar-type perturbations $\lim_{x\to\frac{\pi}{2}}\frac{\mathrm{d}\Phi}{\mathrm{d}x}=0$

Perturbations of the metric can be calculated from these functions by taking derivatives and algebraic manipulations

Periodic solutions at linear order

At order ε^1 there is no inhomogeneous source term: $\bar{\Phi} = 0$

Search solutions in the form $\Phi = p(x)\cos(\omega t)$

Centrally regular and asymptotically AdS solutions only exist:

– scalar-type perturbations: $\boxed{\omega = l + 1 + 2n}$, $n \geq 0$ integer

$$p(x) = \sin^{l+1} x \frac{n!}{(l+\frac{3}{2})_n} P_n^{(l+\frac{1}{2},-\frac{1}{2})}(\cos(2x))$$

– vector-type perturbations: $\boxed{\omega = l + 2 + 2n}$, $n \geq 0$ integer

$$p(x) = \sin^{l+1} x \cos x \frac{n!}{(l+\frac{3}{2})_n} P_n^{(l+\frac{1}{2},\frac{1}{2})}(\cos(2x))$$

where the Pochhammer's Symbol is $(c)_n = \Gamma(c+n)/\Gamma(c)$
and $P_n^{\alpha,\beta}(z)$ are Jacobi polynomials

n gives the number of radial nodes (zero crossings)

For each (l, m, n), where $l \geq 2$, $|m| \leq l$, $n \geq 0$ integers, there is a scalar- and a vector-type linear geon mode with arbitrary amplitude

The frequency for scalar-type: $\omega = l + 1 + 2n$
for vector-type: $\omega = l + 2 + 2n$

Since all frequencies are integers, an arbitrary linear combination of these modes is still a time-periodic solution with $\omega = 1$
$\longrightarrow$ infinite-parameter family of linear geons

The nonlinear system only has one-parameter families of AdS geon solutions

- true for all cases studied by the nonlinear expansion formalism – started with finite number of parameters
- supported by direct numerical search for time-periodic solutions of the Einstein's equations
- proof ???

Inhomogeneous scalar equation at higher orders in ε

$$-\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial^2\Phi}{\partial x^2}-\frac{l(l+1)}{\sin^2 x}\Phi \quad = \quad \frac{\bar{\Phi}}{\sin^2 x}$$

homogeneous solutions with frequency ω

scalar-type: $\omega = l+1+2n$

vector-type: $\omega = l+2+2n$

sum of source terms of the type

$\bar{\Phi} = p_0(x)\sin(\omega_s t)$ or

$\bar{\Phi} = p_0(x)\cos(\omega_s t)$

If $\omega \neq \omega_s$ for all $n \geq 0$ integers, there are always time-periodic solutions which are asymptotically AdS and have a regular center

If $\omega = \omega_s$ for some n , then $\bar{\Phi}$ is a resonant source term

- generally, regular asymptotically AdS solutions for Φ are blow-up solutions of the type $t\cos(\omega t)$
- time-periodic solution for a resonant source term only exists if a consistency condition holds

Consistency conditions

Time-periodic centrally regular asymptotically AdS geon solution can only exist, if for each resonant source term, having the form $\bar{\Phi} = p_0(x)\sin(\omega_s t)$, a consistency condition holds

$$\int_0^{\frac{\pi}{2}} \frac{p_{l,n}(x)p_0(x)}{\sin^2 x}\mathrm{d}x = 0$$

where $p_{l,n}(x)$ is the regular solution of the homogeneous equation

The consistency conditions determine

- the change of physical frequency $\bar{\omega}$ as a function of ε
- ratio of the modes included at linear order

If the consistency conditions cannot be satisfied

$\Longrightarrow$ terms with linearly increasing amplitude $t\cos(\omega t)$

$\longrightarrow$ shift of energy to higher frequency modes

$\dashrightarrow$ turbulent instability $\rightsquigarrow$ black hole formation

Natural simplest case: start with only one mode at linear order

There is a scalar and a vector mode for each $l \geq 2$, $|m| \leq l$, $n \geq 0$

