

Large-N $\mathbb{C}P^{N-1}$ sigma model on a Euclidean torus: uniqueness and stability of the vacuum

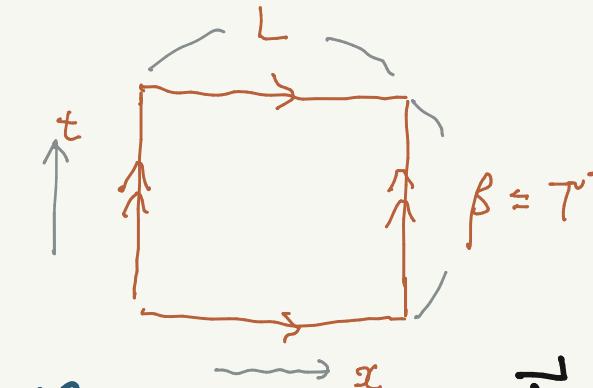
Keisuke Ohashi @ Keio University, Japan

based on **arXiv:1905.10555**

in collaboration with

S.Bolognesi, S.B.Gudnason, K.Konishi

§1. Introduction & Motivation



Large- N $\mathbb{C}P^{N-1}$ sigma model on a Euclidean torus

$$\vec{n} = \vec{n}(x, t) \in \mathbb{C}^N$$

$$S_E = \int_0^\beta dt \int_0^L dx \left[D_\mu \vec{n}^+ \cdot D_\mu \vec{n}^- + \lambda (\vec{n}^+ \cdot \vec{n}^- - r) - p_{uv} \right]$$

↑ Lagrange multiplier
↓ bare pressure const.
 $\hookrightarrow D_\mu = \partial_\mu + i A_\mu$
↓ bare size of $\mathbb{C}P^{N-1}$

the gap equation = the saddle point eq. of λ

$$\langle \vec{n}^+(x, t) \cdot \vec{n}^-(x, t) \rangle = r \equiv \frac{N}{2\pi} \ln \frac{\Lambda_{uv}}{\Lambda} + \dots$$

Mass gap in the confinement phase for large L and $\beta = T^{-1}$

$$\lim_{L, \beta \rightarrow \infty} \langle \lambda(x, t) \rangle \equiv \Lambda^2 > 0$$

ref. D'Adda et. al ('78) Witten ('79), ...

Recent works

- S. Monin, M. Shifman and A. Yung, Phys. Rev. D 92 (2015) no.2, 025011:

"deconfinement" phase ($\langle \lambda \rangle = 0$) for $\beta = \infty$, small $L < L_*$

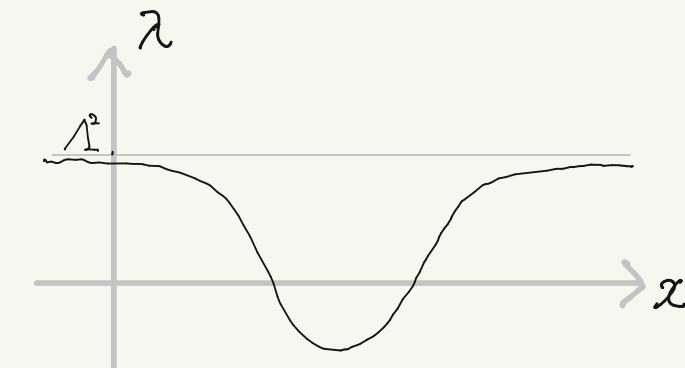
- M. Nitta and R. Yoshii, JHEP 1712, 145 (2017): for $L, \beta = \infty$

soliton like "solution" by using a map from the chiral Gross-Neveu model to the $\mathbb{C}P^{N-1}$ model

- A. Gorsky, A. Pikalov and A. Vainshtein, arXiv:1811.05449. for $L, \beta = \infty$

$$\text{"E}_{\text{soliton}} < E_{\text{vacuum}}$$

\Rightarrow Instability of the vacuum!?



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" $E_{\text{soliton}} < E_{\text{vacuum}}$ " \Rightarrow Instability of the vacuum !!

There is no problem,

if the soliton-like "solution" is not a solution of the saddle point eq. for λ .

