

CLASSICAL QUANTUM DUALITY FROM PATH INTEGRAL

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Some (apparently) unrelated problems

- 1 Wave function collapse, EPR, spooky action at a distance (quantum non-locality)
- 2 Observables: self adjoint operators. Questions: how measure $A + B$?, superselection sectors, observables with sharp values?
- 3 QM is an effective theory. However, there is not a natural limit to get QM from QFT.
- 4 In QFT there are no observables in the QM sense. For example, the position operator does not exist
- 5 4D QFT works only perturbatively (asymptotics series)

What is Classical Mechanics?

Dirac (1933):

$$\langle x' t' | x t \rangle = \int D x e^{\frac{i}{\hbar} \int_t^{t'} d\tau \mathcal{L}(q, \dot{q})} .$$

Dirac: Classical Mechanics is the saddle point approximation of the path integral

see <https://www2.pd.infn.it/matone/QFTCourseNotes.pdf>

We will show

- 1 There is a dual interpretation of Classical Mechanics: expressed as the path integral of a Quantum Theory
- 2 Relation between Fourier and Legendre Transform: leads to Modular transformations in QFT (connection with the Seiberg-Witten duality)
- 3 Nonperturbative from Perturbative (connection with resurgent theory?)

A dual interpretation

Consider the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi , \quad (1)$$

and set

$$\psi = \text{Re}^{i\hbar S} , \quad (2)$$

to get the quantum Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + V - \frac{\hbar^2}{4m} \frac{\Delta R}{R} = 0 , \quad (3)$$

that with respect to the classical Hamilton-Jacobi equation differs by the quantum potential

$$Q = -\frac{\hbar^2}{4m} \frac{\Delta R}{R} . \quad (4)$$

In addition to Eq.(3), the Schrödinger equation implies

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(R^2 \frac{\nabla S}{m} \right) = 0 . \quad (5)$$

Now, consider the Classical Hamilton-Jacobi equation in terms of the modified Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi_c = (H - Q) \psi_c . \quad (6)$$

Replacing ψ_c by (2), we get

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + V = 0 . \quad (7)$$

We then have

$$\lim_{Q \rightarrow 0} \psi_c = \psi . \quad (8)$$



Recall that

$$\begin{aligned}\langle x' t' | x t \rangle &= \int D\mathbf{x} e^{\frac{i}{\hbar} \int_t^{t'} d\tau \mathcal{L}(q, \dot{q})} \\ &= e^{\frac{i}{\hbar} H(t' - t)} \delta(x' - x) .\end{aligned}\tag{9}$$

Similarly, in QFT

$$\langle \phi_b | e^{-iHT} | \phi_a \rangle = e^{-iHT} \langle \phi_b | \phi_a \rangle = e^{-iHT} \delta(\phi_b - \phi_a) ,$$

with $\delta(\phi_b - \phi_a)$ the functional δ distribution whose argument is the difference between two fields. $\delta(\phi_b - \phi_a)$ can be thought of as an infinite product

$$\delta(\phi_b - \phi_a) = \prod_{\mathbf{x} \in \mathbb{R}^3} \delta(\phi_b(\mathbf{x}) - \phi_a(\mathbf{x})) ,$$

The Duality

Consider a scalar theory (extends to all QFT's).

The generating functional is the functional Fourier transform of the classical action

$$Z[J] = \int \mathcal{D}\phi \exp \left[\frac{i}{\hbar} \left(S[\phi] + \int d^4x J(x)\phi(x) \right) \right], \quad (10)$$

it follows that the classical action $S[\phi]$ is expressed in terms of the inverse transform of the generating functional

$$\exp \left(\frac{i}{\hbar} S[\phi] \right) = \int \mathcal{D}J \exp \left[\frac{i}{\hbar} \left(W[J] - \int d^4x J(x)\phi(x) \right) \right], \quad (11)$$

where $Z[J] = \exp \left(iW[J]/\hbar \right)$.

- 1 Key role of the external source
- 2 Same construction for QM
- 3 Extends to the Euclidean formulation
- 4 AdS/CFT correspondence

$$e^{-S_{SG}[\phi_0]} = \langle e^{\int_{S^d} \phi_0 \mathcal{O}} \rangle_{CFT} ,$$

with S_{SG} the classical supergravity action, ϕ_0 is the restriction of ϕ to the boundary of AdS_{d+1} . \mathcal{O} a conformal field coupling to the external source ϕ_0 .

We then have that the classical action is given by the path integral with the quantum action

$$S_Q[J] = W[J] ,$$

that is classical mechanics is seen as a quantum theory with $W[J]$ playing the same role of the classical action in defining $W[J]$. In particular, there is a ϕ - J duality. In (10) J is the source and ϕ the field, whereas in the dual picture (11) $-\phi$ is the source and J the field.

