Lattice study for the $\mathbb{C}P^{N-1}$ models on $\mathbb{R} \times S^1$

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arXiv:1907.06925 and work in progress

Fractional instanton of SU(3) gauge theory

EI, JHEP 1905 (2019) 093

Topological solitons, Nonperturbative Gauge Dynamics and Confinement 2
@ INFN - Pisa and Department of Physics, University of Pisa (2019/7/18)
Motivation: $\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$

$\mathbb{C}P^{N-1}$ on $\mathbb{R}^2$ : a good toy model of QCD

Asymptotic freedom
Nonperturbative properties
(confinement, topological objects)

Several lattice Monte Carlo studies on large $\mathbb{T}^2$

Berg and Luscher (81), Campostrini, Rossi, Vicari (92), Alles, Cosmai, D’Elia, Papa (00), Flynn, Juttner, Lawson, Sanfillippo (15), Abe, Fukushima, Hidaka, Matsueda, Murase, Sasaki (18), Bonanno, Bonati, D’Elia (19)

Reveal its classical solution and topological object
Give some technical developments
(smearing method, autocorrelation problems in MC calculations)
Motivation: $\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$

We can introduce some boundary conditions on $S^1$. Physics on compactified spacetime depends on the b.c., while in any case it goes to the same one in decompactified limit.

PBC for $S^1$ (finite-T system)

- confinement/deconfinement transition?
- How to realize global PSU(N) = SU(N)/$\mathbb{Z}_N$ symmetry in both phases?

(The Coleman theorem in 2-dim. at least finite N)

It will be broken in the large-N limit?

TBC for $S^1$: $\phi(x, \tau + L_\tau) = \Omega \phi(x, \tau)$, $\Omega = \text{diag.}[1, e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2(N-1)\pi i/N}]$


Bolognesi, Gudnason, Konishi, Ohashi: arXiv:1905.10555


T.Fujimori, S.Kamata, T.Misumi, M.Nitta and N.Sakai PRD94 (2016) 105002, PRD95 (2017) 105001, PTEP 2017 no.8 083B02


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novel type of a classical solution(called bion)? Good to see the resurgence structure?
$\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$ with PBC
Lattice setup


Action in continuum

\[ S = \frac{1}{2g^2} \int d^2x |D_\mu \phi|^2 \]
\[ \phi^i, i = 1, \ldots, N \]

\[ |\phi|^2 = 1, D_\mu \phi = (\partial_\mu + iA_\mu)\phi \]

Here, \( A_\mu \) is U(1) gauge field (\( A_\mu = \frac{i}{2} \vec{\phi} \cdot \nabla_\mu \phi \))

Action on the lattice

\[ S_{lat.} = -N\beta \sum_{n,\mu} (\bar{\phi}_{n+\mu} \cdot \phi_n \lambda_{n,\mu} + \bar{\phi}_n \cdot \phi_{n+\mu} \bar{\lambda}_{n,\mu} - 2) \]

U(1) link variable: \( \lambda_{n,\mu} = e^{iA_\mu(n)} \)

Note that \( \beta \) denotes the coupling constant

\[ N\beta = \frac{1}{g^2} \]

Weak coupling = large \( \beta \)

Strong coupling = small \( \beta \)

Over heat-bath algorithm is adopted (local updation)
**$\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$ with PBC**

(1) confinement/deconfinement transition?

