Complexity for Warped AdS black holes

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Based on [R. Auzzi, S.B., G. Nardelli, arXiv: 1804.07521], [R. Auzzi, S.B., M. Grassi, G. Nardelli, N. Zenoni, arXiv: 1806.06216], [R. Auzzi, S.B., A. Mitra, G. Nardelli, N. Zenoni, arXiv: 1906.09345]

19^{th} July 2019

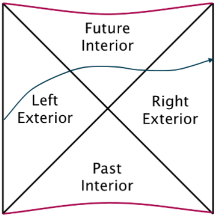
Outline

1. Complexity=Volume or Action conjectures
2. Black holes in Warped AdS_3
3. Computation of the volume
4. Computation of the action
5. Conclusions and perspectives

ER=EPR

Consider the Kruskal extension of the AdS black hole.
The dual interpretation is the existence of a thermofield double state

$$|\Psi_{TFD}\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-E_n\beta/2 - iE_n(t_L+t_R)} |E_n\rangle_R |E_n\rangle_L \, .$$



Correlators between the two CFTs are non-zero due to entanglement:

$$\langle \Psi_{TFD} | \mathcal{O}_L \mathcal{O}_R | \Psi_{TFD} \rangle \neq 0 \tag{1}$$

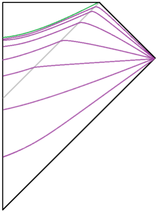
Boundaries are disconnected; the only way to communicate is through the interior regions $\Rightarrow$ the existence of the Einstein-Rosen bridge allows spacelike correlations (ER=EPR) [Maldacena, Susskind, 2013].

Evolution of Einstein-Rosen bridge

The Einstein-Rosen bridge grows with time far after the black hole reaches thermal equilibrium.
In order to follow the history of the interior region, we foliate spacetime with global spacelike slices [Susskind, 2014]:

- Geodesically complete causal curves must intersect these slices once
- Slices must stay away from curvature singularities
- The entire region outside the horizon must be foliated by these slices

Given the set of spacelike slices anchored on a spatial sphere with infinite radius, we choose the one with maximum volume.
Varying t, we foliate the spacetime with maximal slices.



What represents in the dual theory the growth of the Einstein-Rosen bridge?

Computational complexity

Consider a space of states and the concepts of simple state and simple operation. Example: a system composed of K classical bits

- Simple state: $(00000000\dots)$
- Generic state: $(0010111001\dots)$
- Simple operation: flip a single bit $(0 \leftrightarrow 1)$

Computational complexity is the least number of simple operations needed to obtain a generic final state starting from a simple one.

Classical physical quantities:

- Maximum entropy $S = K \log 2$
- Thermalization time $t_{\text{therm}} \sim K^p$
- Maximum complexity $C = K/2$
- Time to get maximally complex $t_{\text{compl}} \sim K^p$

Quantum complexity

Quantum mechanically, we assume the existence of an Hilbert space.
Example: a system of K qubits

- Simple state $|0\rangle = |00000\dots\rangle$
- Generic state $|\psi\rangle = \sum_{i=1}^{2^K} \alpha_i |i\rangle$
- Simple operation: act on 2 qubits

Complexity is the minimum number of simple unitary operators required to transform a simple state into a generic one.
Quantum physical quantities:

- Maximum entropy $S = K \log 2$
- Thermalization time $t_{\text{therm}} \sim K^p$
- Maximum complexity $C = e^K$
- Time to get maximally complex $t_{\text{compl}} \sim e^K$

Complexity=Volume conjecture

Conjecture (Stanford, Susskind, 2014)

The complexity of the boundary state is proportional to the spatial volume V of the Einstein-Rosen bridge anchored at the boundary:

$$C_V \sim \frac{\mathrm{Max}(V)}{Gl} \tag{2}$$

Requirements about complexity from the gravity side:

- Complexity is extensive and proportional to the degrees of freedom of the system [Stanford, Susskind, 2014]:

$$\frac{dC}{dt} \sim TS \tag{3}$$

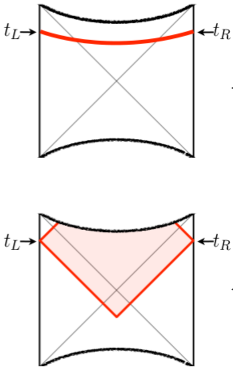
- Extremal black holes are ground states and therefore static $\Rightarrow$ they have vanishing complexity rate

Complexity=Action conjecture

Conjecture (Brown, Roberts, Susskind, Swingle, Zhao, 2016)

The complexity of the boundary state is proportional to the classical action I computed in the Wheeler-de Witt patch associated to a boundary section:

$$C_A = \frac{I}{\pi \hbar} \tag{4}$$



Aspects of the conjecture:

- It is more universal: the normalization is independent from the background
- It passes the same tests of the Complexity=Volume proposal for late times
- For intermediate times, Volume and Action conjectures give different results

Black holes in Warped AdS_3

Black holes metric in Warped AdS_3 [Anninos, Padi, Song, Strominger, 2008]:

$$\frac{ds^2}{l^2} = dt^2 + \frac{dr^2}{(\nu^2+3)(r-r_+)(r-r_-)} + \left(2\nu r - \sqrt{r_+ r_- (\nu^2+3)}\right) dt d\theta + \frac{r}{4}\left[3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_-(\nu^2+3)}\right] d\theta^2 \,, \tag{5}$$

with r_-, r_+ inner and outer horizons.

- If $\nu = 1$ we recover the BTZ black hole in AdS spacetime. A change of coordinates recasts the metric in the standard form (in Poincaré patch)

$$ds^2 = -\frac{\tilde{r}^2 - \tilde{r}_+^2 - \tilde{r}_-^2}{l^2} d\tilde{t}^2 + \frac{l^2 \tilde{r}^2 d\tilde{r}^2}{(\tilde{r}^2 - \tilde{r}_-^2)(\tilde{r}^2 - \tilde{r}_+^2)} - 2\frac{\tilde{r}_+ \tilde{r}_-}{l} d\tilde{t} d\tilde{\theta} + \tilde{r}^2 d\tilde{\theta}^2$$

- If $\nu^2 < 1$ the solution admits closed timelike curves [Banados, Barnich, Compère, Gomberoff, 2005]

Eddington-Finkelstein coordinates

The metric of the Warped black hole is put in Arnowitt-Deser-Misner form

$$ds^2 = -N^2 dt^2 + \frac{l^4 dr^2}{4R^2 N^2} + l^2 R^2 (d\theta + N^\theta dt)^2 \,, \tag{6}$$

with an appropriate choice of $\{R, N, N^\theta\}$.
We consider a set of null geodesics satisfying $(d\theta + N^\theta dt) = 0$ to introduce coordinates à la Eddington-Finkelstein

$$du = dt - \frac{l^2}{2RN^2} dr \,, \qquad dv = dt + \frac{l^2}{2RN^2} dr \,, \tag{7}$$

where null geodesics are described by constant values of u, v.
We define the finite coordinate transformation as

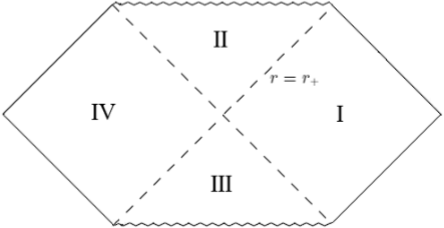
$$u = t - r^*(r) \,, \qquad v = t + r^*(r) \,, \tag{8}$$

where

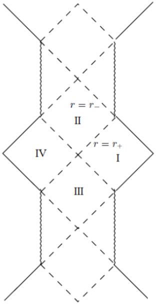
$$\frac{dr^*}{dr} = \frac{l^2}{2RN^2} \,. \tag{9}$$

Causal structure

Using null coordinates, we build the Kruskal extension of the Warped black hole and then we find the causal structure of spacetime.
The Penrose diagrams are the same of Schwarzschild (non-rotating case) and Reissner-Nordstrom (rotating case) black holes in 3+1 dimensions [Jugeau, Moutsopoulos, Ritter, 2010].



