

Lattice study for the $\mathbb{C}P^{N-1}$ models on $\mathbb{R} \times S^1$

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arXiv:1907.06925 and work in progress

Fractional instanton of SU(3) gauge theory

EI, JHEP 1905 (2019) 093

Motivation: $\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$

$\mathbb{C}P^{N-1}$ on \mathbb{R}^2 :a good toy model of QCD

Asymptotic freedom

Nonperturbative properties

(confinement, topological objects)

Several lattice Monte Carlo studies on large \mathbb{T}^2

Berg and Luscher(81), Campostrini, Rossi, Vicari.(92), Alles, Cosmai, D'Elia, Papa(00),
Flynn,Juttner, Lawson, Sanfilippo(15), Abe, Fukushima, Hidaka, Matsueda, Murase, Sasaki(18),
Bonanno, Bonati, D'Elia (19)

Reveal its classical solution and topological object

Give some technical developments

(smearing method, autocorrelation problems in MC calculations)

Motivation: $\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$

We can introduce some boundary conditions on S^1 .

Physics on compactified spacetime depends on the b.c.,
while in any case it goes to the same one in decompactified limit.

[S. I. Hong and J. K. Kim, Phys. Rev. D 50, 2942 \(1994\).](#) [J. Phys. A 27, 1557 \(1994\).](#)

PBC for S^1 (finite-T system)

[Monin, Shifman, Yung: Phys. Rev. D 92, 025011 \(2015\)](#)

[Bolognesi, Gudnason, Konishi, Ohashi: arXiv:1905.10555](#)

confinement/deconfinement transition?

How to realize global $PSU(N) = SU(N)/\mathbb{Z}_N$ symmetry in both phases?

(The Coleman theorem in 2-dim. at least finite N)

It will be broken in the large-N limit?

TBC for S^1 : $\phi(x, \tau + L_\tau) = \Omega \phi(x, \tau)$, $\Omega = \text{diag.}[1, e^{2\pi i/N}, e^{4\pi i/N}, \dots, e^{2(N-1)\pi i/N}]$

[G.V.Dunne and M.Unsal JHEP1211\(2012\) 170, PRD87\(2013\)025015, PRD89\(2014\)0141701, JHEP1612\(2016\)002](#)

[T.Misumi, M.Nitta and N.Sakai, JHEP1509,157 \(2015\)](#)

[T.Fujimori, S.Kamata, T.Misumi, M.Nitta and N.Sakai PRD94 \(2016\) 105002, PRD95 \(2017\) 105001, PTEP 2017 no.8 083B02](#)

novel type of a classical solution(called bion)?

Good to see the resurgence structure?

$\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$ with PBC

Lattice setup

M.Campostrini, P.Rossi and E.Vicari, Phys. Rev. D46 (1992) 2647.

Action in continuum

$$|\phi|^2 = 1, D_\mu \phi = (\partial_\mu + iA_\mu)\phi$$

$$S = \frac{1}{2g^2} \int d^2x |D_\mu \phi|^2 \quad \phi^i, i = 1, \dots, N$$

Here, A_μ is U(1) gauge field ($A_\mu = \frac{i}{2}\bar{\phi} \cdot \overleftrightarrow{\partial}_\mu \phi$)

Action on the lattice

$$S_{lat.} = -N\beta \sum_{n,\mu} (\bar{\phi}_{n+\mu} \cdot \phi_n \lambda_{n,\mu} + \bar{\phi}_n \cdot \phi_{n+\mu} \bar{\lambda}_{n,\mu} - 2)$$

U(1) link variable: $\lambda_{n,\mu} = e^{iA_\mu(n)}$

Note that β denotes the coupling constant

$$N\beta = \frac{1}{g^2}$$

Weak coupling = large β
Strong coupling = small β

Over heat-bath algorithm is adopted (local updation)

$\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$ with PBC

(1) confinement/deconfinement transition?

Monin, Shifman, Yung: Phys. Rev. D 92, 025011 (2015)
Bolognesi, Gudnason, Konishi, Ohashi: arXiv:1905.10555

Large-N studies based on the gap equation do not reach a consensus
(Next talk by Ohashi-san)

Lattice study does not need any gauge fixing
Good to see the information of ground state

(this work)

We consider only finite-N and show the N-dependence of the transition

By analogy with QCD, the Polyakov loop of link variables coupled to matter fields is a good order parameter for confinement or deconfinement

Polyakov loop in continuum spacetime

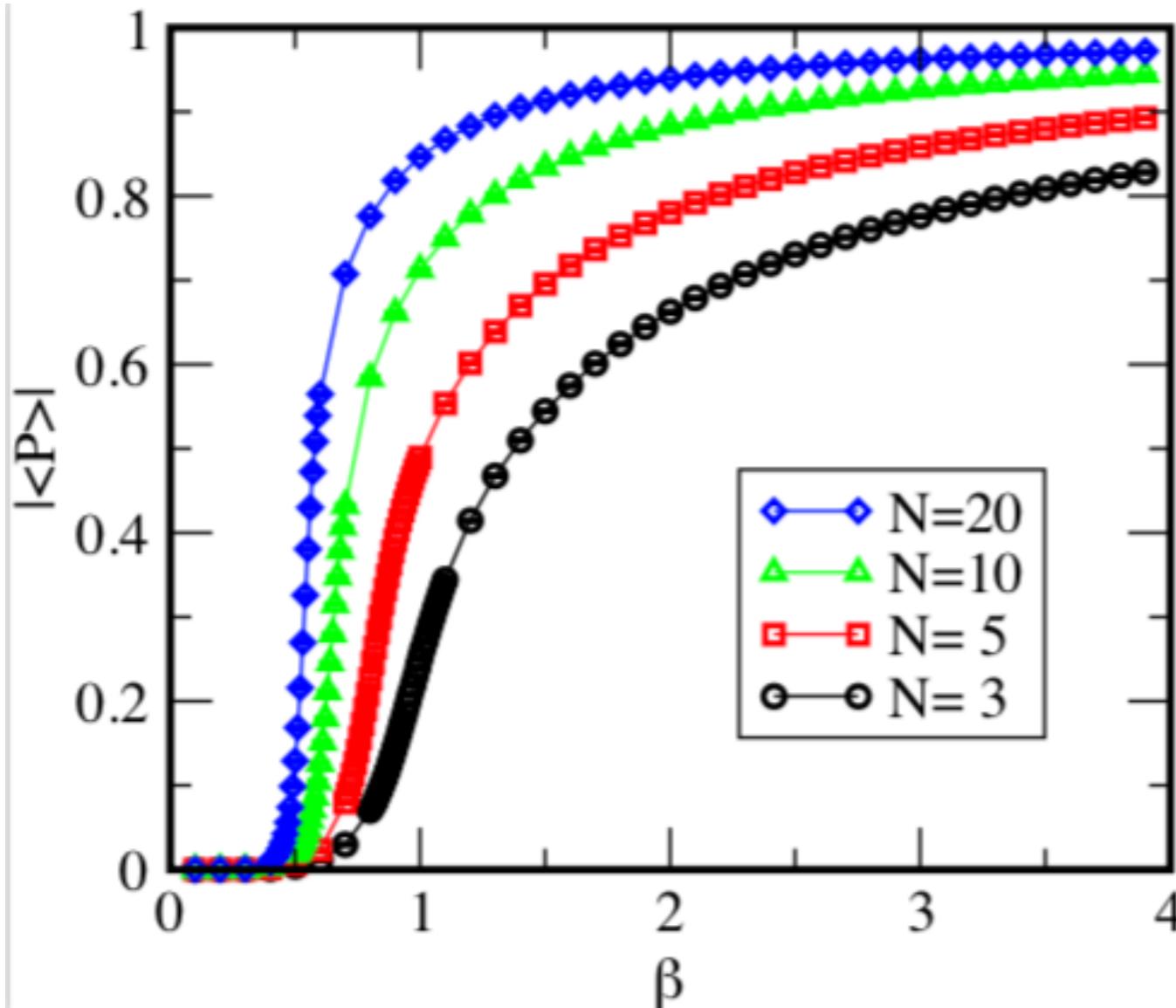
$$P = e^{i \int d\tau A_\tau}$$

Polyakov loop on the lattice

$$P = \frac{1}{N_s} \sum_{n_x} \prod_{n_\tau} \lambda_\tau(n_x, n_\tau)$$

Confinement/Deconfinement

Polyakov loop $\langle P \rangle \propto e^{-F_\phi L_\tau}$



$(N_s, N_\tau) = (200, 8)$

Nsweep = 200,000

$\langle P \rangle \approx 0 \leftrightarrow F_\phi \rightarrow \infty \text{ in } V \rightarrow \infty : \text{confinement}$

$\langle P \rangle \neq 0 \leftrightarrow F_\phi \text{ is finite} : \text{deconfinement}$