The above duality extends to several structures holding in the standard path integral formulation of both quantum mechanics and quantum field theory. In particular, the vacuum to vacuum amplitude

$$\langle \Omega | \Omega \rangle = \exp \left(iW[0]/\hbar \right) ,$$

has a classical counterpart

$$\langle \Omega_C | \Omega_C \rangle = \exp \left(iS[0]/\hbar \right) .$$

The above duality is rather different with respect to previous formulations of classical mechanics in terms of a quantum formalism

- ① Koopman - von Neumann classical mechanics in terms of Hilbert space (1932, 1933)
- ② Path integral with classical trajectories

Effective action from saddle point approximation

In the saddle point approximation, the leading contribution to $Z[J]$ is given by

$$S[\phi, J] = S[\phi] + \int d^4x J(x)\phi(x)$$

evaluated on its minimum ϕ_0 , that is

$$W[J] = S[\phi_0[J], J] + \mathcal{O}(\hbar) . \quad (12)$$

In the dual formulation, we have that the exponent in the integrand of (11)

$$S_Q[J, \phi] = W[J] - \int d^4x J(x)\phi(x) ,$$

evaluated on the J for which it takes the minimum, that is such that

$$\varphi = \frac{\delta W[J]}{\delta J(x)} ,$$

is just the effective action.

Therefore, the dual version of

$$W[J] = S[\phi_0[J], J] + \mathcal{O}(\hbar) ,$$

is the identification of $S_Q[J, \varphi]$ with the effective action $\Gamma[\varphi]$.
In particular, treating φ as independent variable

$$S_Q[J[\varphi], \varphi] = \Gamma[\varphi] .$$

A Hidden Invariance

The above results show a relationship between

- 1 Fourier transform and its inverse
- 2 Legendre transform.

Such a relationship implies an invariance which holds in general and will be applied to the generating functional.

We want to extend the modular transformations found in Seiber-Witten theory to arbitrary theories. To explain this point we start with the relation between the $u = \langle \text{Tr } \phi^2 \rangle$ modulus and the prepotential \mathcal{F} in Seiberg-Witten theory. Such a relation shows that u is proportional to the Legendre transform of \mathcal{F} with respect to a^2

$$u = \pi i \left(\mathcal{F} - \frac{a}{2} \frac{\partial \mathcal{F}}{\partial a} \right). \quad (13)$$

Set

$$a^D = \frac{\partial \mathcal{F}}{\partial a}, \quad (14)$$

and consider the transformation

$$\begin{pmatrix} a^D \\ a \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{a}^D \\ \tilde{a} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a^D \\ a \end{pmatrix}. \quad (15)$$

Eqs.(13), (14) and (15) imply

$$\tilde{\mathcal{F}}(\tilde{a}) = \mathcal{F}(a) + \frac{AC}{2}a_D^2 + \frac{BD}{2}a^2 + BCaa_D. \quad (16)$$

This means that the function

$$\mathcal{G}(a) = \pi i \left(\mathcal{F}(a) - \frac{1}{2}a\partial_a\mathcal{F}(a) \right), \quad (17)$$

is modular invariant, that is

$$\tilde{\mathcal{G}}(\tilde{a}) = \mathcal{G}(a). \quad (18)$$

Taking the second derivative with respect to u we have

$$\frac{a''}{a} = \frac{a_D''}{a_D} ,$$

that is a and a^D are two linearly independent solutions of a second order differential equation. If there is a subgroup such that $\tilde{G}(\tilde{a}) = G(\tilde{a})$, so that

$$G(\tilde{a}) = G(a) ,$$

then we have a full symmetry that, in principle, fixes $G(a)$. In $N = 2$ SYM with gauge group $SU(2)$

$$\left[(1 - u^2) \partial_u^2 - \frac{1}{4} \right] a_D = \left[(1 - u^2) \partial_u^2 - \frac{1}{4} \right] a = 0. \quad (19)$$

The above properties, extend to the case of $SU(N_c)$, $SO(N_c)$ and $Sp(N_c)$.
Let b_1 the one-loop contribution to the β -function

$$u = -\frac{4\pi i}{b_1} \left(\mathcal{F} - \frac{a_k}{2} \frac{\partial \mathcal{F}}{\partial a_k} \right), \quad (20)$$

is invariant under symplectic transformations. In particular, the transformation of \mathcal{F} now reads

$$\tilde{\mathcal{F}}(\tilde{a}) = \mathcal{F}(a) + \frac{1}{2} a^T B^T D a + \frac{1}{2} a_D^T C^T A a_D + a^T B^T C a_D, \quad (21)$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \in Sp(2r, \mathbb{Z}), \quad (22)$$

with r the rank of the gauge group.