Large-N studies based on the gap equation do not reach a consensus  
(Next talk by Ohashi-san)

Lattice study does not need any gauge fixing  
Good to see the information of ground state

(this work)  
We consider only finite-N and show the N-dependence of the transition  
By analogy with QCD, the Polyakov loop of link variables coupled to matter fields is a good order parameter for confinement or deconfinement

Polyakov loop in continuum spacetime

$$\mathcal{P} = e^{i \int d\tau A_\tau}$$

Polyakov loop on the lattice

$$\mathcal{P} = \frac{1}{N_s} \sum_{n_x} \prod_{n_\tau} \lambda_\tau(n_x, n_\tau)$$

Bolognesi, Gudnason, Konishi, Ohashi: arXiv:1905.10555
Confinement/Deconfinement

Polyakov loop $\langle P \rangle \propto e^{-F_\phi L_\tau}$

$(N_s, N_\tau) = (200, 8)$

$N_{\text{sweep}} = 200,000$

$\langle P \rangle \approx 0 \iff F_\phi \to \infty$ in $V \to \infty$: confinement

$\langle P \rangle \neq 0 \iff F_\phi$ is finite: deconfinement
Confinement/Deconfinement

To study strength of the transition, volume scaling

$$\chi_{\langle |P|\rangle,\text{max}} = a + cV^p$$

$p=1$: 1st order

$0<p<1$: 2nd order or crossover

N= 3: $p=0.056(7)$
N= 5: $p=0.058(7)$
N=10: $p=0.052(7)$
N=20: $p=0.043(8)$

indicate it is crossover for finite $N$
Confinement/Deconfinement

N dependence of $\chi<|P|>$ with fixed $T(= N \tau)$ simulation


The peak is quite broad for small $N$. It gets shaper as $N$ increases. The result suggests the 2nd order transition occurs in large-$N$ limit? But $\beta_c \to 0$ at that time. Careful analysis is necessary.
Global PSU(N) symmetry

(2) PSU(N) symmetry breaking is associated with deconfinement in the large-N analysis? What is happen in finite N in deconfinement phase?


$N \times N$ matrix: $P^{ij} = \sum_n \bar{\phi}_n^i \phi_n^j - \frac{1}{N} \delta^{ij}$

If all $\langle P^{ij} \rangle = 0$, then PSU(N) is preserved.

Confinement phase

$(N, \beta, N_s, N_\tau) = (3, 0.1, 200, 8)$

Deconfinement phase

$(N, \beta, N_s, N_\tau) = (3, 3.9, 200, 8)$
Global PSU(N) symmetry

(2) PSU(N) symmetry breaking is associated with deconfinement in the large-N analysis? What is happen in finite N in deconfinement phase?


\[ N \times N \text{ matrix: } P^{ij} = \sum_n \bar{\phi}_n^i \phi_n^j - \frac{1}{N} \delta^{ij} \]

If all \( \langle P^{ij} \rangle = 0 \), then PSU(N) is preserved.

Confined phase

\( (N, \beta, N_s, N_\tau) = (3, 0.1, 200, 8) \)

Deconfinement phase

\( (N, \beta, N_s, N_\tau) = (3, 3.9, 200, 8) \)

Ensemble average of each component \( \langle P^{ii} \rangle \) is zero within stat. error

In the large-N limit, it is still open question.
Thermal entropy and free energy

All N fields are equivalent. But there is one constraint $|\phi|^2 = 1$

Let us count the actual d.o.f by the entropy

Free energy density can be calculated by the one for massive free complex scalar system on $S^1_s \times S^1_t$

$$ f = \frac{1}{L_s L_t} \sum_{n=-\infty}^{\infty} \log 4 \sinh^2 \frac{\alpha_n L_t}{2} - f_0 $$

Massless and thermodynamic limit: $f(L_t) = -\frac{\pi}{3L_t^2}$ Here, $T = 1/L_t$

Entropy density: $s/T = -\frac{1}{T} \frac{\partial f}{\partial T} = \frac{2\pi}{3}$ for one free massless complex scalar field

Our lattice approach: entropy from energy-momentum-tensor (EMT)

Entropy can be obtained by EMT

$$ s/T = (\epsilon + P)/T^2 = \frac{\langle T_{xx} \rangle - \langle T_{tt} \rangle}{T^2} $$