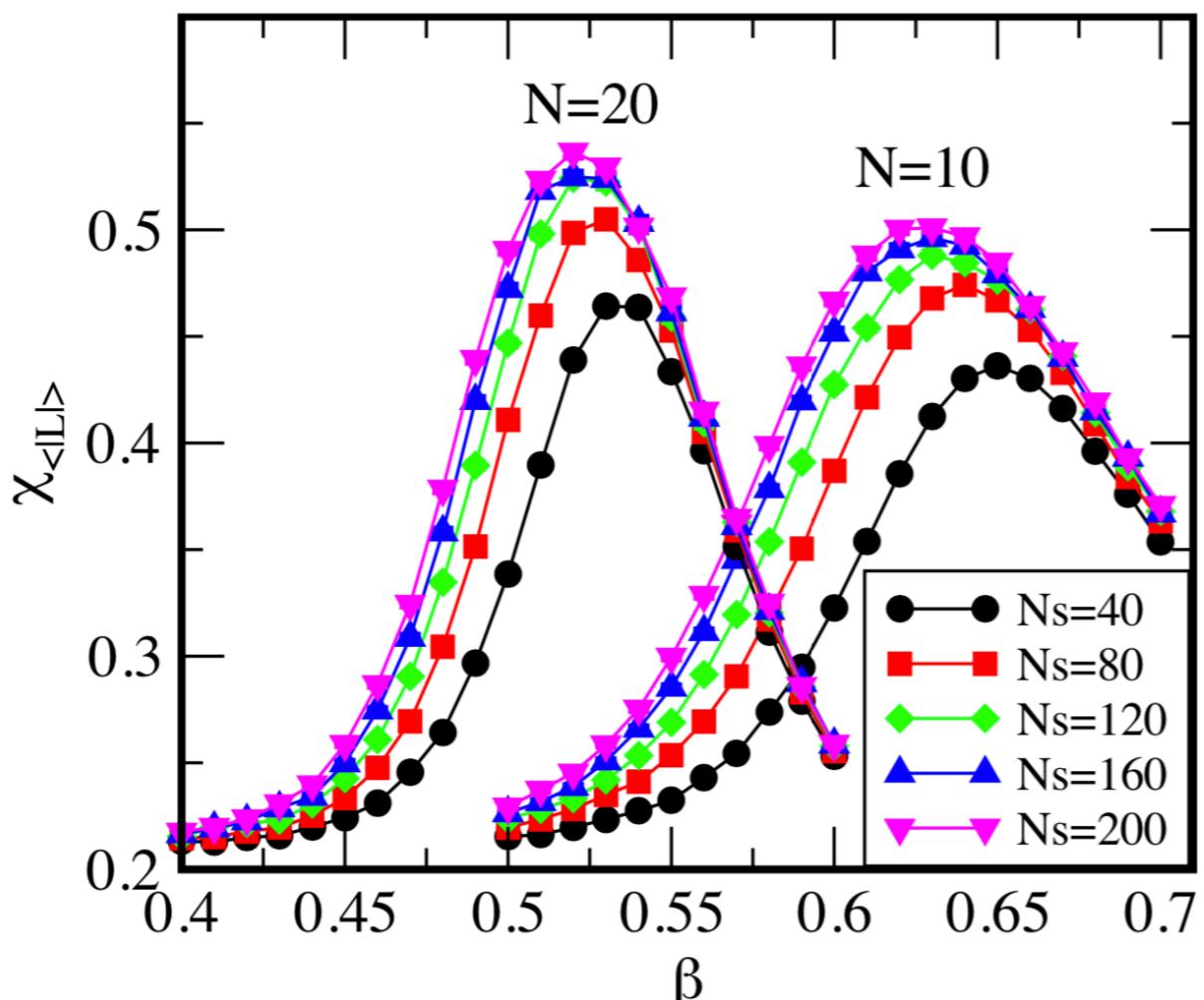
Confinement/Deconfinement

To study strength of the transition, volume scaling

$$\chi_{\langle |P| \rangle, \text{max}} = a + cV^p$$

p=1: 1st order

0<p<1: 2nd order or crossover

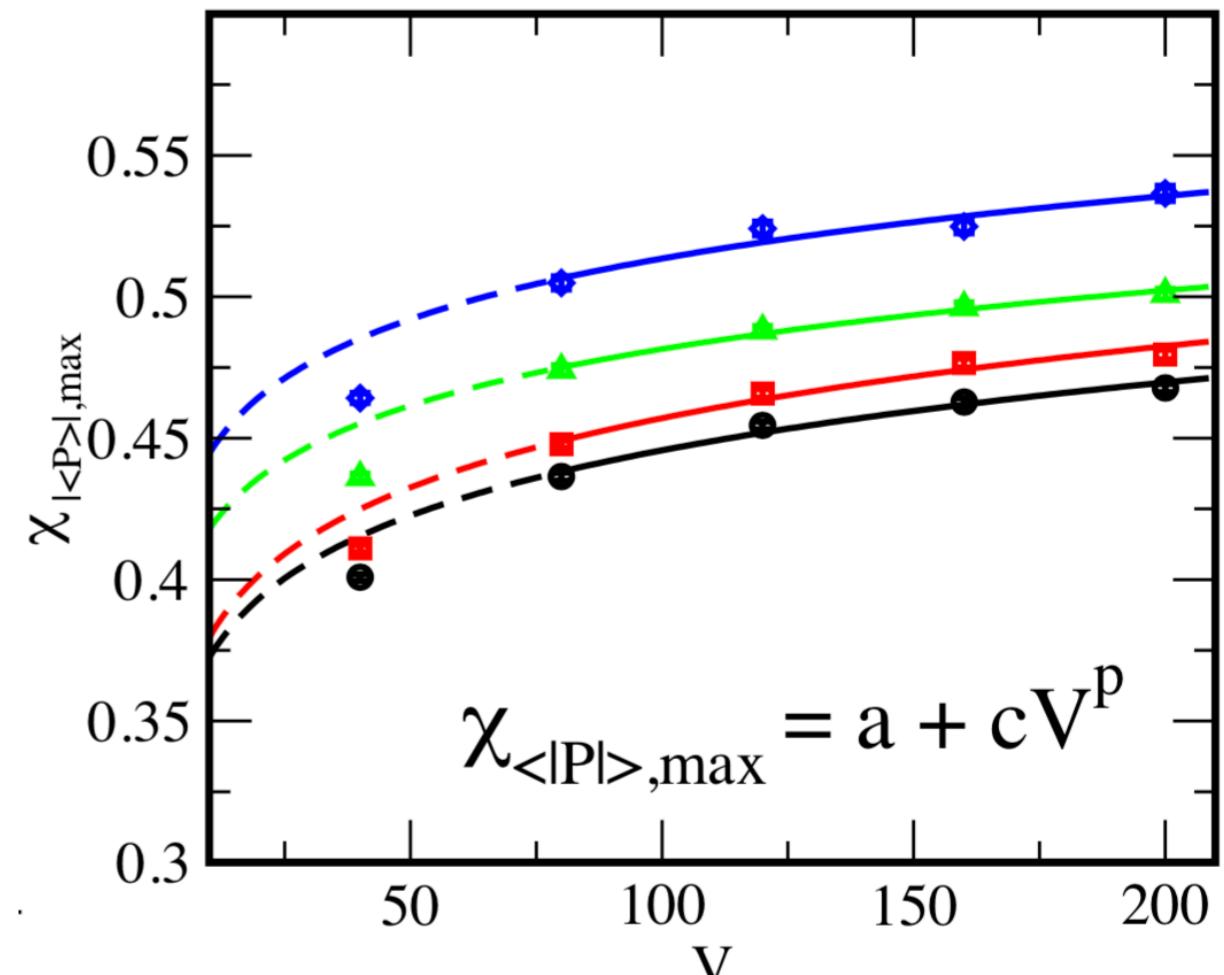


$N=3$: $p=0.056(7)$

$N=5$: $p=0.058(7)$

$N=10$: $p=0.052(7)$

$N=20$: $p=0.043(8)$

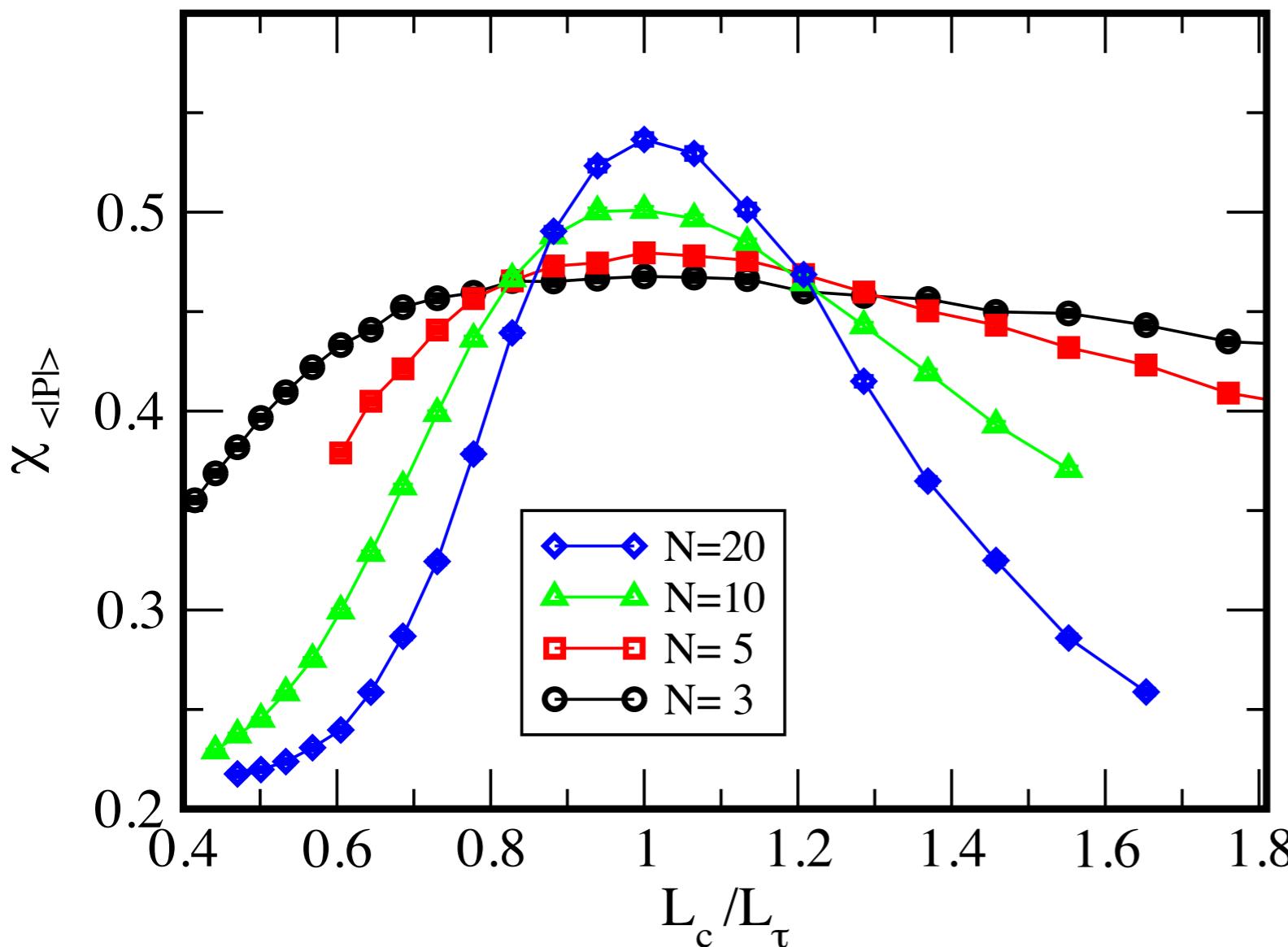


indicate it is crossover for finite N

Confinement/Deconfinement

N dependence of $\chi_{\langle |P| \rangle}$ with fixed $T(=N_\tau)$ simulation

M.Campostrini, P.Rossi and E.Vicari,
Phys. Rev. D46 (1992) 2647.



reference scale:
N-dependent lattice Λ parameter

$$\Lambda_{lat}a = \frac{1}{\sqrt{32}}(2\pi\beta)^{\frac{2}{N}}e^{-2\pi\beta-\frac{\pi}{2N}}$$

$$T = \frac{1}{L_\tau} = \frac{1}{aN_\tau}$$

The peak is quite broad for small N. It gets sharper as N increases
The result suggests the 2nd order transition occurs in large-N limit?
But $\beta_c \rightarrow 0$ at that time. Careful analysis is necessary.

Global PSU(N) symmetry

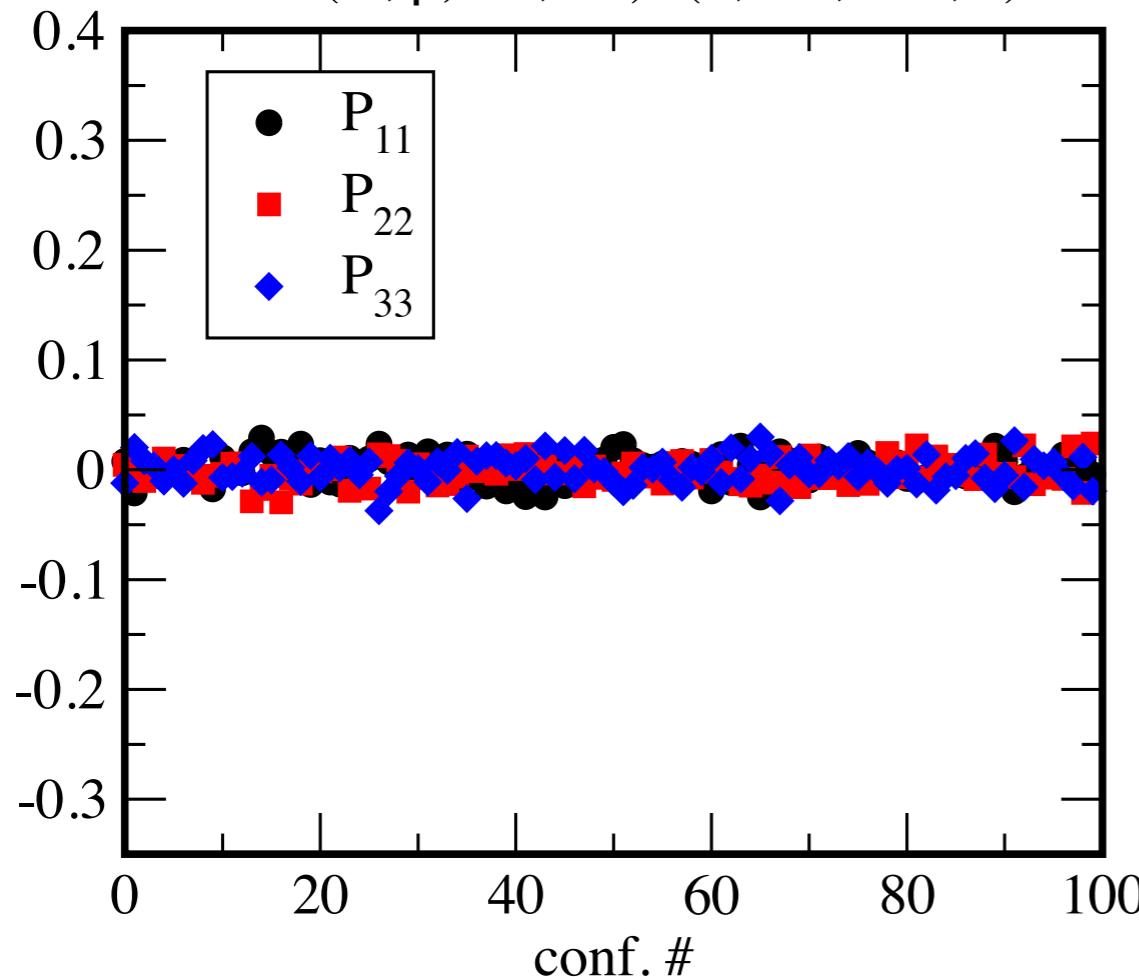
(2) PSU(N) symmetry breaking is associated with deconfinement in the large-N analysis?
What is happen in finite N in deconfinement phase?

Monin, Shifman, Yung: Phys. Rev. D 92, 025011 (2015)

$N \times N$ matrix : $P^{ij} = \sum_n \bar{\phi}_n^i \phi_n^j - \frac{1}{N} \delta^{ij}$ If all $\langle P^{ij} \rangle = 0$, then PSU(N) is preserved.