Non-rotating case



Rotating case

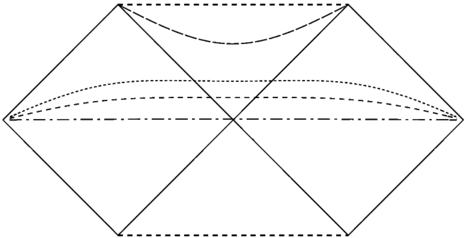
Extremal volume

Time translation symmetry in Schwarzschild coordinates corresponds to invariance under the time evolution in the boundary WCFT with Hamiltonian $H = H_L - H_R$

$$t_L \to t_L + \Delta t \,, \qquad t_R \to t_R - \Delta t \,. \tag{10}$$

It is not restrictive to consider for the extremal volume the symmetric configuration

$$t_L = t_R = \frac{t_b}{2} \tag{11}$$



Computation of the volume

We follow the strategy of [Carmi, Chapman, Marrochio, Myers, Sugishita, 2017]. The volume functional is taken along the angular direction giving

$$V = 2 \times 2\pi \int_{\lambda_{\min}}^{\lambda_{\max}} d\lambda \, \mathcal{L}(r, \dot{r}, \dot{v}) \, ,$$

with conserved quantity

$$E = \frac{1}{l^2} \frac{\partial \mathcal{L}}{\partial \dot{v}} \, . \tag{12}$$

The volume can be written as

$$\begin{aligned} \frac{V}{4\pi l^2} \;=\; & \int_{r_{\min}}^{r_{\max}} dr \left[\frac{\sqrt{4E^2 + (\nu^2+3)(r-r_-)(r-r_+)}}{2(\nu^2+3)(r-r_-)(r-r_+)} \times \right. \\ & \times \sqrt{\left(2\nu r - \sqrt{r_+ r_- (\nu^2+3)}\right)^2 - (\nu^2+3)(r-r_-)(r-r_+)} \\ & \left. - E \frac{2\nu r - \sqrt{r_+ r_- (\nu^2+3)}}{(\nu^2+3)(r-r_-)(r-r_+)} \right] + E(v(r_{\max}) - v(r_{\min})) \, . \end{aligned} \tag{13}$$

Differentiating with respect to time we get

$$\frac{1}{2l}\frac{dV}{dt_R} = \frac{dV}{d\tau} = 2\pi l E\,. \tag{14}$$

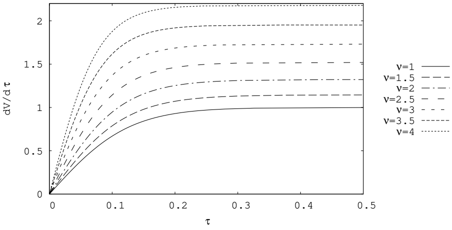


Figure 1: Time dependence of $dV/d\tau$ in units of πl, for $r_0 = 1$ and various values of the warping parameter ν

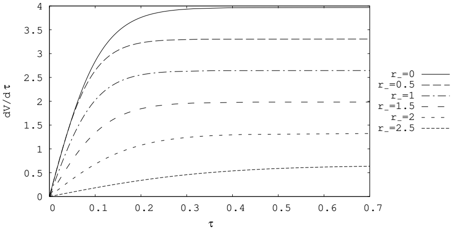


Figure 2: Time dependence of $dV/d\tau$ in units of πl, for $r_+ = 3, \nu = 2$ and various values of the inner radius r_-