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~~soliton like "solution"~~ by using a map from the chiral Gross-Neveu model to the $\mathbb{C}P^{N-1}$ model

This is **not true solution!!**

- A. Gorsky, A. Pikalov and A. Vainshtein, arXiv:1811.05449. for $L, \beta = \infty$

" $E_{\text{soliton}} < E_{\text{vacuum}}$



The vacuum is stable

Recent works

- S. Monin, M. Shifman and A. Yung, Phys. Rev. D 92 (2015) no.2, 025011:
"deconfinement" phase ($\langle \lambda \rangle = 0$)
for $\beta = \infty$, small $L < L_*$

The same wrong technic are used.
- M. Nitta and R. Yoshii, JHEP 1712, 145 (2017) : for $L, \beta = \infty$
~~soliton like "solution"~~ by using a map from the chiral Gross-Neveu model to the $(\mathbb{C}P^{N-1})_{\text{model}}$
- A. Gorsky, A. Pikelov and A. Vainshtein, arXiv:1811.05449. for $L, \beta = \infty$
"E soliton" $<$ E vacuum

Today's talk

The vacuum is unique and stable for $\forall L, \forall \beta$.

arXiv:1905.10555

§2. The gap equation and uniqueness of solutions

Partition function

$$Z = \int dA_\mu d\lambda Z_{\text{pre}}[\lambda, A_\mu],$$

with

$$Z_{\text{pre}}[\lambda, A_\mu] = \int d\vec{n} d\vec{n}^+ e^{-S_E} = Z_0 e^{\int dx r \lambda \left[\prod_{\vec{k} \in \mathbb{Z}^2} w_{\vec{k}}^{-2} \right]^N}$$

$\left\{ w_{\vec{k}}, f_{\vec{k}}(x, t) \mid \vec{k} \in \mathbb{Z}^2, w_{\vec{k}}^2 > 0 \right\}$: eigen system defined by

$$[-\partial_\mu \partial_\mu + \lambda(x, t)] f_{\vec{k}}(x, t) = w_{\vec{k}}^2 f_{\vec{k}}(x, t)$$

$$\int dx f_{\vec{k}}(x, t) \bar{f}_{\vec{k}}(x, t) = \delta_{\vec{k}, \vec{k}}$$

- Large- N limit

a saddle point

$$Z \sim Z_{\text{pre}}[\lambda^{\text{sp}}, A_{\mu}^{\text{sp}}]$$

(the free energy: $F = -T \ln Z \sim -T \ln Z_{\text{pre}}|_{\text{sp}}$)

- Saddle point equations

$$0 = \frac{\delta}{\delta \lambda(x,t)} \ln Z_{\text{pre}}[\lambda, A_{\mu}] \Big|_{\text{sp}} = r - \sum_{\vec{k} \in \mathbb{Z}^2} \frac{|\vec{f}_{\vec{k}}(x,t)|^2}{w_{\vec{k}}^2} \Big|_{\text{sp}}$$

↓ regularization : gap equation

$w_{\vec{k}} \neq 0$

renormalization

At the saddle point, a spectrum can contain no zero-mode.

$$0 = \frac{\delta}{\delta A_{\mu}(x,t)} \ln Z_{\text{pre}}[\lambda, A_{\mu}] \Big|_{\text{sp}} = \sum_{\vec{k} \in \mathbb{Z}^2} \frac{i(\vec{f}_{\vec{k}} \partial_{\mu} \vec{f}_{\vec{k}} - \vec{f}_{\vec{k}} \partial_{\mu} \vec{f}_{\vec{k}})}{w_{\vec{k}}^2} \Big|_{\text{sp}}$$