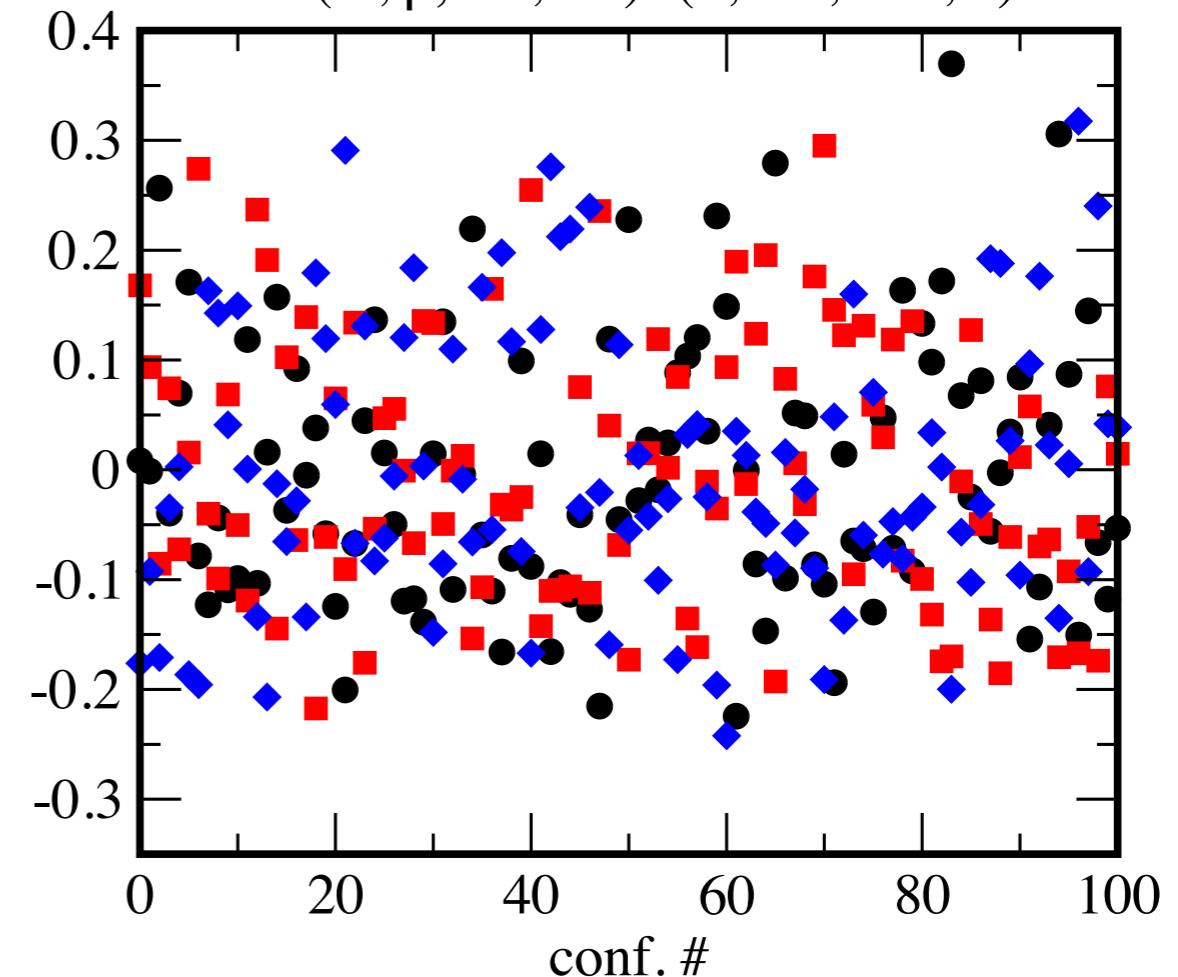
Confinement phase

$(N, \beta, N_s, N\tau) = (3, 0.1, 200, 8)$



Deconfinement phase

$(N, \beta, N_s, N\tau) = (3, 3.9, 200, 8)$



Global PSU(N) symmetry

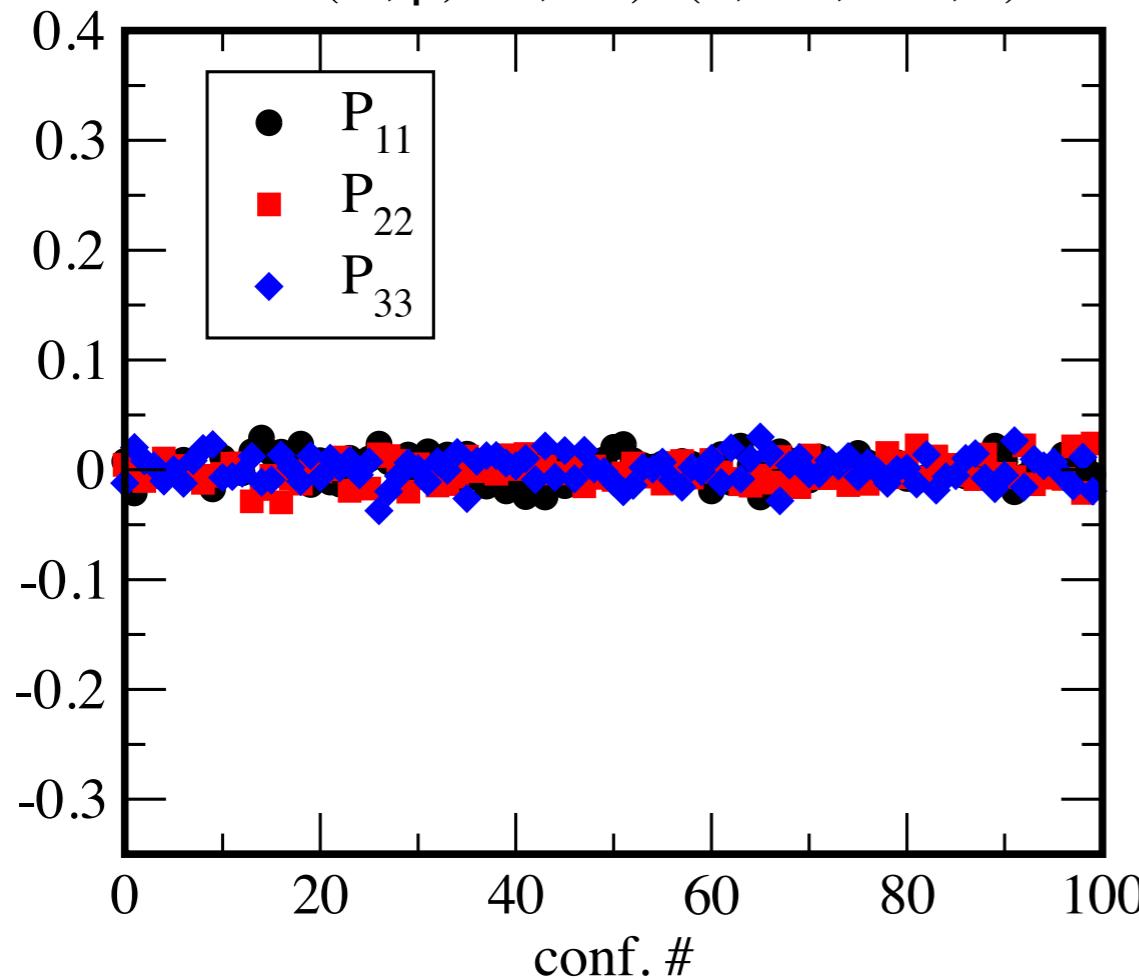
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Confinement phase

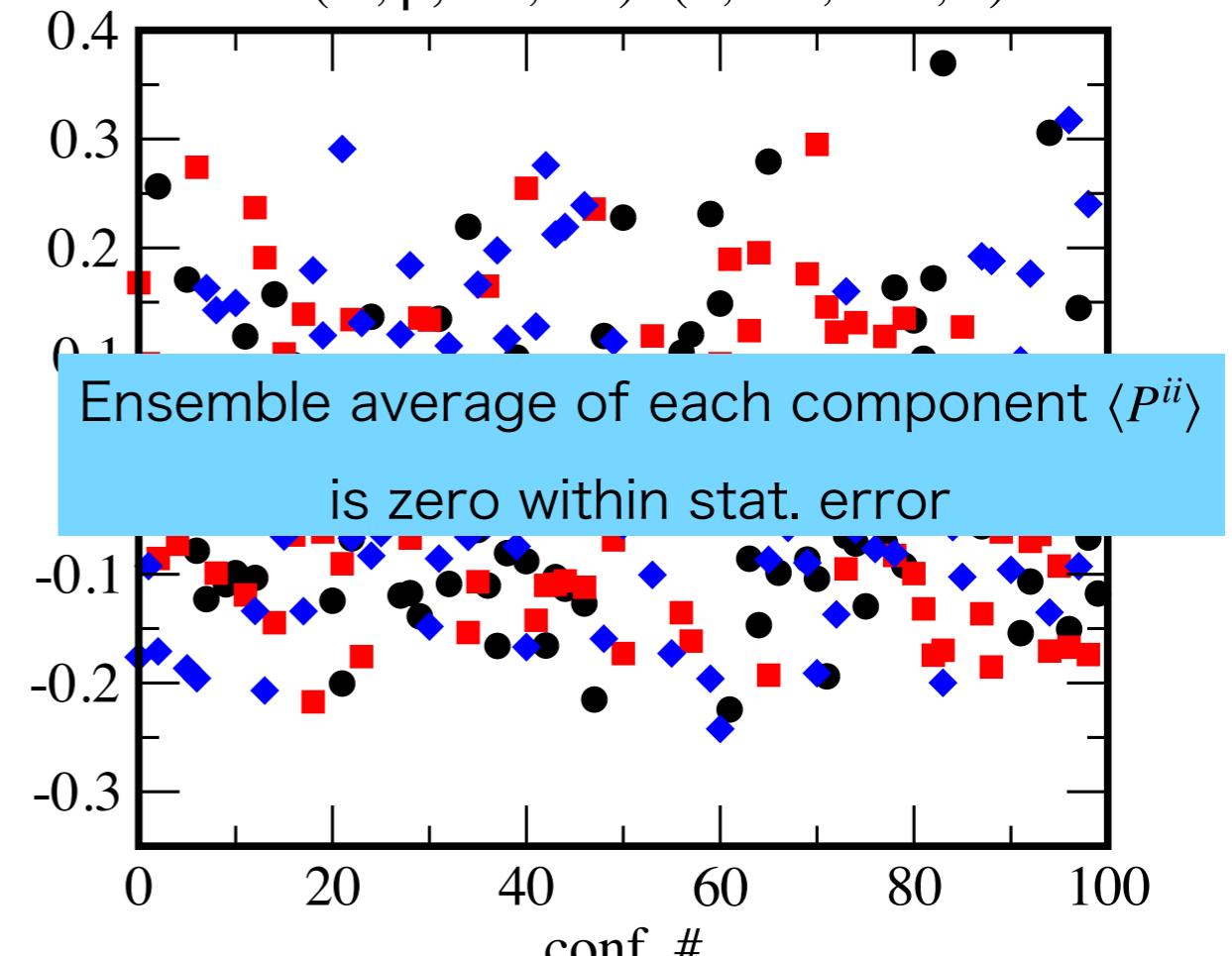
$(N, \beta, N_s, N\tau) = (3, 0.1, 200, 8)$



PSU(N) symm. at “classical level”

Deconfinement phase

$(N, \beta, N_s, N\tau) = (3, 3.9, 200, 8)$



PSU(N) symm. at “quantum level”

In the large-N limit, it is still open question.

Thermal entropy and free energy

All N fields are equivalent. But there is one constraint $|\phi|^2 = 1$

Let us count the actual d.o.f by the entropy

Free energy density can be calculated by the one for massive free complex scalar system

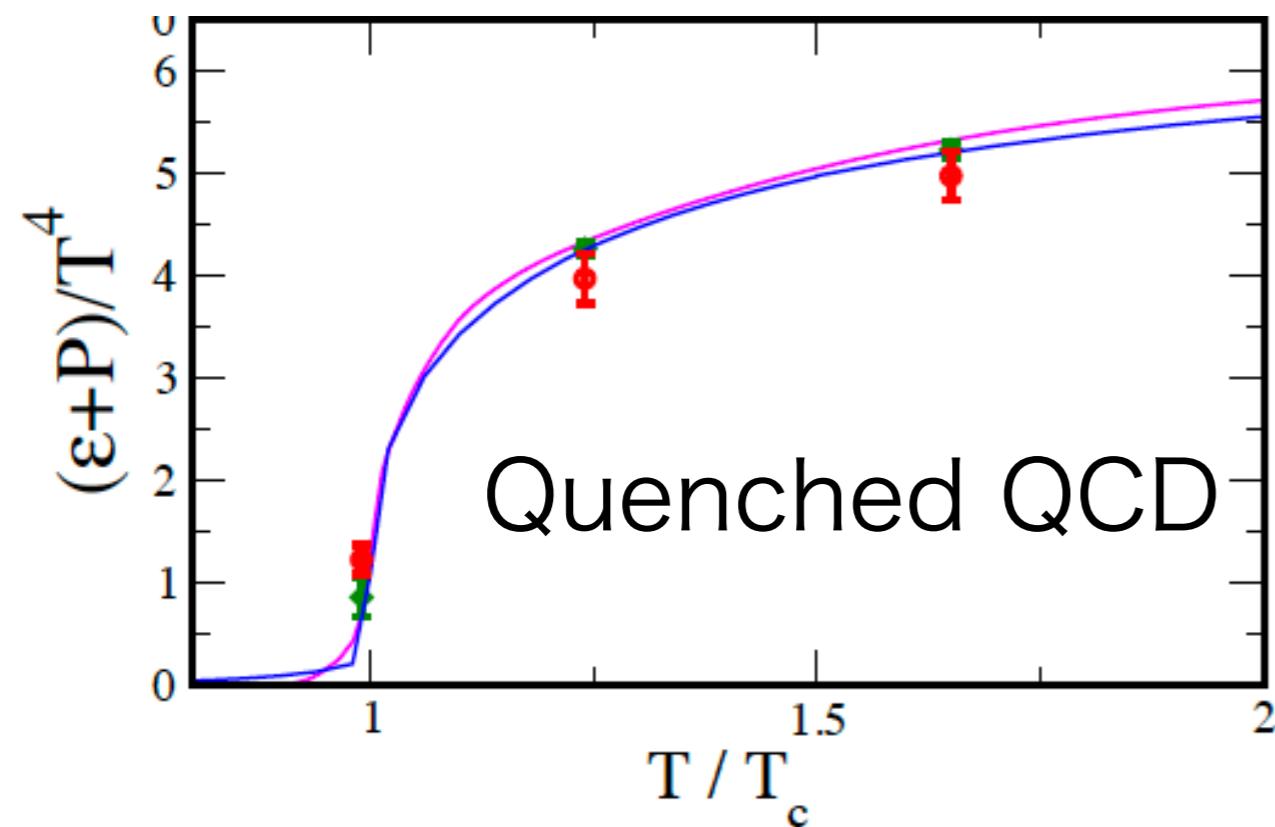
on $S_s^1 \times S_\tau^1$

$$f = \frac{1}{L_s L_\tau} \sum_{n=-\infty}^{\infty} \log 4 \sinh^2 \frac{\omega_n L_\tau}{2} - f_0$$

Massless and thermodynamic limit: $f(L_\tau) = -\frac{\pi}{3L_\tau^2}$ Here, $T = 1/L_\tau$

Entropy density: $s/T = -\frac{1}{T} \frac{\partial f}{\partial T} = \frac{2\pi}{3}$ for one free massless complex scalar field

Our lattice approach: entropy from energy-momentum-tensor(EMT)

