Topological properties of QCD-like theories

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Topological Solitons, Nonperturbative Gauge Dynamics and Confinement 2 - Pisa 18-20 July 2019

OUTLINE

After a brief review of θ-dependence in Yang-Mills theories (analytic predictions vs lattice results), I will discuss 2 main topics:

- **θ-dependence and Z_N realization: an analysis in trace deformed Yang-Mills theories**

- **precision tests of the large-N limit: the case of CP^{N-1} models in two dimensions**

QCD presents various non-perturbative features: confinement, χSB, ...

An important role is played by the presence of gauge configurations with non-trivial topology, labelled by an integer winding number $Q = \int d^4x\, q(x)$

$$q(x) = \frac{g^2}{64\pi^2} G^a_{\mu\nu}(x) \tilde{G}^a_{\mu\nu}(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G^a_{\mu\nu}(x) G^a_{\rho\sigma}(x)$$

$$GG \propto \vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a \;\; ; \quad G\tilde{G} \propto \vec{E}^a \cdot \vec{B}^a$$

Homotopy group: $\pi_3(SU(N_c)) = \mathbb{Z}$

$G\tilde{G}$ **is renormalizable and a possibile coupling to it is a free parameter of QCD**

$$Z(\theta) = \int [\mathcal{D}A][\mathcal{D}\bar{\psi}][\mathcal{D}\psi]\, e^{-S_{QCD}}\, e^{i\theta Q}$$

the theory at $\theta \neq 0$ **is well defined, but presents explicit breaking of** CP **symmetry.**

$|\theta| < 10^{-10}$ **(strong CP-problem), however** θ**-dependence is related to essential theoretical and phenomenological aspects anyway.**

QCD at non-zero θ

The free energy density $f(\theta) = -T \log Z/V$ is a periodic even function of θ

It is connected to the probability distribution $P(Q)$ at $\theta = 0$ via Taylor expansion:

$$f(\theta) - f(0) = \frac{1}{2} f^{(2)} \theta^2 + \frac{1}{4!} f^{(4)} \theta^4 + \dots \quad ; \qquad f^{(2n)} = \left. \frac{d^{2n} f}{d\theta^{2n}} \right|_{\theta=0} = -(-1)^n \frac{\langle Q^{2n} \rangle_c}{V}$$

A common parametrization is the following

$$f(\theta, T) - f(0, T) = \frac{1}{2} \chi(T) \theta^2 \Big[1 + b_2(T) \theta^2 + b_4(T) \theta^4 + \cdots \Big]$$

$$\chi = \frac{1}{V} \langle Q^2 \rangle_0 = f^{(2)} \qquad b_2 = - \left. \frac{\langle Q^4 \rangle - 3 \langle Q^2 \rangle^2}{12 \langle Q^2 \rangle} \right|_{\theta=0} \qquad b_4 = \left. \frac{\langle Q^6 \rangle - 15 \langle Q^4 \rangle \langle Q^2 \rangle + 30 \langle Q^2 \rangle^3}{360 \langle Q^2 \rangle} \right|_{\theta=0}$$

$P(Q)$ is non-perturbative: a lattice investigation is the ideal first-principle approach

however various analytic predictions exist, working well in different regimes

Predictions about θ-dependence - I

Dilute Instanton Gas Approximation (DIGA) for high T (Gross, Pisarski, Yaffe 1981)

IDEA: semi-classical integration around classical solutions with $Q \neq 0$: instantons

1-loop one-instanton contribution (ρ is the instanton radius):

$$\exp\left(-\frac{8\pi^2}{g^2(\rho)}\right)$$

topological fluctuations exponentially suppressed $\Longrightarrow$ dilute instanton gas approximation

- by asymptotic freedom, works well only for small instantons
- breaks down for large instantons ($1/\rho \lesssim \Lambda_{QCD}$)

Finite-T acts as an IR cut-off to ρ, making the 1-loop result more and more reliable

- **instantons - antiinstantons treated as uncorrelated (non-interacting) objects**
  Poisson distribution with an average probability density p per unit volume

$$Z_\theta \simeq \sum \frac{1}{n_+!n_-!}(V_4 p)^{n_+ + n_-} e^{i\theta(n_+ - n_-)} = \exp\left[2V_4 p\cos\theta\right]$$

$$F(\theta,T) - F(0,T) \simeq \chi(T)(1-\cos\theta) \implies \quad b_2 = -1/12\,; \quad b_4 = 1/360\,; \ldots$$

- **The prefactor $\chi(T)$ can also be computed in the 1-loop approximation:**

$$\chi(T) \sim T^4 \left(\frac{m}{T}\right)^{N_f} e^{-8\pi^2/g^2(T)} \sim m^{N_f} T^{4-\frac{11}{3}N_c - \frac{1}{3}N_f}$$

Notice: the $(1 - \cos\theta)$ prediction is just related to diluteness and might be good before reaching the asymptotic perturbative behavior

Predictions about θ-dependence - II

Chiral Perturbation Theory (χPT) for low T

In the presence of quarks, θ can be moved to light quark masses (if any!) by $U(1)$ axial rotations. Then, at low T, χPT can be applied as usual.