Late time complexity

In the late time limit, the volume is invariant under translations in t and rotations in $\theta \Rightarrow$ the maximal slice sits at constant $r = \hat{r}$ [Susskind, 2014].
Extremizing the volume, the only possible constant-r slice sits at

$$\hat{r} = \frac{r_+ + r_-}{2} \Rightarrow \lim_{\tau \to \infty} \frac{dV}{d\tau} = \frac{\pi l}{2}(r_+ - r_-)\sqrt{3 + \nu^2}\,. \tag{15}$$

Consistency checks:

- It vanishes in the extremal case
- It is proportional to the product

$$TS = \frac{(r_+ - r_-)(3 + \nu^2)}{16G}\,. \tag{16}$$

- It satisfies a bound involving the conserved charges of the black hole [Cai, Ruan, Wang, Yang, Peng, 2016]

$$\frac{dV}{d\tau} \leq [(M - \Omega J)_+ - (M - \Omega J)_-] = TS\,. \tag{17}$$

Holographic dictionary for complexity

In AdS_D case the standard dictionary is

$$\lim_{\tau \to \infty} \frac{dV}{d\tau} = \frac{8\pi G l}{D-1} T S \,, \qquad C = (D-1) \frac{V}{G l} \,. \tag{18}$$

In the warped AdS_3 case we obtain

$$\lim_{\tau \to \infty} \frac{dV}{d\tau} = 4\pi G l \, \eta \, T S \,, \qquad \eta = \frac{2}{\sqrt{3+\nu^2}} \,. \tag{19}$$

Possible interpretations:

- Complexity approaches at late times $\eta T S$ with $\eta \leq 1 \Rightarrow$ warping would make the complexity rate decrease
- The holographic dictionary is

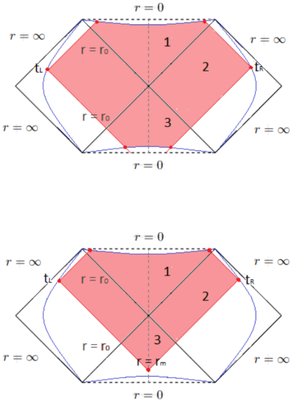
$$C = \frac{2}{G l \eta} V \,, \tag{20}$$

  and the rate always saturates at TS

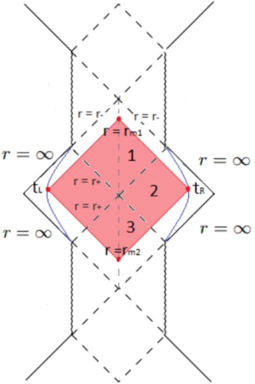
Wheeler-de Witt patch

We put a cutoff surface at $r = \Lambda$, where we fix the boundary slices.
The Wheeler-de Witt patch changes after a critical time t_C in the non-rotating case, while it is the same at all times in the rotating case.

Non-rotating case



Rotating case



Computation of the action

The action on the Wheeler-de Witt patch is decomposed as

$$I = I_{\mathcal{V}} + I_{\mathcal{B}} + I_{\mathcal{J}} + I_{\text{ct}} \tag{21}$$

- $I_{\mathcal{V}}$ is the bulk contribution
- $I_{\mathcal{B}}$ is the contribution from boundaries
- $I_{\mathcal{J}}$ is the contribution from joints
- I_{ct} is a counterterm

Bulk action [Banados, Barnich, Compère, Gomberoff, 2005]:

$$I_{\mathcal{V}} = \frac{1}{16\pi G} \int_{\text{WdW}} d^3x \left\{ \sqrt{g} \left[\left(R + \frac{2}{L^2} \right) - \frac{\kappa}{4} F^{\mu\nu} F_{\mu\nu} \right] - \frac{\alpha}{2} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \right\} , \tag{22}$$

which admits regular black hole solutions without closed timelike curves if $\kappa = -1$.

Gibbons-Hawking-York term for timelike ($\varepsilon = 1$) and spacelike boundaries ($\varepsilon = -1$) [York, 1972; Gibbons, Hawking, 1977]:

$$I_{\text{GHY}} = \frac{\varepsilon}{8\pi G} \int_{\mathcal{B}} d^2x \sqrt{h}\, K \, , \tag{23}$$

where h_{ij} is the indiced metric and K is the extrinsic curvature.