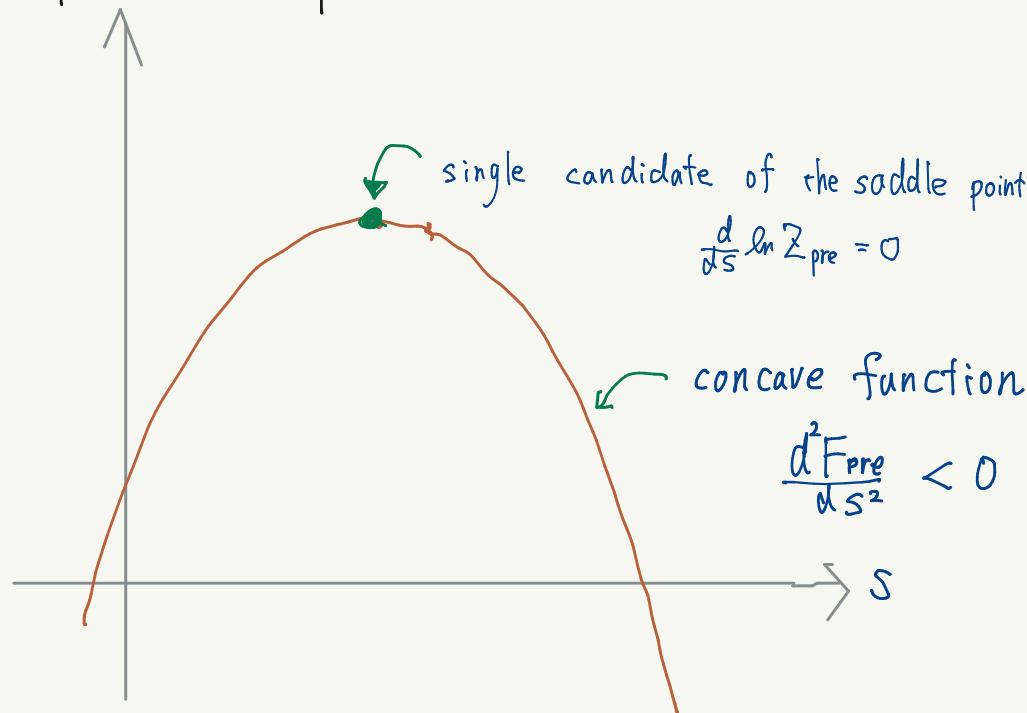
• with 1-para. family config. $\lambda^{(s)}(x,t) \equiv s \lambda_1(x,t) + (1-s) \lambda_2(x,t)$, $\dot{\lambda}^{(s)} = \frac{d\lambda^{(s)}}{ds} \in \mathbb{R}$

$$\frac{d^2}{ds^2} \ln Z_{\text{pre}}[\lambda^{(s)}, A_\mu] \stackrel{\text{fix}}{=} \sum_{k, \ell \in \mathbb{Z}^2} \frac{\left| \int dx \dot{\lambda}^{(s)} f_k^\dagger \tilde{f}_\ell \right|^2}{w_k^2 w_\ell^2} > 0 \quad (\because w_k^2 > 0)$$

positive definite !!
 Well-defined quantity without using regularization
 for 2-dim $\mathbb{C}P^{N-1}$ model

"pseudo free energy"