Consider matrices with continuous indices. Matrix multiplication is defined as usual, the only difference is that the summation over the indices is replaced by integrals. For example, the product of two matrices is defined by

$$(AB)(x, y) = \int d^D z A(x, z) B(z, y) . \quad (23)$$

Continuous Symplectic Group (CSG) $Sp(2\infty, \mathbb{C})$

$$M(x_1 x_2, y_1 y_2) = \begin{pmatrix} A(x_1, y_1) & B(x_1, y_2) \\ C(x_2, y_1) & D(x_2, y_2) \end{pmatrix} . \quad (24)$$

satisfying the continuous analogue of

$$M^T \Omega M = \Omega , \quad (25)$$

that is

$$\int d^D z_1 d^D z_2 d^D w_1 d^D w_2 M^T(x_1 x_2, z_1 z_2) \Omega(z_1 z_2, w_1 w_2) M(w_1 w_2, y_1 y_2) \\ = \Omega(x_1 x_2, y_1 y_2) ,$$

where

$$\Omega(xy, zw) = \begin{pmatrix} 0 & \delta(x - w) \\ -\delta(y - z) & 0 \end{pmatrix} . \quad (26)$$

Note that

$$\int d^D z_1 d^D z_2 \Omega(x_1 x_2, z_1 z_2) \Omega(z_1 z_2, y_1 y_2) = -\mathbb{I}(x_1 x_2, y_2 y_2) , \quad (27)$$

where

$$\mathbb{I}(x_1 x_2, y_2 y_2) = \begin{pmatrix} \delta(x_1 - y_1) & 0 \\ 0 & \delta(x_2 - y_2) \end{pmatrix} . \quad (28)$$

INTERMEZZO

The above construction may be used to define Riemann surfaces with infinite genus

A Modular Invariant Functional

Consider the following functional

$$\begin{aligned}U[J] &= W[J] - \frac{1}{2} J \frac{\delta W[J]}{\delta J} \\&= W[J] - \frac{1}{2} J \varphi .\end{aligned}\tag{29}$$

Note that

$$U[J] = \Gamma[\varphi] + \frac{1}{2} J \varphi .\tag{30}$$

Consider the $Sp(2\infty, \mathbb{C})$ transformation

$$\begin{aligned}\tilde{\varphi}(x) &= (A\varphi)(x) + (BJ)(x) , \\ \tilde{J}(x) &= (C\varphi)(x) + (DJ)(x) .\end{aligned}\tag{31}$$

Since $\varphi(x) = \delta W[J]/\delta J(x)$, the above transformations imply that the generating functional of connected Green functions transforms to

$$\tilde{W}[\tilde{J}] = W[J] + \frac{1}{2} J B^T D J + \frac{1}{2} \varphi C^T A \varphi + J B^T C \varphi .\tag{32}$$

It follows that the effective action transforms to

$$\Gamma[\varphi] \longrightarrow \tilde{\Gamma}[\tilde{\varphi}] = \Gamma[\varphi] + \frac{1}{2}JB^TDJ + \frac{1}{2}\varphi C^TA\varphi + JB^TC\varphi . \quad (34)$$

Recalling that

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi} = -J , \quad (35)$$

we have the identity

$$\tilde{\Gamma}[\tilde{\varphi}] = \Gamma[\varphi] + \frac{1}{2}\frac{\delta\Gamma[\varphi]}{\delta\varphi}B^TD\frac{\delta\Gamma[\varphi]}{\delta\varphi} + \frac{1}{2}\varphi C^TA\varphi - \frac{\delta\Gamma[\varphi]}{\delta\varphi}B^TC\varphi . \quad (36)$$

Search for transformations that leave invariant the functional structure of $U[J]$, that is

$$U[\tilde{J}] = U[J] . \quad (37)$$

This fixes $U[J]$ and therefore $\Gamma[\varphi]$. It corresponds to the functional analogue of the modular invariance in Seiberg-Witten theory

$$u(\tilde{a}) = u(a) , \quad (38)$$

that implies the differential equation

$$(1 - u^2) u'' + \frac{1}{4} a u'^3 = 0 , \quad (39)$$

fixing all the instanton contributions

$$u(a) = \sum_{k=0}^{\infty} \mathcal{G}_k a^{2-4k}, \quad \mathcal{G}_0 = \frac{1}{2}, \quad \mathcal{G}_k = 2\pi i k \mathcal{F}_k, \quad (40)$$



by recursion relation.

Conclusions

1 Classical-Quantum Duality

$$\exp\left(\frac{i}{\hbar}S[\phi]\right) = \int \mathcal{D}J \exp\left[\frac{i}{\hbar}\left(W[J] - \int d^4x J(x)\phi(x)\right)\right],$$

2 Fourier and Legendre transforms

3 Continuous Modular transformations in QFT.

$$U[J] = \Gamma[\varphi] + \frac{1}{2}J\varphi,$$

is modular invariant