– denote them by $(l, m, n, \omega_s)_S \quad , \quad (l, m, n, \omega_v)_V$
where $\omega_s = l + 1 + 2n$ and $\omega_v = l + 2 + 2n$

For some single linear modes there is no corresponding nonlinear AdS geon solution (Dias, Horowitz, Santos)

– consistency conditions cannot be solved at ε^3 order

– examples: $(2, 0, 1, 5)_S$, $(4, 0, 0, 5)_S$, $(3, 2, 0, 4)_S$, $(2, 2, 0, 4)_V$

Resolution: take the linear combination of same frequency modes at linear order in ε (Rostworowski)

$$\left.\begin{array}{l}(2,0,1,5)_S \text{ with amplitude } \alpha \\ (4,0,0,5)_S \text{ with amplitude } \beta\end{array}\right\} \longrightarrow \frac{\alpha}{\beta} \approx 0.12909 \text{ or } -152.52$$

$\longrightarrow$ two one-parameter families with frequency $\omega = 5$
($m = 0$ corresponds to axial symmetry)

There are non-rotating non-axially-symmetric geons

Example: $(l, m, n, \omega_s)_S = (2, 2, 0, 3)_S$

– angular dependence of the linearized solution is $\cos(2\phi)$

– there is a corresponding nonlinear solution

– it has zero angular momentum

Taking identical-amplitude linear combination of $(2, 2, 0, 3)_S$ and $(2, -2, 0, 3)_S$ with a shift in time phase, we get a rotating linearized solution, which corresponds to a rotating nonlinear geon with a helical Killing vector

After solving the consistency conditions at ε^3 order, two-parameter families of solutions may remain. They split into two one-parameter families because of the conditions at ε^5 order

Example: the one-parameter family of non-rotating geons generated by $(l, m, n, \omega_s)_S = (2, 2, 0, 3)_S$, and the axially symmetric one-parameter family generated by $(2, 0, 0, 3)_S$, appear to be a single two-parameter family at ε^3 order

$\longrightarrow$ it is important to go as high as ε^5 order in the expansion

It is necessary to use algebraic manipulation programs (Maple, Mathematica)

Same-frequency linear modes should be treated together

Lowest frequency is $\omega = 3$, belonging to $l = 2$, $n = 0$ scalar modes

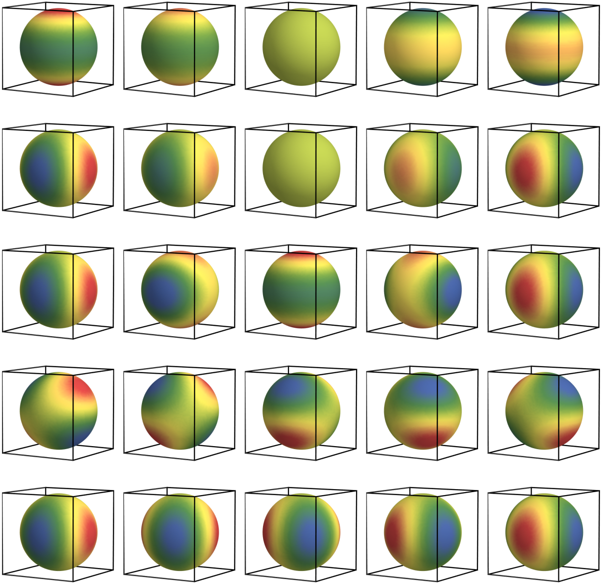
We have constructed all AdS geon solutions that in the small-amplitude limit reduce to $\omega = 3$ modes only

There are five such modes, belonging to $m = -2, -1, 0, 1, 2$, each of them can have $\cos(3t)$ or $\sin(3t)$ time dependence
$\longrightarrow$ there are 10 independent amplitude constants

Solutions are considered equivalent if they can be transformed into each other by time shift and spatial rotation