$$F_{\text{pre}} = -T \ln Z_{\text{pre}}$$



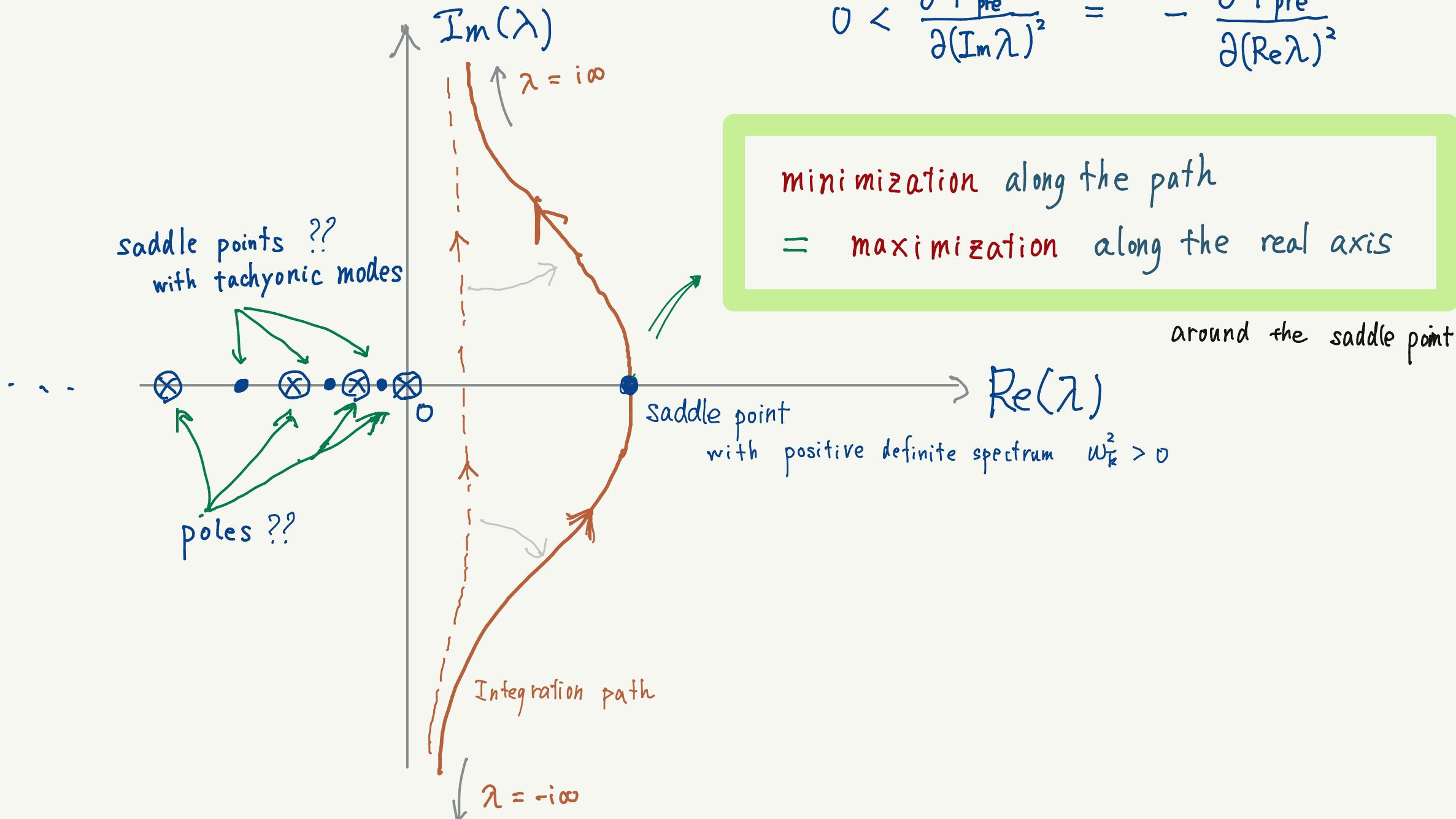
- On the real axis of λ , with a given A_μ
positive
 the saddle point must be unique if it exists.
- At the saddle point, along the real axis of λ
 "pseudo free energy" $F_{\text{pre}} = -T \ln Z_{\text{pre}}$ is maximized.

Q Why "pseudo free energy" F_{pre} is maximized?

Answer :

Integration path of $\lambda \perp$ the real axis of λ

$$0 < \frac{\partial^2 F_{\text{pre}}}{\partial (\text{Im} \lambda)^2} = - \frac{\partial^2 F_{\text{pre}}}{\partial (\text{Re} \lambda)^2}$$



§3. Existence of the solution: the homogeneous vacuum for λ_L, λ_β .

Under the assumption of translational invariance

turns out to be...