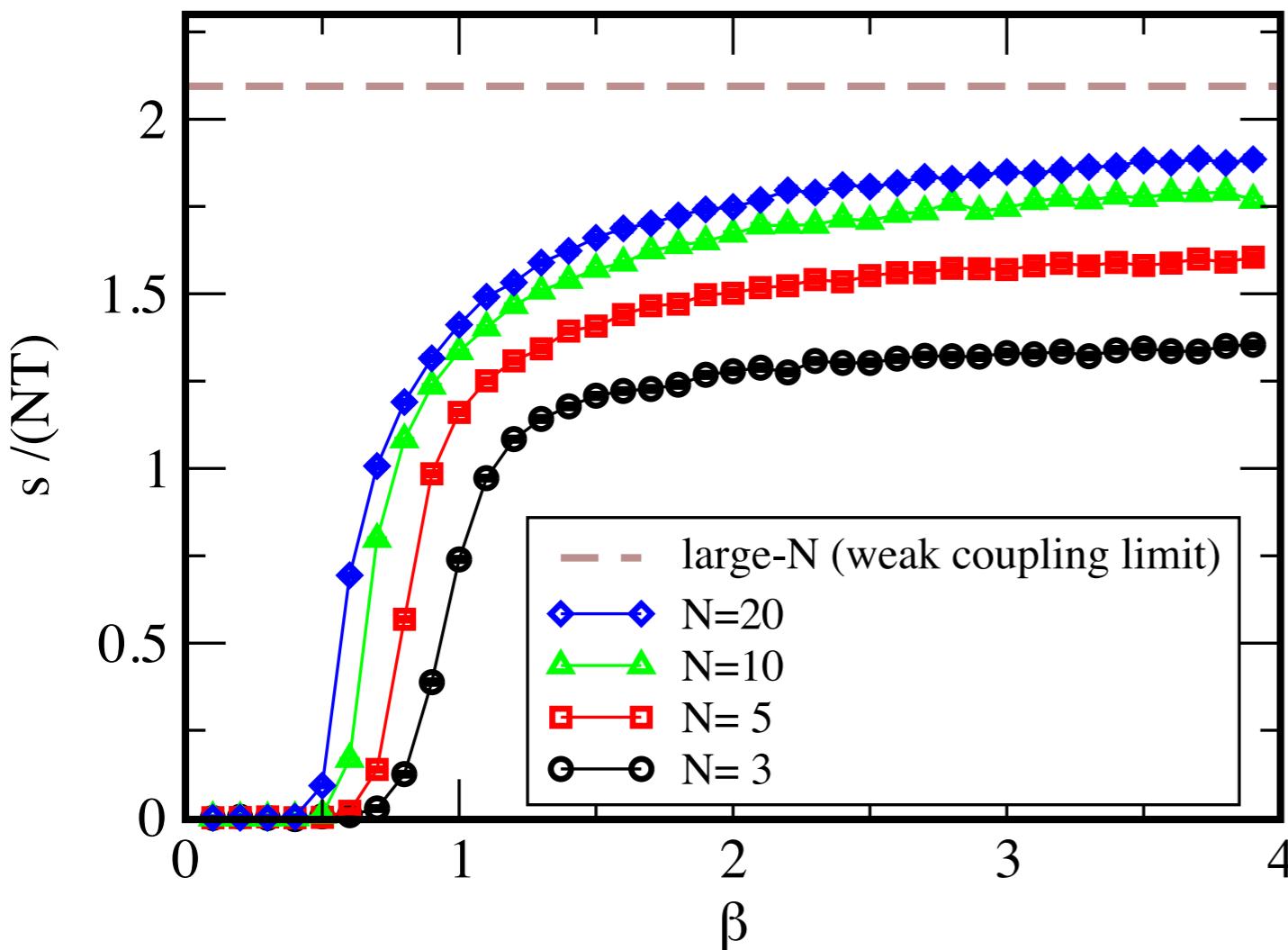


Entropy can be obtained by EMT
 $s/T = (\epsilon + P)/T^2 = (\langle T_{xx} \rangle - \langle T_{\tau\tau} \rangle) / T^2$

In quenched QCD,
Two approaches give a consistent result
Asakawa, Hatsuda, EI, Kitazawa, Suzuki :
Phys. Rev. D 90, 011501 (2014)

Thermal entropy and free energy

s/T : entropy density for $\mathbb{C}P^{N-1}$ model



Our lattice results show

$$s/T = \frac{2\pi}{3}(N - 1)$$

in weak coupling limit ($\beta \rightarrow \infty$)

cf) one scalar: $s/T = -\frac{1}{T} \frac{\partial f}{\partial T} = \frac{2\pi}{3}$

Our entropy data is consistent
with the one for free $N-1$
complex scalar

It is also consistent with entropy density: $s/T = -\frac{1}{T} \frac{\partial f}{\partial T} = \frac{2\pi N}{3}$ in the large- N analysis

Monin, Shifman, Yung: Phys. Rev. D 92, 025011 (2015)

All N components are equivalent, but the actual d.o.f. on the lattice is correctly $N-1$

$\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$ with TBC

All results are preliminary

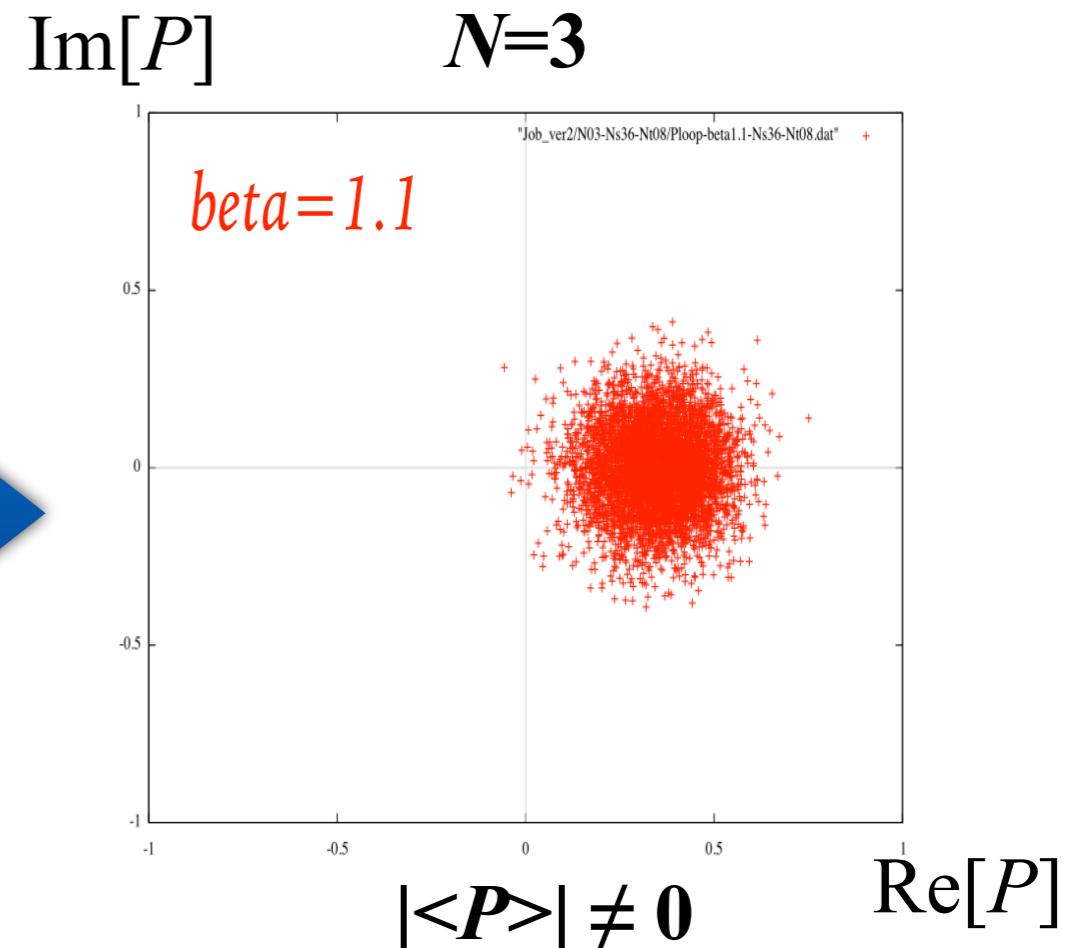
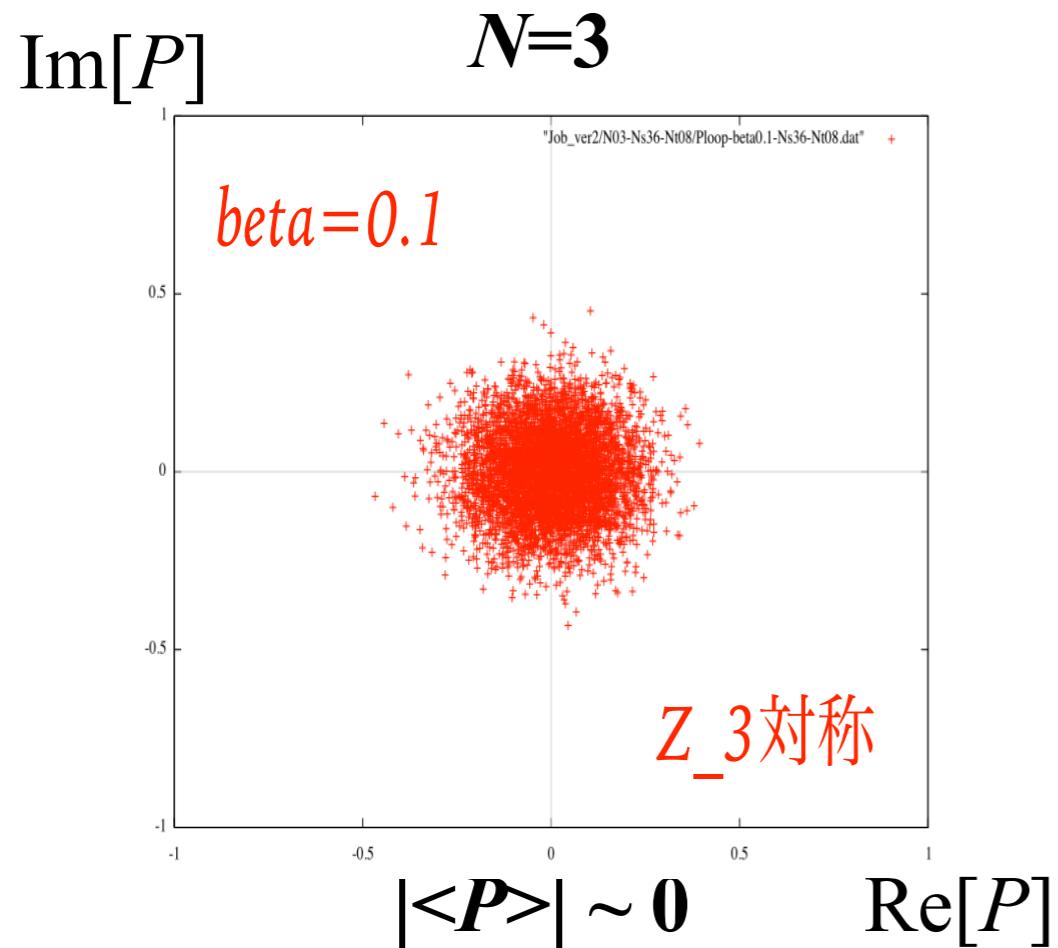
TBC for S^1 :

$$\phi(x, \tau + L_\tau) = \Omega \phi(x, \tau), \quad \Omega = \text{diag.}[1, e^{2\pi i/N}, e^{4\pi i/N}, \dots, e^{2(N-1)\pi i/N}]$$

b.c. has flavor dependence

Polyakov loop : scatter plot

PBC case: The action does not have \mathbb{Z}_N symmetry



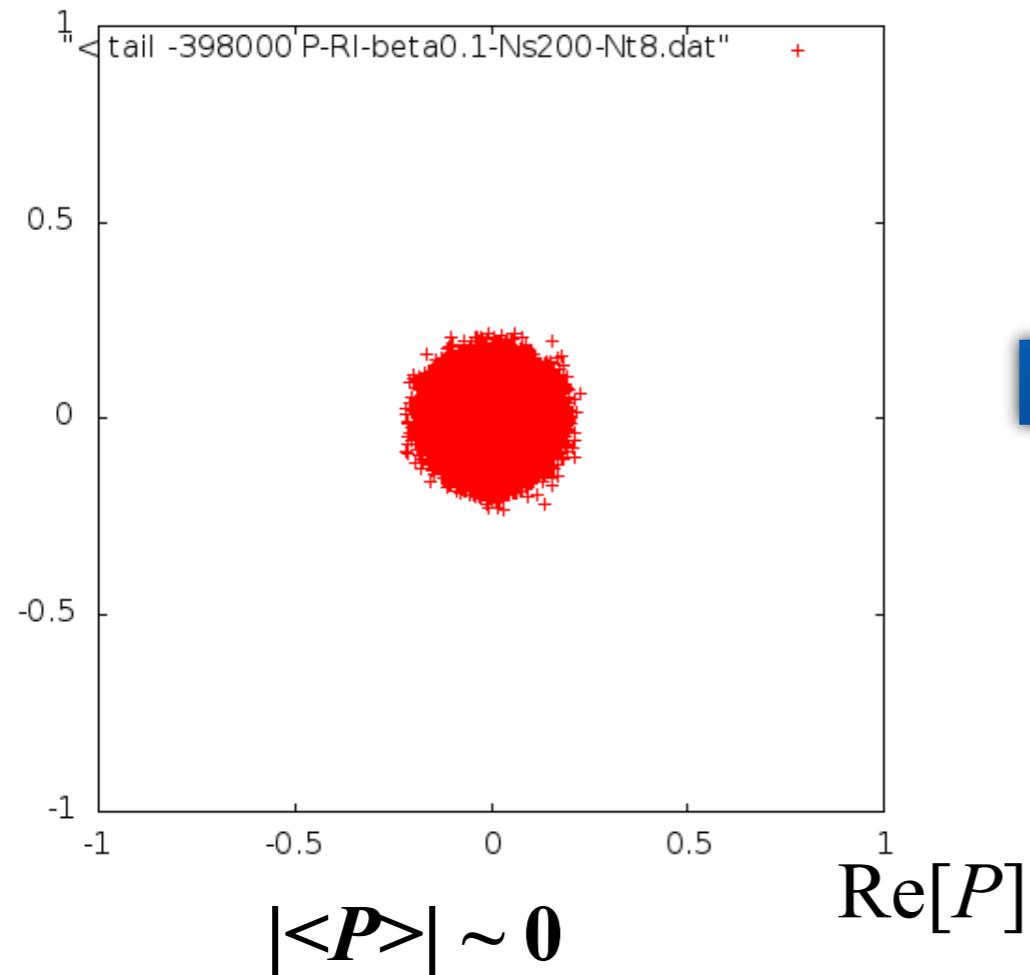
Low- β : around the origin
→ (approximately) \mathbb{Z}_N symmetric

High- β : moves to one of \mathbb{Z}_N vacua
→ (manifestly) \mathbb{Z}_N breaking