In quenched QCD,

Two approaches give a consistent result

Asakawa, Hatsuda, EI, Kitazawa, Suzuki:
Thermal entropy and free energy

$s/T$: entropy density for $\mathbb{C}P^{N-1}$ model

Our lattice results show

$$s/T = \frac{2\pi}{3}(N - 1)$$

in weak coupling limit ($\beta \to \infty$)

cf) one scalar:

$$s/T = -\frac{1}{T} \frac{\partial f}{\partial T} = \frac{2\pi}{3}$$

Our entropy data is consistent with the one for free $N-1$ complex scalar

It is also consistent with entropy density:

$$s/T = -\frac{1}{T} \frac{\partial f}{\partial T} = \frac{2\pi N}{3}$$

in the large-N analysis


All $N$ components are equivalent, but the actual d.o.f. on the lattice is correctly $N-1$
\( \mathbb{C}P^{N-1} \) **sigma model on** \( \mathbb{R} \times S^1 \) **with TBC**

All results are preliminary

**TBC for** \( S^1 \) :

\[
\phi(x, \tau + L_\tau) = \Omega \phi(x, \tau), \quad \Omega = \text{diag.}[1, e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2(N-1)\pi i/N}]
\]

b.c. has flavor dependence
Polyakov loop : scatter plot

PBC case: The action does not have $Z_N$ symmetry

Low-$\beta$ : around the origin
$\rightarrow$ (approximately) $Z_N$ symmetric

High-$\beta$ : moves to one of $Z_N$ vacua
$\rightarrow$ (manifestly) $Z_N$ breaking
Polyakov loop : scatter plot

TBC case: The action has $\mathbb{Z}_N$ symmetry

Low-$\beta$ :
around the origin $\rightarrow$
$\mathbb{Z}_N$ symmetric at “classical level”

Intermediate-$\beta$ :
Transition between $N$ vacua $\rightarrow$
$\mathbb{Z}_N$ symmetric at “quantum level”
Polyakov loop : scatter plot

further high $\beta$ regime

$N=3$

- Low $\beta \to |\langle P \rangle| = 0$ : distribution around origin
- Mid $\beta \to |\langle P \rangle|$ highly fluctuates : distribution forms polygons
- High $\beta \to |\langle P \rangle| \neq 0$ : (but it seems that more stat. can form polygon?)

Adiabatically continue to $\beta \to \infty$?
Polyakov loop : scatter plot

N dependence

N=5

• Low $\beta \rightarrow |<P>|=0$: distribution around origin

• Mid $\beta \rightarrow |<P>|$ highly fluctuates: distribution forms polygons

• High $\beta \rightarrow |<P>|\neq 0$: (but it seems that more stat. can form polygon?)

Adiabatically continue to $\beta \rightarrow \infty$?
higher $\beta$ and larger volume

$N=3$, $\beta=4.0$, $(400 \times 12)$: Polygon-shaped distribution appears

Weak coupling expansion is valid in this $\beta$

$|\langle P \rangle| \sim$ small

Very high-$\beta$: quantum $Z_N$ symmetric case found with certain probability

Let us see which type of configurations appears inside and the perimeter of the polygon.
Pick up two configurations and look into the x-dependence of \( \text{arg}[P] \)

conf. at the perimeter of Polygon

conf. inside Polygon

\[
\text{arg}[P]
\]
Polyakov loop and topology

Pick up two configurations and look into the $x$-dependence of $\text{arg}[P]$.