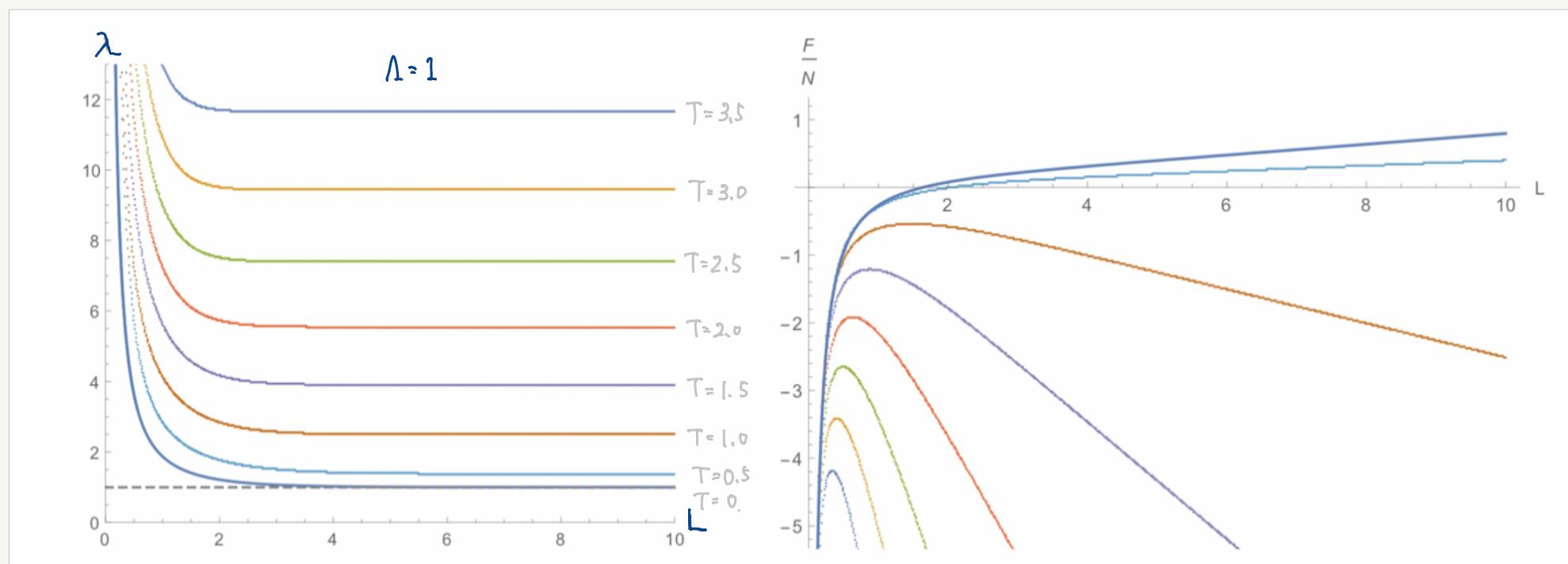
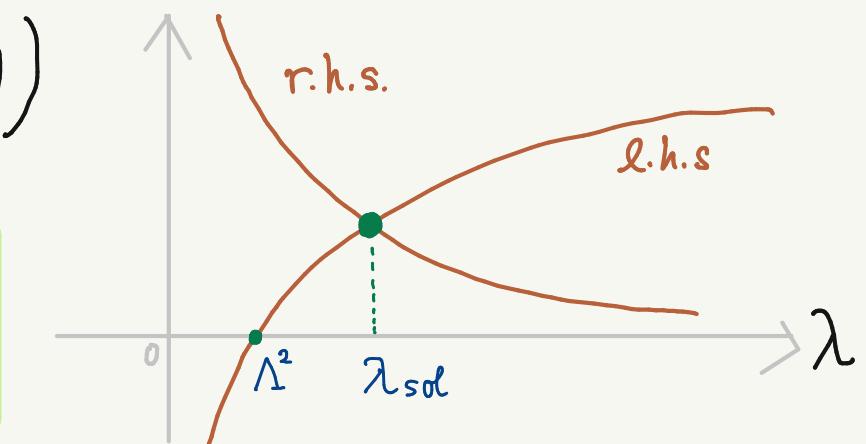
$$\langle \lambda(x, t) \rangle = \lambda : \text{const.} \quad \langle A_\mu(x, t) \rangle = \text{const.} \Rightarrow 0$$

the gap equation

the 2-nd kind of the modified Bessel func.

$$\Rightarrow \frac{N}{4\pi} \ln \frac{\lambda}{\Lambda^2} = \frac{N}{2\pi} \sum_{n, m \in \mathbb{Z}^2 \setminus \{0, 0\}} K_0(\sqrt{\lambda((m\beta)^2 + (nL)^2)})$$

\Rightarrow There always exists an unique solution $\lambda = \lambda_{\text{sol}} \geq \Lambda^2$



§4. Deconfinement phase for small L?

Let us separate $\vec{n}(x,t)$ as

$$n_i(x,t) = \underbrace{\sigma_i}_{\text{zero-modes}} + \text{massive modes}$$

$\lambda \neq 0 !!$

$$Z_{\text{pre}}[\lambda, A_\mu] = \underbrace{Z_\sigma}_{\text{zero-mode}} \times \underbrace{Z_{\text{mass}}}_{\text{massive-mode}} \quad \text{with} \quad Z_\sigma = \prod_{i=1}^N \int d\sigma_i d\bar{\sigma}_i e^{-\beta L \lambda |\sigma_i|^2} = \left(\frac{\pi}{\beta L \lambda} \right)^N$$

- Wilsonian-effective-action like treatment in Monin-Shifman-Yung (2015) with $\beta = \infty$

with fixing $\sigma_i = \delta_{i1} \sigma$

σ is treated classically

"zero-mode eq."

$$\lambda \sigma = 0$$

\oplus

gap eq.

$$\sigma^2 - \lim_{\beta \rightarrow \infty} \frac{1}{\beta L} \frac{\partial}{\partial \lambda} \ln Z_{\text{mass}} = 0$$

→ They found "deconfinement" phase: $\lambda = 0, \sigma > 0$, for $L < L_*$

\uparrow
This solution is pathological !!