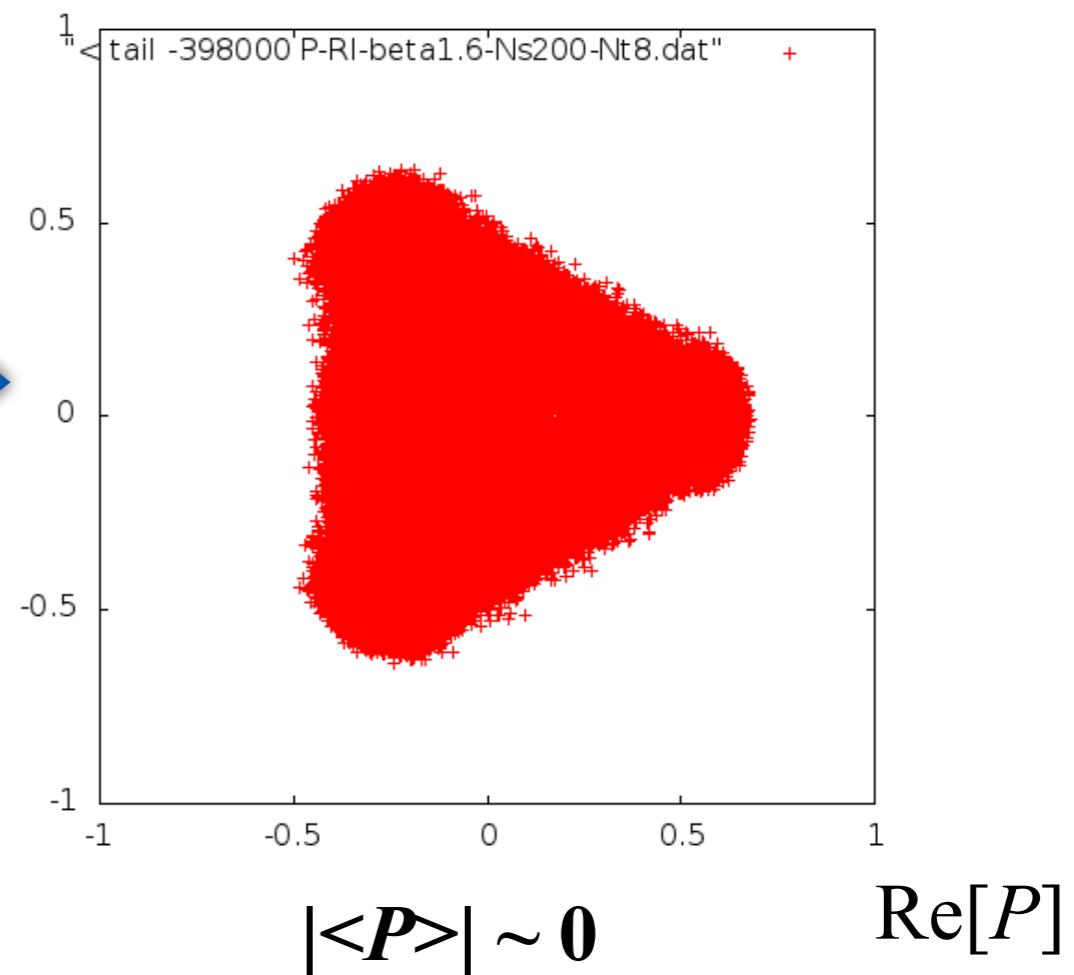
Polyakov loop : scatter plot

TBC case: The action has \mathbb{Z}_N symmetry

$\text{Im}[P]$ $N=3, \beta=0.1$



$\text{Im}[P]$ $N=3, \beta=1.6$



Low- β :

around the origin \rightarrow
 \mathbb{Z}_N symmetric at “classical level”

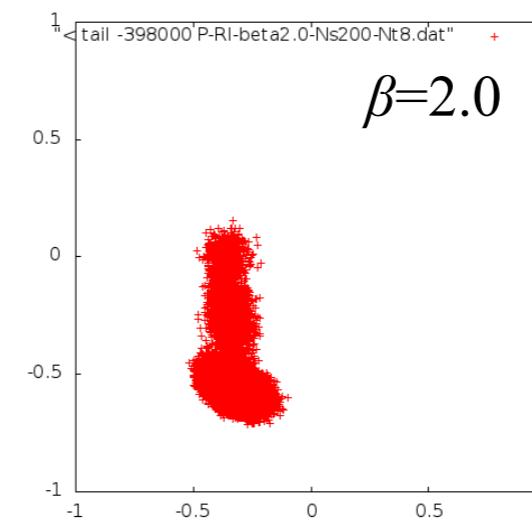
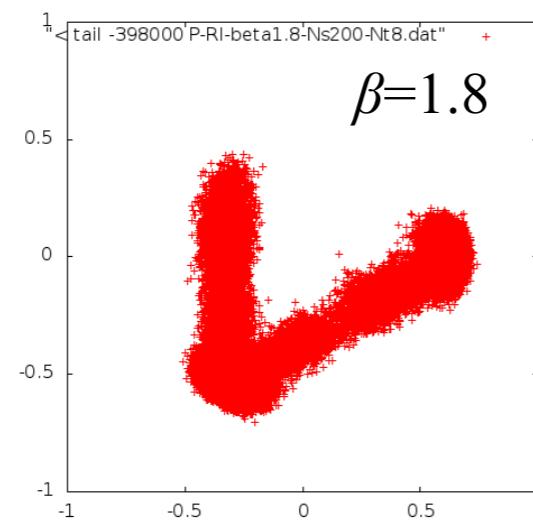
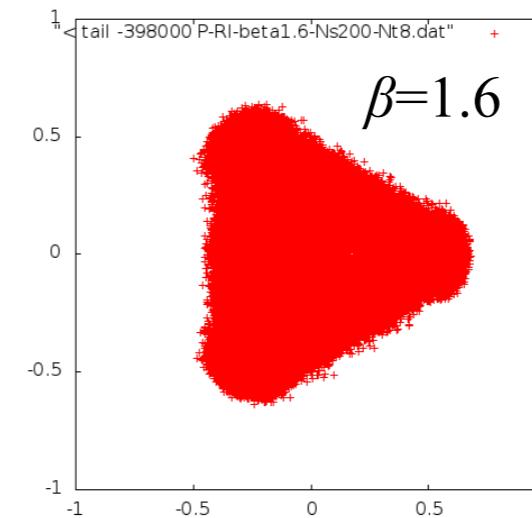
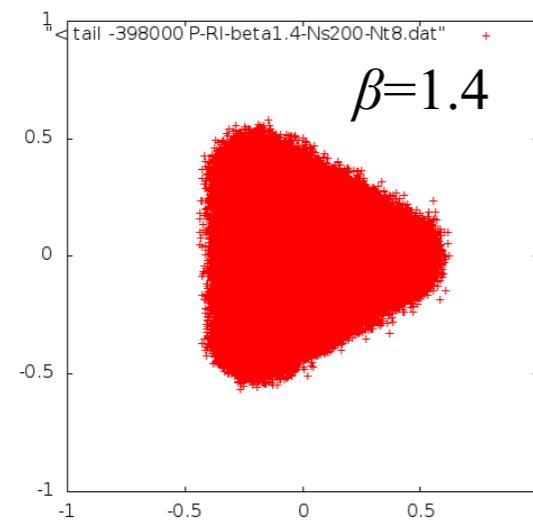
Intermediate- β :

Transition between N vacua \rightarrow
 \mathbb{Z}_N symmetric at “quantum level”

Polyakov loop : scatter plot

further high β regime

$N=3$

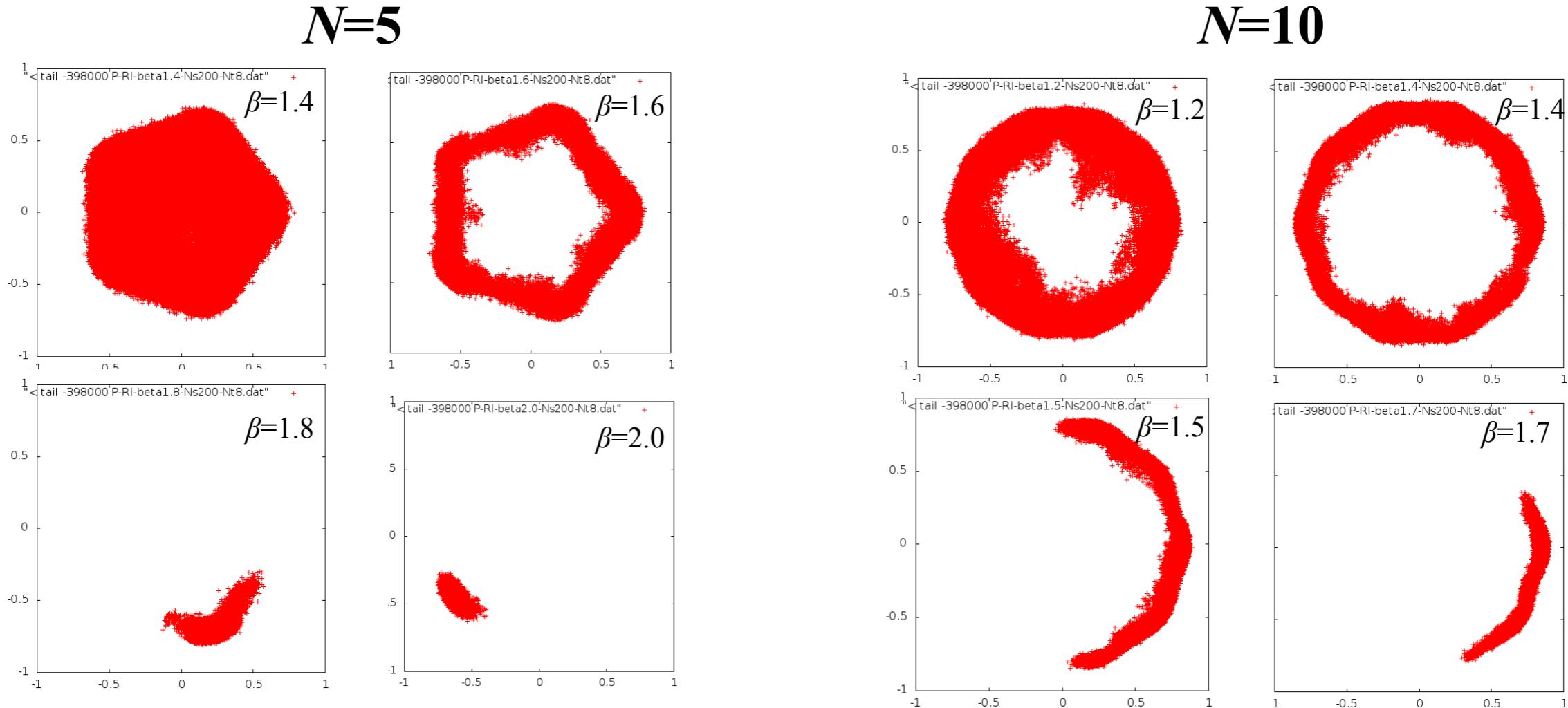


- Low $\beta \rightarrow |\langle P \rangle| = 0$: distribution around origin
- Mid $\beta \rightarrow |\langle P \rangle|$ highly fluctuates : distribution forms **polygons**
- High $\beta \rightarrow |\langle P \rangle| \neq 0$: (but it seems that more stat. can form polygon?)

Adiabatically continue to $\beta \rightarrow \infty$?

Polyakov loop : scatter plot

N dependence

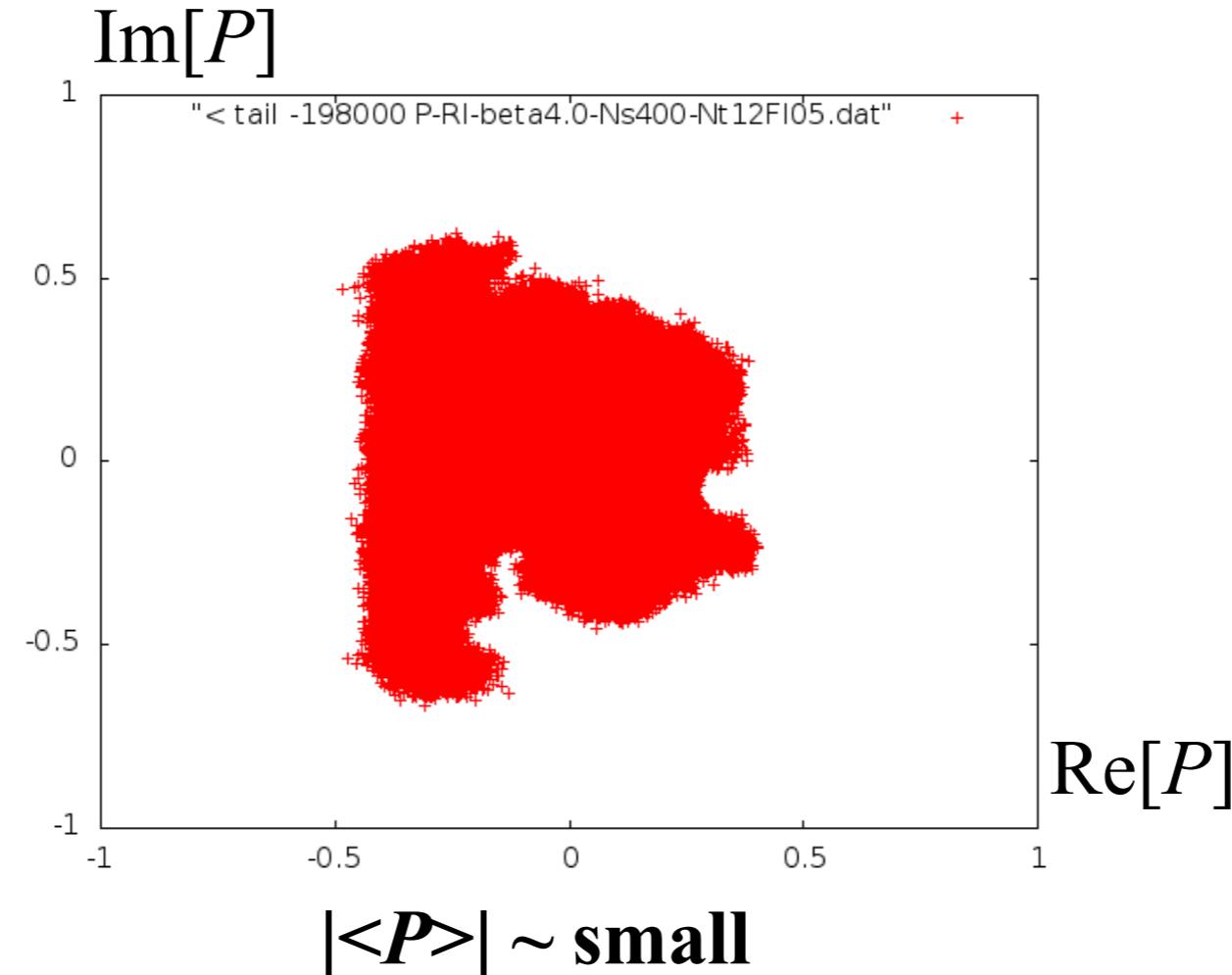


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- High $\beta \rightarrow |\langle P \rangle| \neq 0$: (but it seems that more stat. can form polygon?)