Surface term for null boundaries [Lehner, Myers, Poisson, Sorkin, 2016]:

$$I_{\mathcal{N}} = \frac{1}{8\pi G} \int_{\mathcal{B}} d\lambda dS \, \tilde{\kappa} \, , \tag{24}$$

where λ parametrizes the null direction of the surface and $\tilde{\kappa}$ measures the failure of λ to be an affine parameter.

Joint terms [Hayward, 1993; Lehner, Myers, Poisson, Sorkin, 2016]:

$$I_{\mathcal{J}} = \frac{1}{8\pi G} \int_{\Sigma} d\theta \sqrt{\sigma} \, \mathfrak{a} \, , \tag{25}$$

where σ_{ab} is the induced metric and $\mathfrak{a}$ depends from the joint.

Counterterm [Lehner, Myers, Poisson, Sorkin, 2016]:

$$I_{\text{ct}} = \frac{1}{8\pi G} \int d\theta \, d\lambda \, \sqrt{\sigma} \, \Theta \log |\tilde{L}\Theta| \, , \tag{26}$$

where $\tilde{L}$ is a length scale and Θ is the expansion of the geodesics.

The late time behaviour is given by

$$\frac{1}{2l}\frac{dI}{dt_R} = \frac{dI}{d\tau} = \frac{\nu^2+3}{16G}(r_+ - r_-) = TS\,. \tag{27}$$

Complexity approaches TS without additional factors of (ν, l).

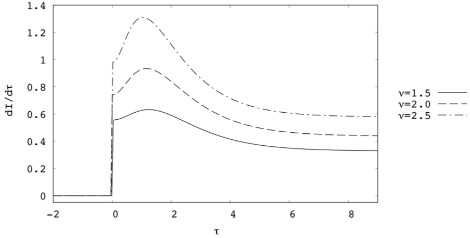


Figure 3: Time dependence of $dI/d\tau$ for $r_0 = 1$ and various values of the warping parameter ν. The critical time corresponds to $\tau = 0$.

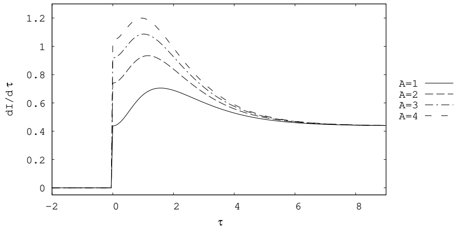


Figure 4: Time dependence of the WDW action in the non-rotating case for different values of the parameter A. We set $G = 1, l = 1, r_0 = 1$ and $\nu = 2$.

Subregion complexity

Notion of subsystem complexity for a mixed state on the boundary[Carmi, Myers, Rath, 2016]:

Conjecture (Subregion Complexity=Volume)

The complexity of a subregion is dual to the volume of the extremal codimension-1 slice anchored to the boundary and its Ryu-Takayanagi surface.

Conjecture (Subregion Complexity=Action)

The complexity of a subregion is dual to the gravitational action computed on the intersection of the WDW patch and the entanglement wedge associated to the region.

We can distinguish between the two conjectures by studying

- The structure of UV divergences
- The subadditivity or superadditivity properties
- The temperature dependence

Conclusions and perspectives

Conclusions

- Complexity is a monotonically increasing function of time
- Volume: The rate of growth is a monotonically increasing function of time and saturates to a constant value at late times
  Action: The rate of growth increases up to a maximum, then decreases until it reaches a constant value for late times
- Complexity rate is proportional to TS at late times

Perspectives

- Study of complexity in the boundary WCFT [Caputa, Kundu, Miyaji, Takayanagi, Watanabe, 2017]
- Study of the complexity for general subregions [Carmi, Myers, Rath, 2017],[Erdmenger et al., 2018],[Swingle et al, 2018],[Alishahiha et al., 2018]

Thank you for the attention!