→ We cannot omit a finite-temperature effect.

\uparrow as a reference energy scale μ

• Correct Willsonian-effective-action like treatment.

Let us treat only "effective" size of $\mathbb{C}P^{N-1}$, $\hat{\sigma}^2 = \sum_{i=1}^N |\sigma_i|^2$, as a classical quantity.

Rewriting of Z_σ

$$Z_\sigma = \pi \int d\sigma_i |^2 \underbrace{\int d\hat{\sigma}^2 \delta(\hat{\sigma}^2 - \sum |\sigma_i|^2)}_{\text{insertion of 1.}} e^{-\beta L \lambda \hat{\sigma}^2} = \int d\hat{\sigma}^2 e^{-S_\sigma}$$

$$\text{with } S_\sigma \equiv N \ln \left(\frac{N}{\pi e \hat{\sigma}^2} \right) + \beta L \lambda \hat{\sigma}^2$$

saddle point eq. of $\hat{\sigma}^2$

$$\lambda \hat{\sigma} = \frac{N}{\beta L \hat{\sigma}}$$

$$\text{modified gap eq. } \hat{\sigma}^2 - \frac{1}{\beta L} \frac{\partial}{\partial \lambda} \ln Z_{\text{mass}} = 0$$

volum effect

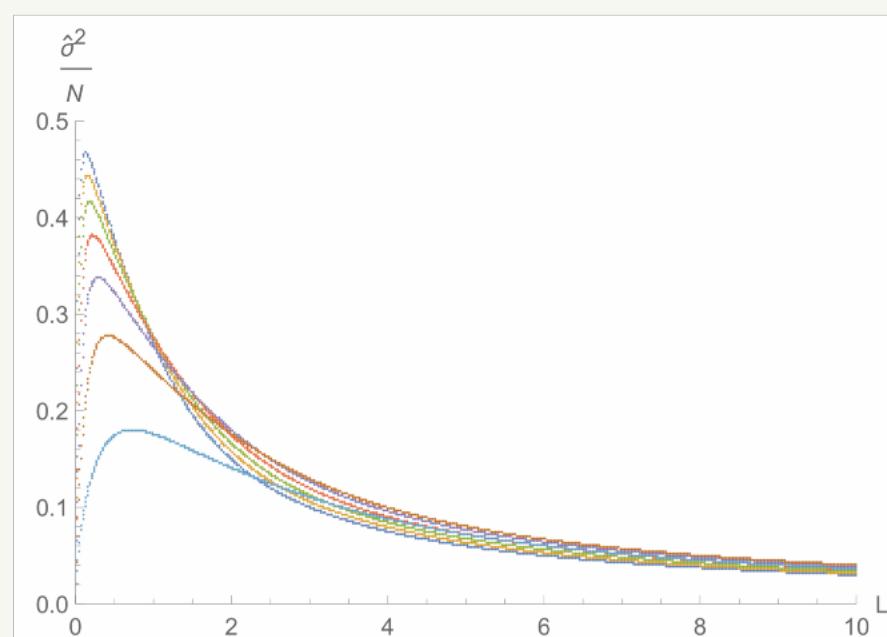
equivalent

the original gap eq.

$$\frac{\partial}{\partial \lambda} \ln Z[\lambda, A_r] = 0$$

Confinement phase $\lambda \geq \lambda^*$

not deconfinement phase!!



• Zero-temperature limit.

$$\hat{\sigma}^2 = \frac{TN}{L\lambda} \leq \frac{TN}{L\Lambda^2} \Rightarrow \lim_{T \rightarrow 0} \hat{\sigma}^2 = 0$$

§5. Summary

We proved that Large-N $\mathbb{C}\mathbb{P}^{N-1}$ sigma model on a torus has an unique stable homogeneous vacuum in the confinement phase for arbitrary L and T.

Actually, with correct Willsonian-effective-action like treatment the deconfinement phase proposed by Monin-Shifman-Yung is disappeared.

Similarly, soliton like solutions proposed by Nitta-Yoshii are not true solutions.