Adiabatically continue to $\beta \rightarrow \infty$?

higher β and larger volume

$N=3, \beta=4.0, (400 \times 12)$: Polygon-shaped distribution appears
Weak coupling expansion is valid in this β



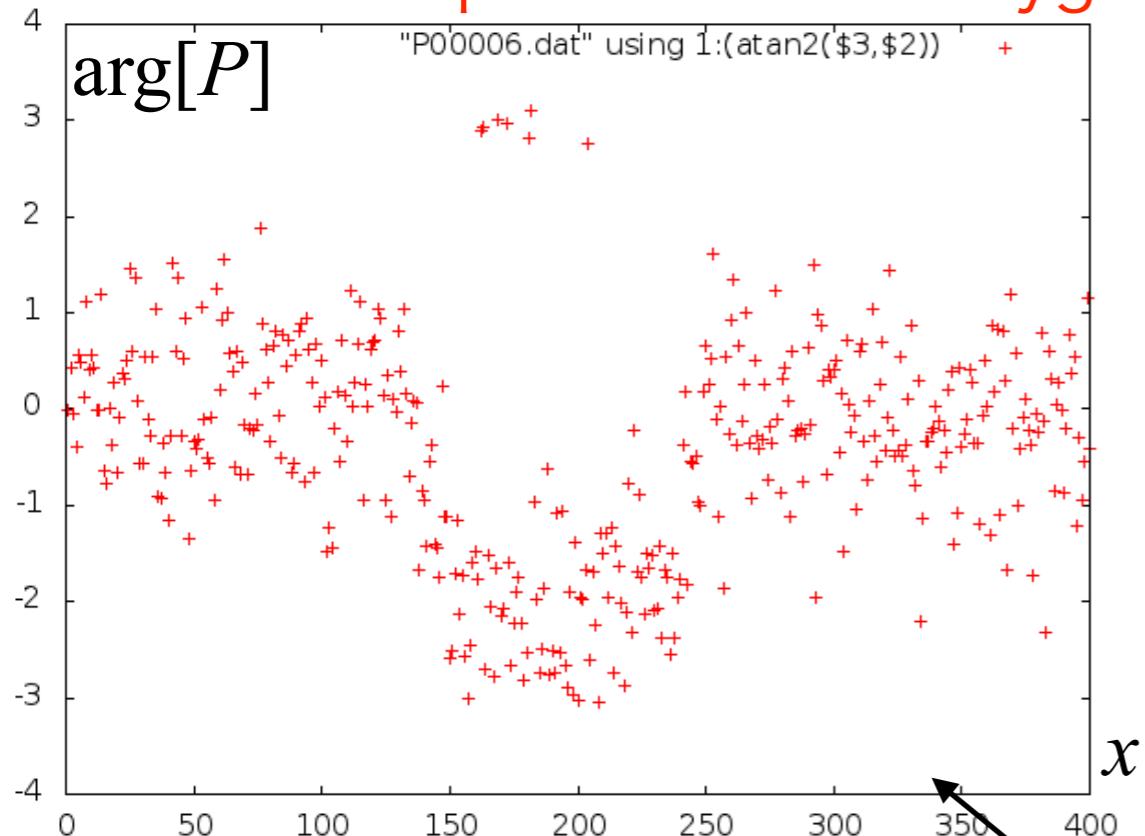
Very high- β : quantum Z_N symmetric case found with certain probability

Let us see which type of configurations appears inside and the perimeter of the polygon.

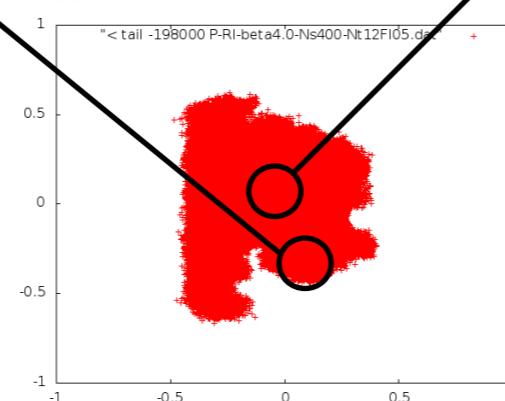
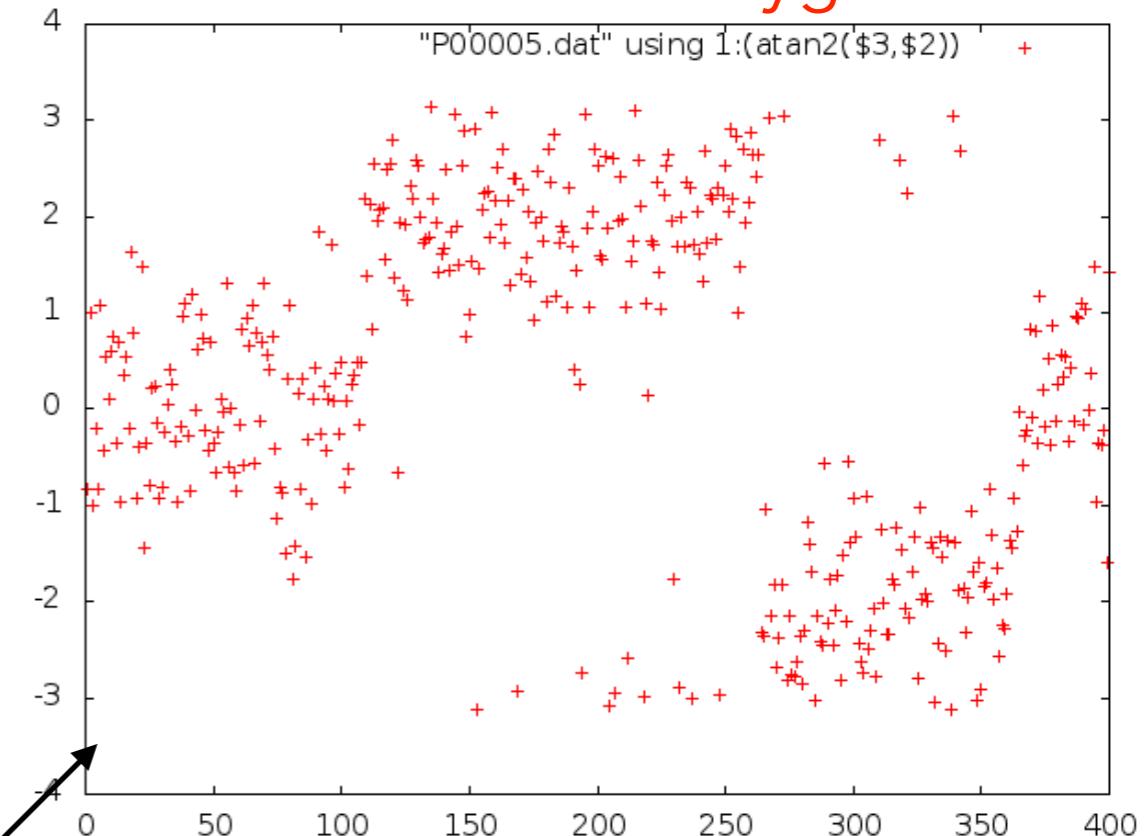
Polyakov loop and topology

Pick up two configurations and look into the x-dependence of $\arg[P]$

conf. at the perimeter of Polygon



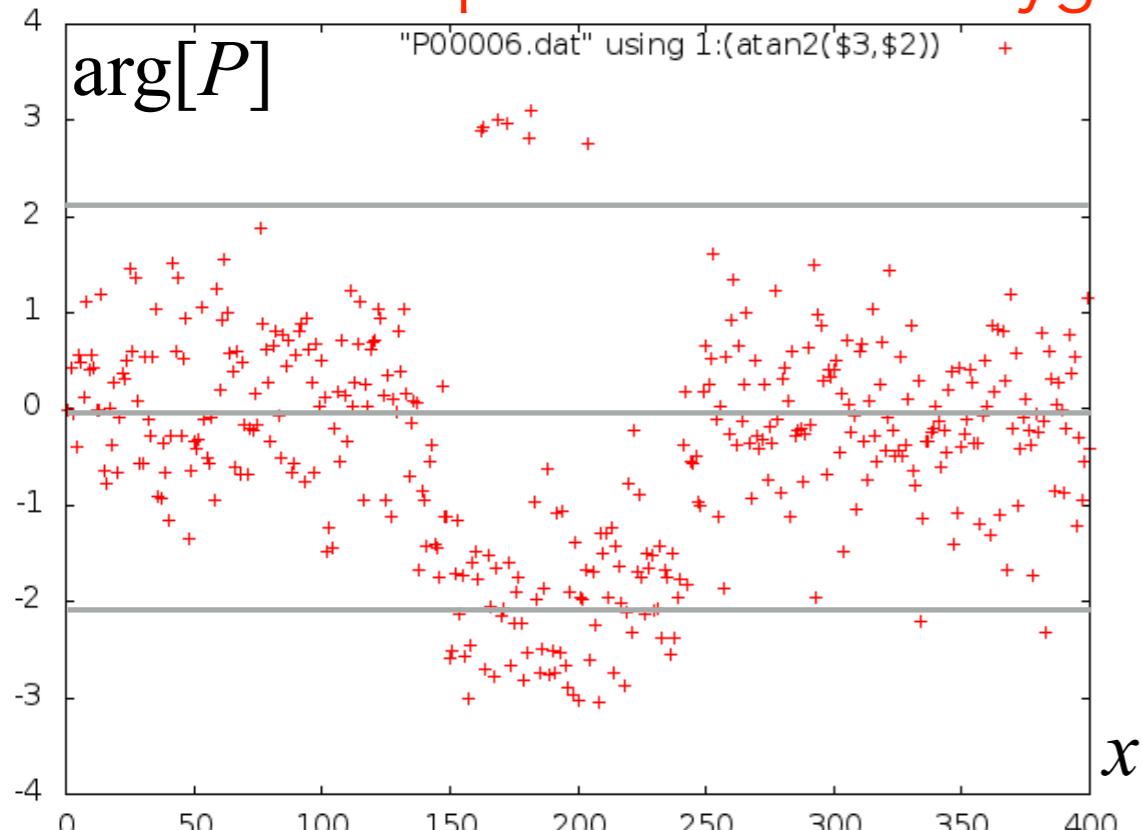
conf. inside Polygon



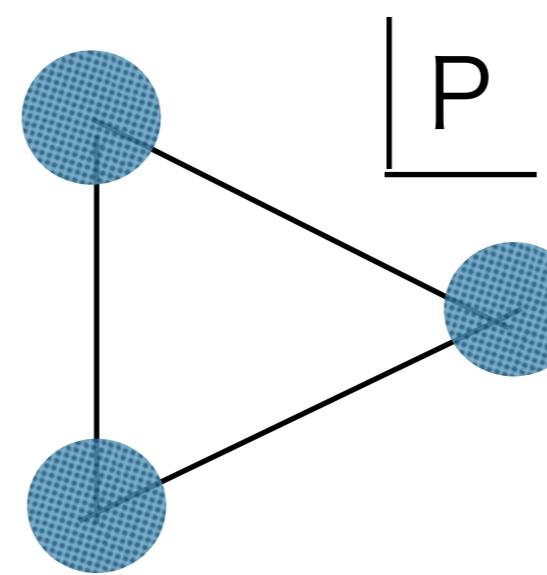
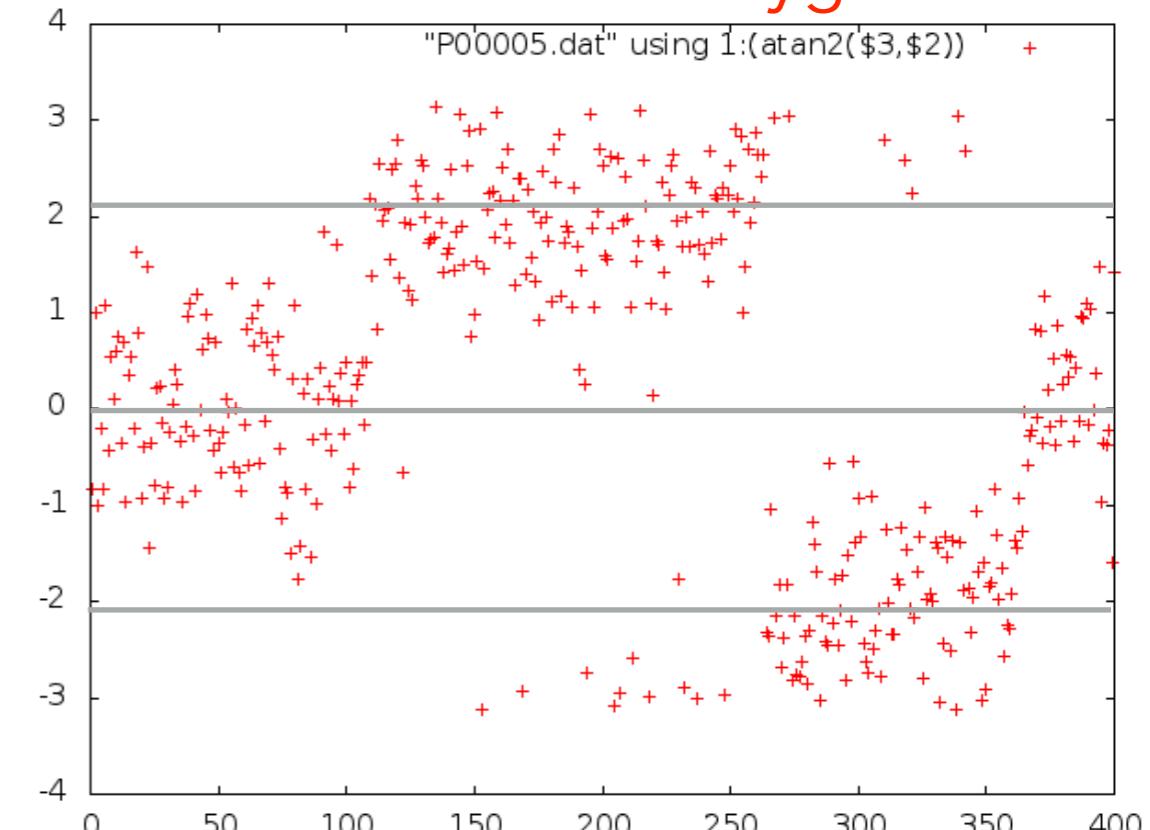
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Pick up two configurations and look into the x -dependence of $\arg[P]$