**conf. at the perimeter of Polygon**

**conf. inside Polygon**
Polyakov loop and topology

Pick up two configurations and look into the x-dependence of arg[P]

conf. at the perimeter of Polygon

\[
\begin{align*}
\text{arg}[P] \quad & \quad \text{conf. inside Polygon}
\end{align*}
\]

1/3 fractional anti-instanton +
1/3 fractional instanton

= bion (Q=0)

In 2-dim, the Polyakov loop phase is directly related with the topological charge

M. Hongo, T. Misumi, Y. Tanizaki: JHEP 1902 (2019) 070

Polygon-shaped distribution of Polyakov loop on TBC lattice
includes fractional instanton
Fractional instanton in Yang-Mills theory

On $\mathbb{T}_{L_s}^3 \times S_{L_\tau}^1$ where $L_s \ll L_\tau$

with center-twisted boundary condition to (only) two-dimensions in spatial directions

B.C. for $(x,y,z,\tau) = (TBC,TBC,PBC,PBC)$

Lattice parameters, $\beta = 16, (N_s, N_\tau) = (12,60)$, set to be in weak coupling regime

$$q(\tau) = \frac{1}{32\pi^2} \sum_{x,y,z} \text{Tr} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(x,y,z,\tau)$$

Type-$\Pi(b)$
Relationship between $q$ and P-loop

EI, JHEP 1905 (2019) 093

Complex phase of Pz for a configuration including fractional instanton in Yang-Mills theory in the twisted spacetime

$$\tilde{P}_z(x, y, \tau) = \frac{1}{N_c} \text{Tr} \left[ \prod_j U_z(x, y, z = j, \tau) \right]$$

$$\equiv |\tilde{P}_z(x, y, \tau)| e^{i\varphi(x, y, \tau)}.$$

Sum up them for spatial coordinates

$$\langle \varphi(\tau) \rangle = \sum_{x,y} \varphi(x, y, \tau)$$

Fractional instanton connects $N$ degenerate vacua in SU(N) gauge theory
Summary for $CP^{N-1}$ models on $\mathbb{R} \times S^1$

PBC case

1. confinement-deconfinement crossover exists for finite $N$, where the order parameter is the Polyakov loop

2. Global PSU($N$) symmetry is preserved in both phases
   - Classical in confinement phase and quantum in deconfinement phase

3. thermal entropy is described by the one for $N-1$ free complex scalar fields in weak coupling limit. It is consistent with the large-$N$ analysis.

TBC case

1. $\mathbb{Z}_N$ symmetry is restored even in high $\beta$ regime at quantum level

2. We have to consider what is the definition of the confinement or deconfinement

3. Some bion configurations appear in lattice simulation
Backup
Property of classical solution on $\mathbb{T}^3 \times \mathbb{R}$

Consider classical solution and its gauge equivalent
Gauge transf. under twisted b.c. $U_\mu(n) \to \Lambda(n) U_\mu(n) \Lambda^\dagger(n + \hat{\mu})$

$z$ direction (compact, PBC)

$\Lambda(n + \hat{z}N_S) = e^{2\pi i l_z/N_c} \Lambda(n)$

$\quad l_z = 0, 1, \cdots N_c - 1$

Extended $\mathbb{Z}_{N_c}$ transf. is allowed

**Topological charge**

$$Q = \frac{1}{8\pi^2} \int \text{Tr}(F \wedge F)$$

$$= -\frac{1}{24\pi^2} \int \text{Tr}(\Lambda^{-1} d\Lambda) \wedge (\Lambda^{-1} d\Lambda) \wedge (\Lambda^{-1} d\Lambda)$$

$$= \frac{l_z n'}{N_c} + \text{integer}$$

**Polyakov loop in $z$-direction**

$$P_z = \frac{1}{N_c} \text{Tr} \exp \left[i \int A_z dx \right]$$

Gauge transf. $A_z \to \Lambda^{-1} A_z \Lambda - i \Lambda^{-1} (\partial_z \Lambda)$

$$P_z \to \frac{1}{N_c} \text{Tr} \exp \left[i \int A_z dx + 2\pi l_z / N_c + 2\pi n \right]$$

$$= e^{2\pi i l_z/N_c} P_z$$

If $l_z$ is not a multiple number of $N_c$, then $Q$ can be fractional.
If fractional instanton appears, the $P_z$ rotates in complex plane.
The type of local topological charge is strongly related with the distribution of Pz.