Result of detailed analysis up to ε^5 order :
There are 5 nonequivalent one-parameter families that reduce to $\omega = 3$ frequency modes only

$t = 0$ $\quad t = \frac{\pi}{12}$ $\quad t = \frac{\pi}{6}$ $\quad t = \frac{\pi}{4}$ $\quad t = \frac{\pi}{3}$



$(2,0,0,3)_S \times \cos(3t)$

$(2,2,0,3)_S \times \cos(3t)$

$(2,2,0,3)_S \times \cos(3t)$
$+(2,0,0,3)_S \times \sin(3t)$

$(2,1,0,3)_S \times \cos(3t)$
$+(2,-1,0,3)_S \times \sin(3t)$

$(2,2,0,3)_S \times \cos(3t)$
$+(2,-2,0,3)_S \times \sin(3t)$

Numerical method

KADATH library – multi-domain spectral method
– developed by Philippe Grandclément at Paris Observatory

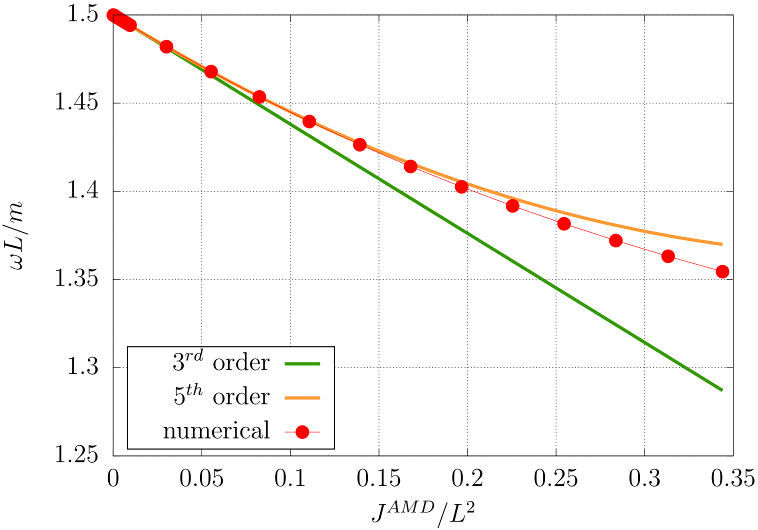
Maximal slicing ($K = 0$) and harmonic coordinates in space
– De-Turck method

Start from a linearized solution – increase the amplitude in steps

– typical resolution: radial 37, angular 9×9

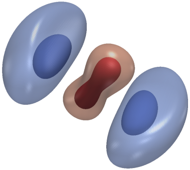
– typical running time: several days on hundreds of processors

Frequency – angular momentum relation for $(l, m, n, \omega_s)_S = (2, \pm 2, 0, 3)_S$ helically rotating geons

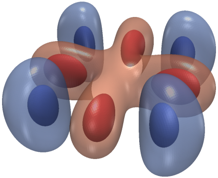


Helically symmetric rotating geons from the PhD Thesis of Gregoire Martinon 2017

isocontours of g_{tt} – (l, m, n, ω)

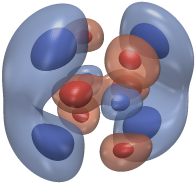


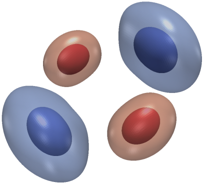
$(2, \pm 2, 0, 3)_S$

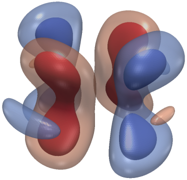


$(4, \pm 4, 0, 5)_S$

Three one-parameter families from the linear combination of $(2, \pm 2, 1, 5)_S$, $(4, \pm 2, 0, 5)_S$, $(3, \pm 2, 0, 5)_V$







Outlook – things to do

- Investigate non-rotating AdS geons
  - with or without axial symmetry
  - analytically and numerically
- Improve numerical method to reach maximal mass geons
- Study the stability of geons
  - $3 + 1$ dimensional time-evolution code
- Construction of asymptotically flat geons ...