conf. at the perimeter of Polygon



conf. inside Polygon

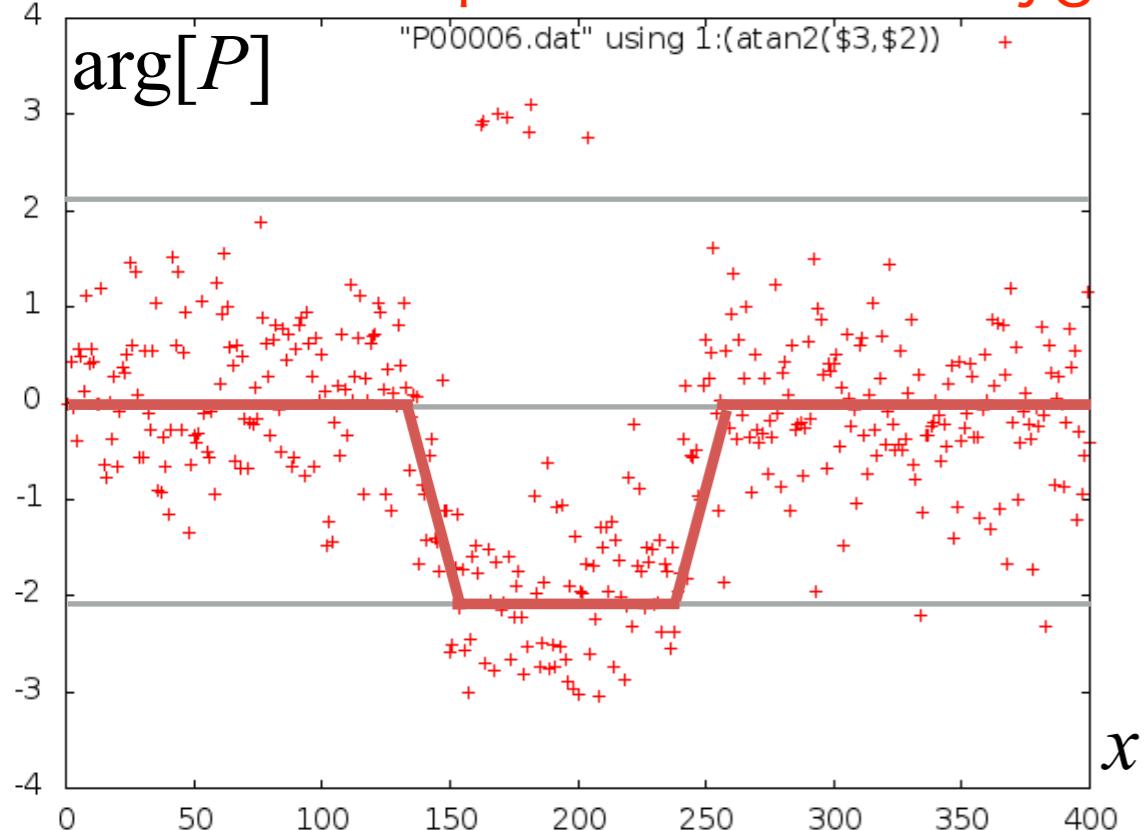


P

Polyakov loop and topology

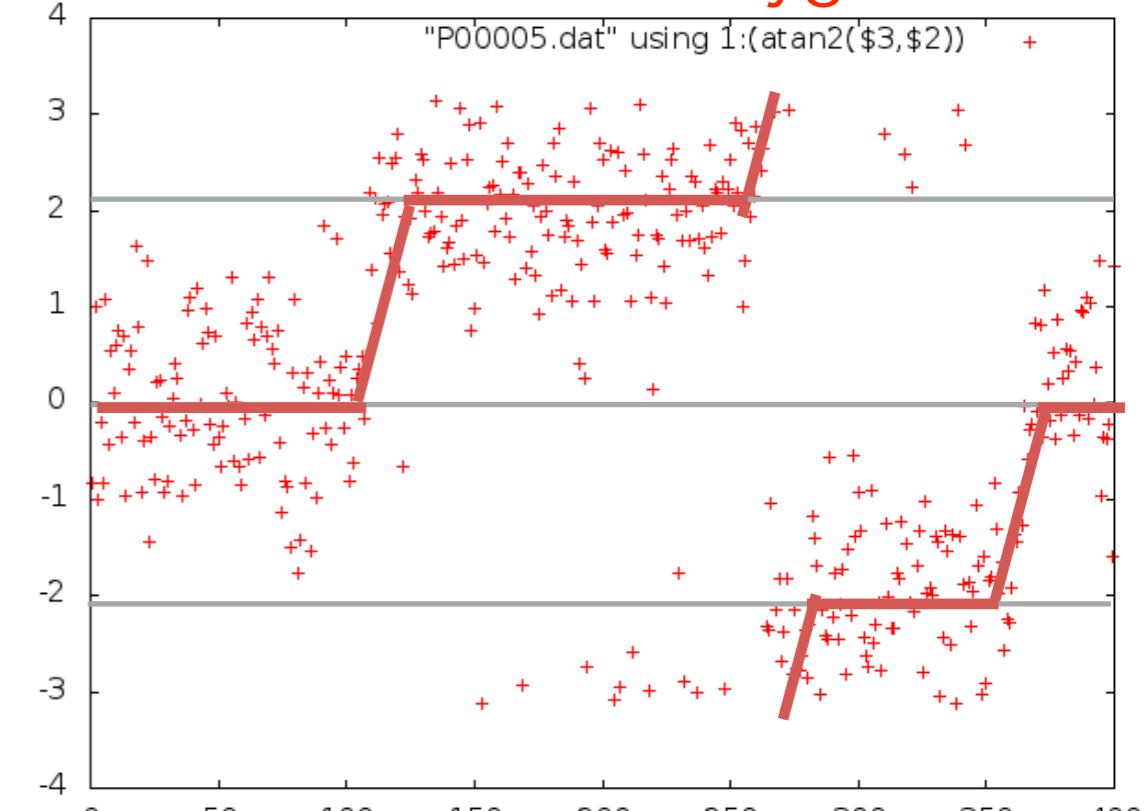
Pick up two configurations and look into the x-dependence of $\arg[P]$

conf. at the perimeter of Polygon



1/3 fractional anti-instanton +
1/3 fractional instanton
= **bion (Q=0)**

conf. inside Polygon



3 × 1/3 fractional instantons
= **instanton (Q=1)**

In 2-dim, the Polyakov loop phase is directly related with the topological charge

M.Hongo,T.Misumi,Y.Tanizaki: JHEP 1902 (2019) 070

Polygon-shaped distribution of Polyakov loop on TBC lattice
includes fractional instanton

Fractional instanton in Yang-Mills theory

EI, JHEP 1905 (2019) 093

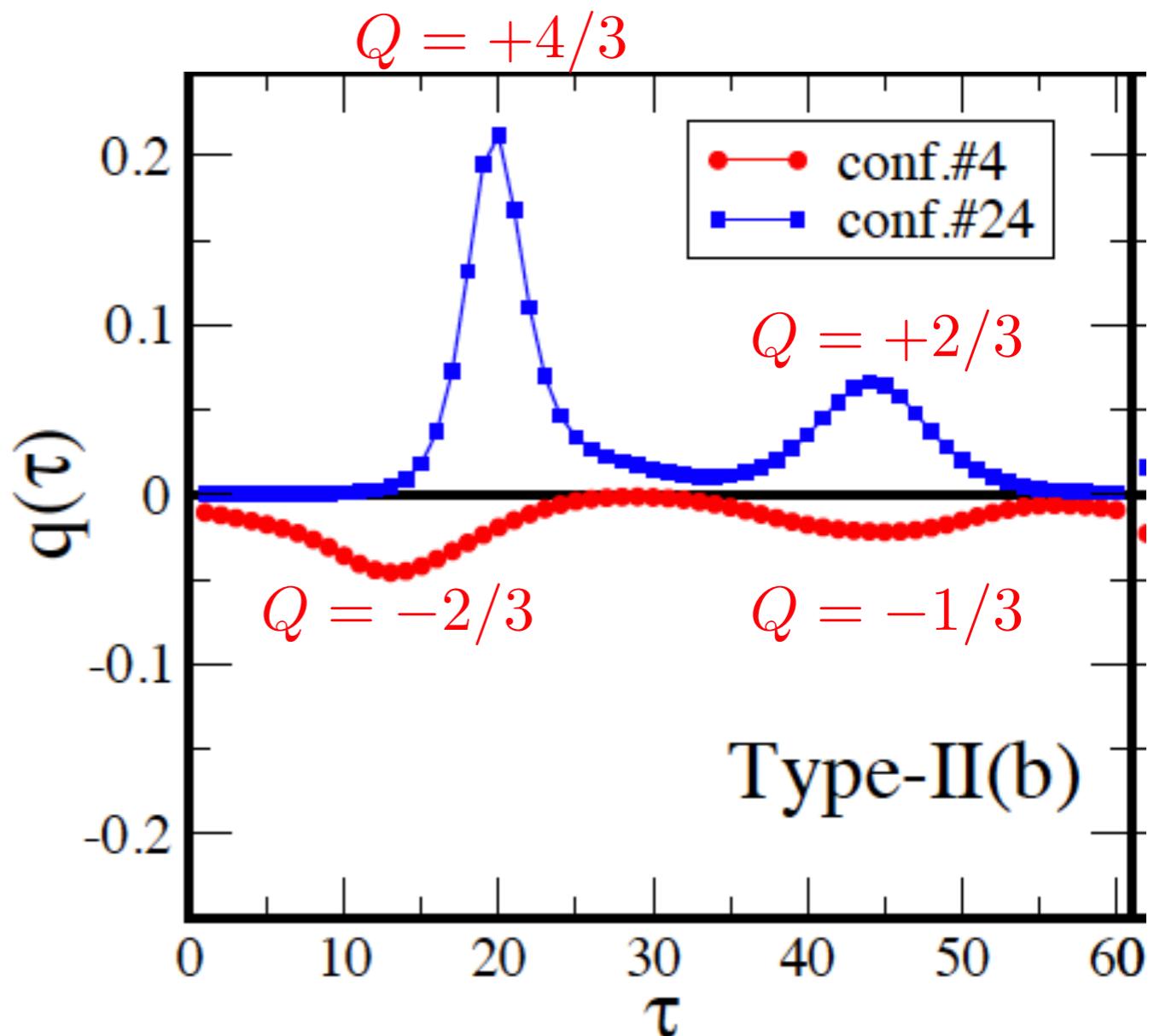
On $\mathbb{T}_{L_s}^3 \times S_{L_\tau}^1$ where $L_s \ll L_\tau$

with center-twisted boundary condition to (only) two-dimensions in spatial directions

B.C. for $(x,y,z, \tau) = (\text{TBC}, \text{TBC}, \text{PBC}, \text{PBC})$

Lattice parameters, $\beta = 16, (N_s, N_\tau) = (12, 60)$, set to be in weak coupling regime

$$q(\tau) = \frac{1}{32\pi^2} \sum_{x,y,z} \text{Tr} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(x, y, z, \tau)$$



Relationship between q and P-loop

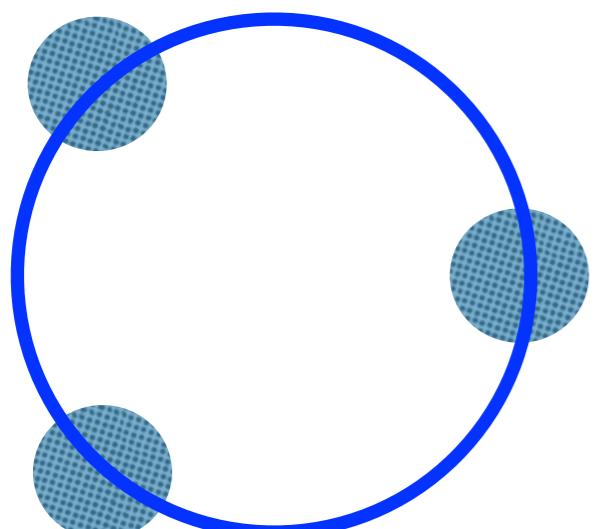
EI, JHEP 1905 (2019) 093

Complex phase of P_z for a configuration including fractional instanton in Yang-Mills theory in the twisted spacetime

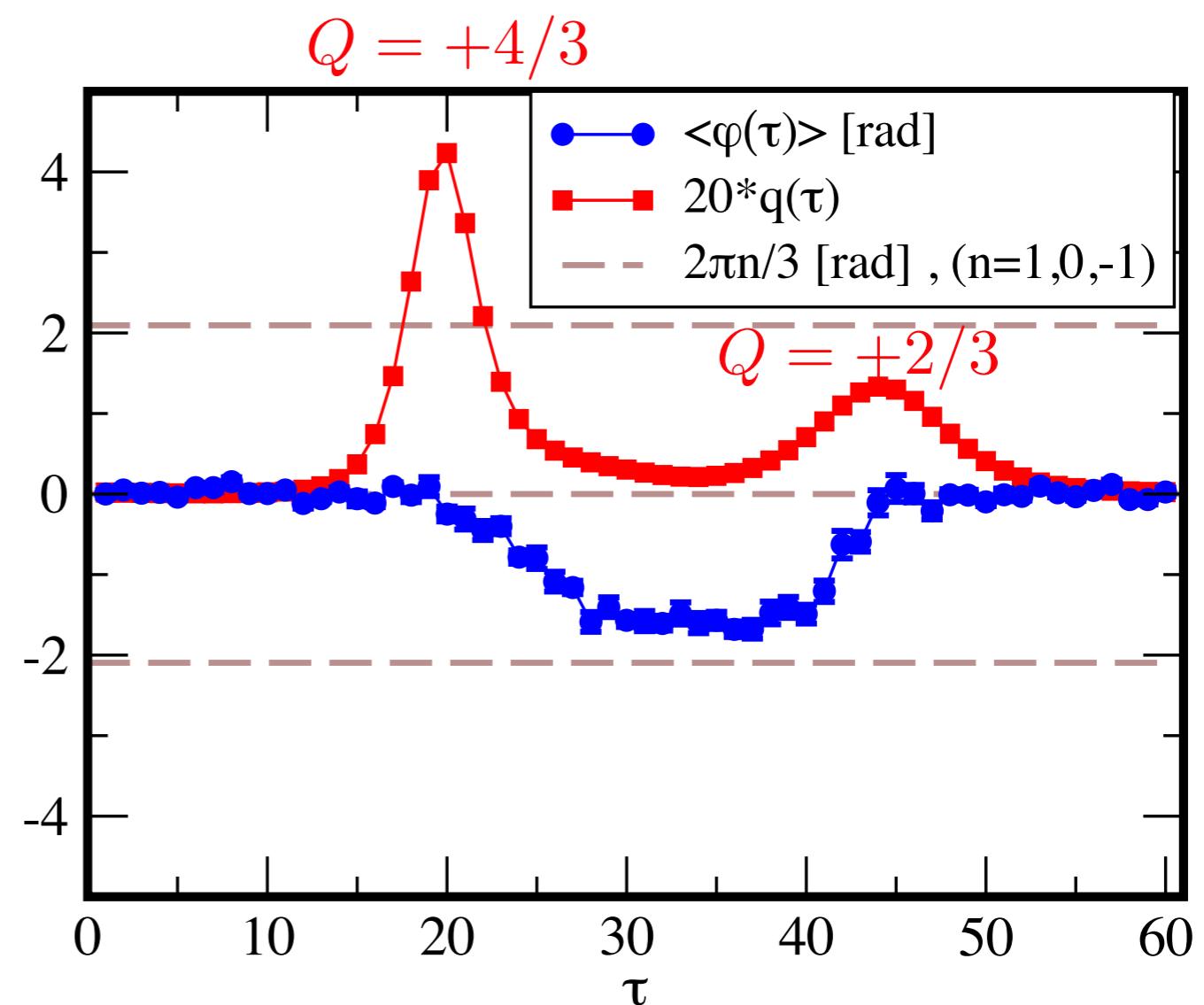
$$\tilde{P}_z(x, y, \tau) = \frac{1}{N_c} \text{Tr} \left[\prod_j U_z(x, y, z=j, \tau) \right]$$
$$\equiv |\tilde{P}_z(x, y, \tau)| e^{i\varphi(x, y, \tau)}.$$

Sum up them for spatial coordinates

$$\langle \varphi(\tau) \rangle = \sum_{x,y} \varphi(x, y, \tau)$$



Fractional instanton connects
 N degenerate vacua in $SU(N)$
gauge theory



Summary for $\mathbb{C}P^{N-1}$ models on $\mathbb{R} \times S^1$

PBC case

- (1) confinement- deconfinement crossover exists for finite N, where the order parameter is the Polyakov loop
- (2) Global PSU(N) symmetry is preserved in both phases
Classical in confinement phase and quantum in deconfinement phase
- (3) thermal entropy is described by the one for N-1 free complex scalar fields in weak coupling limit. It is consistent with the large-N analysis.

TBC case

- (1) \mathbb{Z}_N symmetry is restored even in high β regime at quantum level
- (2) We have to consider what is the definition of the confinement or deconfinement
- (3) Some bion configurations appear in lattice simulation

Backup

Property of classical solution on $\mathbb{T}^3 \times \mathbb{R}$

Witten, NPB202(1982)253 (section7)

Consider classical solution and its gauge equivalent

Gauge transf. under twisted b.c. $U_\mu(n) \rightarrow \Lambda(n)U_\mu(n)\Lambda^\dagger(n + \hat{\mu})$

z direction (compact, PBC)

$$\Lambda(n + \hat{z}N_s) = e^{2\pi i l_z/N_c} \Lambda(n)$$

$$l_z = 0, 1, \dots N_c - 1$$

Extended \mathbb{Z}_{N_c} transf. is allowed

Topological charge

$$\begin{aligned} Q &= \frac{1}{8\pi^2} \int \text{Tr}(F \wedge F) \\ &= -\frac{1}{24\pi^2} \int \text{Tr}(\Lambda^{-1} d\Lambda) \wedge (\Lambda^{-1} d\Lambda) \wedge (\Lambda^{-1} d\Lambda) \\ &= \frac{l_z n'}{N_c} + \text{integer} \end{aligned}$$

Polyakov loop in z-direction

$$P_z = \frac{1}{N_c} \text{Tr} \exp \left[i \int A_z dx \right]$$

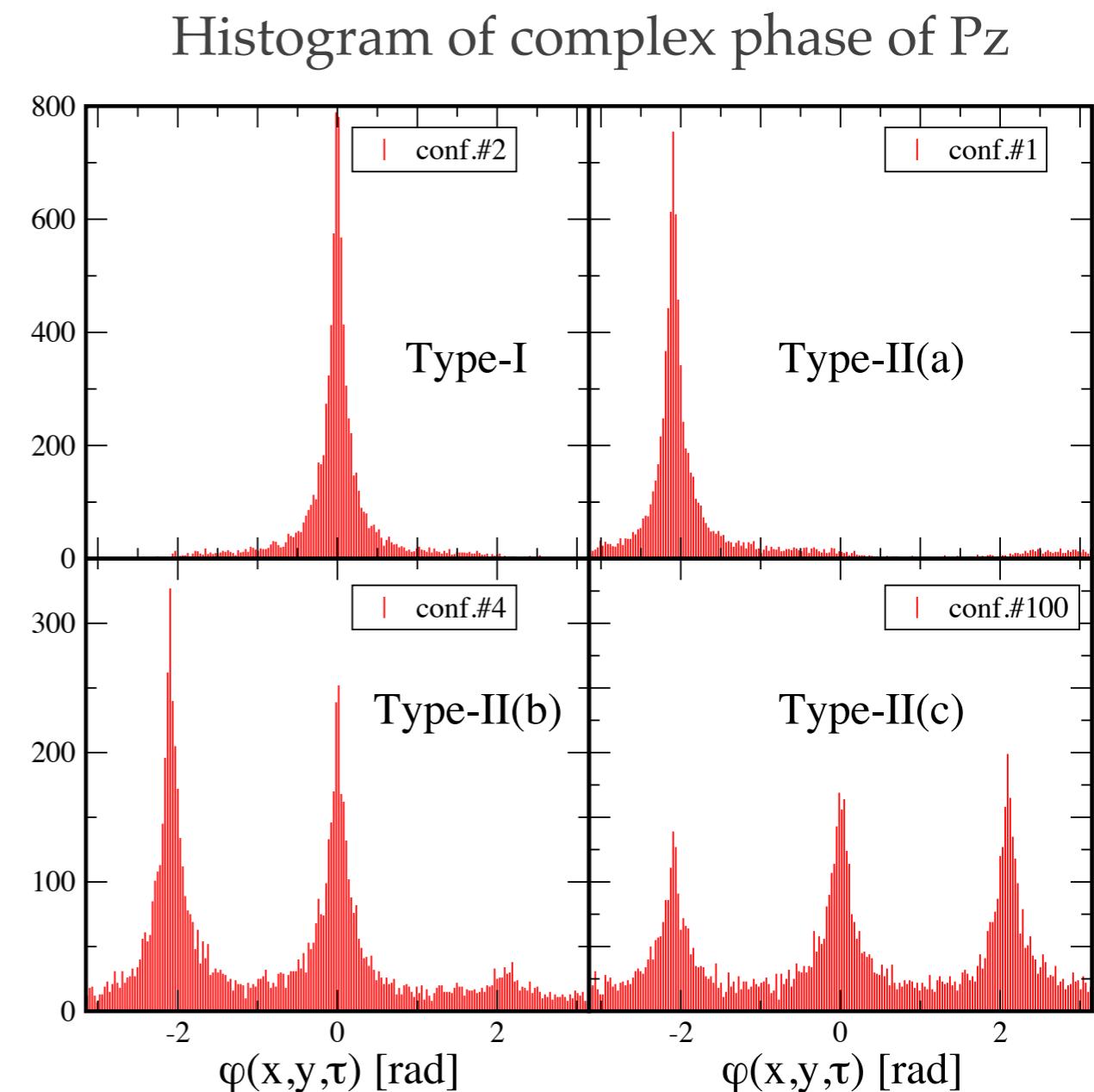
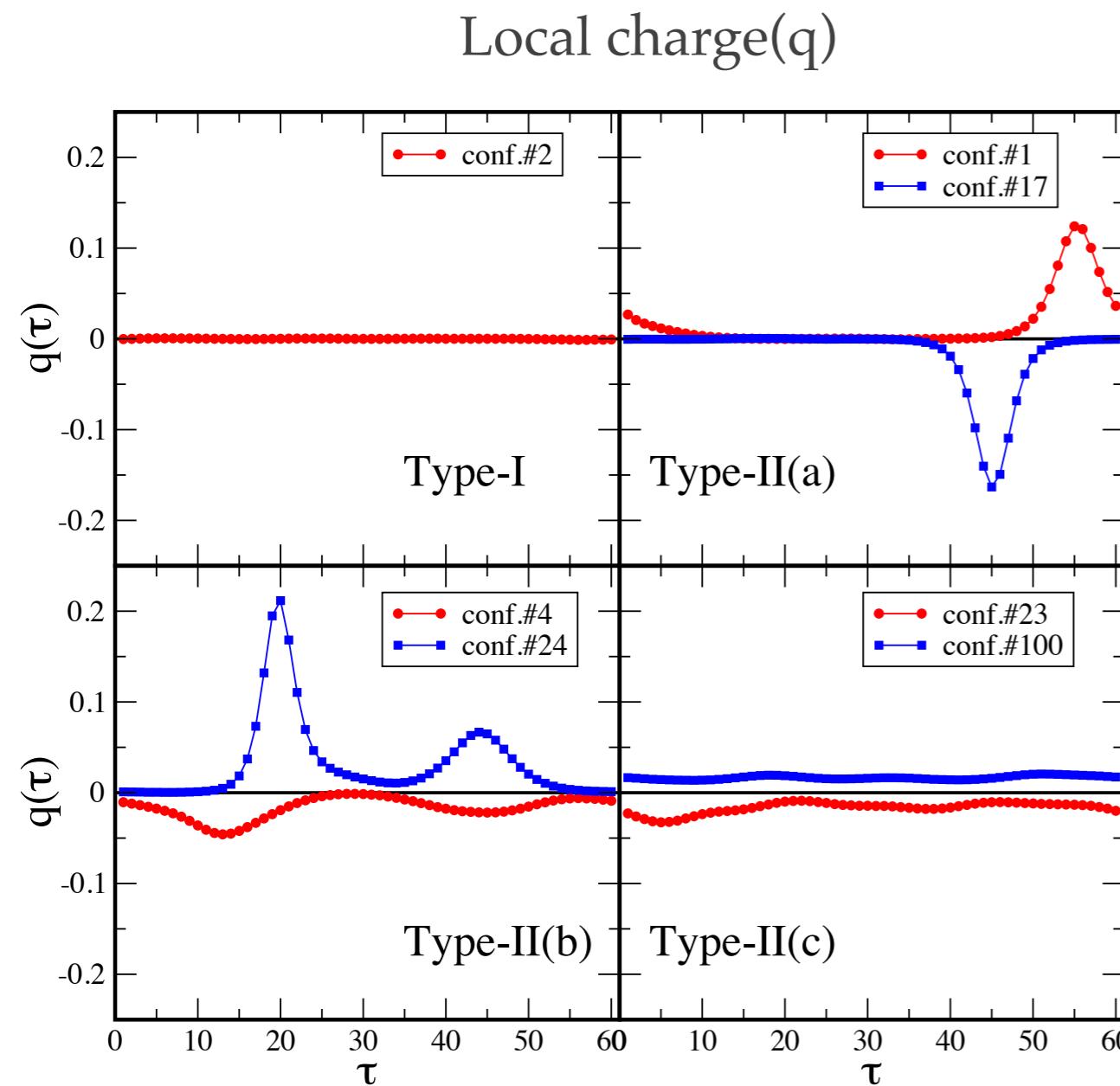
$$\text{Gauge transf. } A_z \rightarrow \Lambda^{-1} A_z \Lambda - i \Lambda^{-1} (\partial_z \Lambda)$$

$$\begin{aligned} P_z &\rightarrow \frac{1}{N_c} \text{Tr} \exp \left[i \int A_z dx + 2\pi l_z/N_c + 2\pi n \right] \\ &= e^{2\pi i l_z/N_c} P_z \end{aligned}$$

If l_z is not a multiple number of N_c , then Q can be fractional.

If fractional instanton appears, the P_z rotates in complex plane.

Relationship between q and Pz



The type of local topological charge is strongly related with the distribution of P_z .