Result for the ground state energy (Di Vecchia, Veneziano 1980)

$$E_0(\theta) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2 \frac{\theta}{2}}$$

$$\chi = \frac{z}{(1+z)^2} m_\pi^2 f_\pi^2, \quad b_2 = -\frac{1}{12} \frac{1+z^3}{(1+z)^3}, \quad z = \frac{m_u}{m_d}$$

Not relevant to pure gauge theories

Predictions about θ-dependence - III

Large-N_c for low T $SU(N_c)$ gauge theories (Witten, 1980)

$g^2 N_c = \lambda$ **fixed as** $N_c \to \infty$ $\Longrightarrow$ **Effective instanton weight** $e^{-8\pi^2 N_c/g^2} \to 0$

Non-trivial θ-dependence persists only if the dependence is on $\bar\theta = \theta/N_c$.

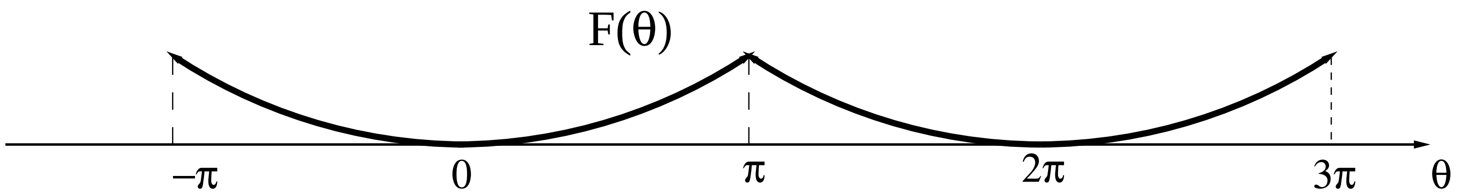
$$f(\theta,T) - f(0,T) = N_c^2 \bar f(\bar\theta, T)$$

$$\bar f(\bar\theta, T) = \frac{1}{2}\bar\chi\bar\theta^2\left[1 + \bar b_2\bar\theta^2 + \bar b_4\bar\theta^4 + \cdots\right]$$

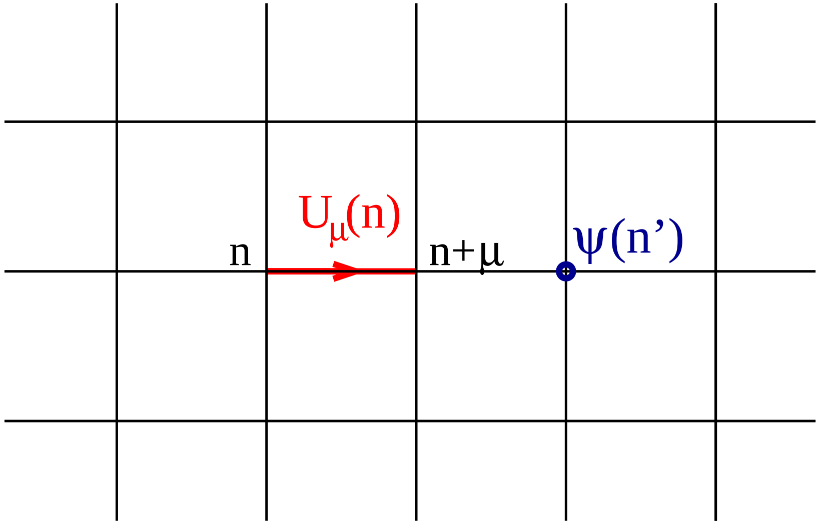
Matching powers of $\bar\theta$ and θ we obtain

$$\chi \sim N_c^0 \;\; ; \quad b_2 \sim N_c^{-2} \;\; ; \quad b_{2n} \sim N_c^{-2n}$$

$P(Q)$ is purely Gaussian in the large N_c limit.



θ-dependence from Lattice QCD simulations



Gauge fields are 3×3 unitary complex matrixes living on lattice links (link variables)

$$U_\mu(n) \simeq \mathcal{P} \exp\left(ig \int_n^{n+\mu} A_\mu dx_\mu \right)$$

Fermion fields live on lattice sites

$$Z(V,T) = \mathrm{Tr}\left(e^{-\frac{H_{\mathrm{QCD}}}{T}} \right) \Rightarrow \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-(S_G[U] + \bar{\psi} M[U] \psi)} = \int \mathcal{D}U e^{-S_G[U]} \det M[U]$$

$$T = \frac{1}{\tau} = \frac{1}{N_t a(\beta, m)}$$

τ is the extension of the compactified time

Numerical Problems in Lattice QCD simulations

main technical issues that one has to face

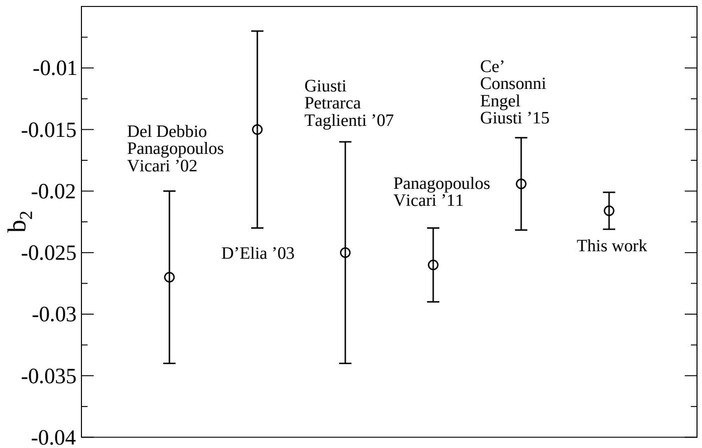
- topological charge renormalizes, naive lattice discretizations are non-integer valued.
  Various methods devised leading to consistent results
  - field theoretic compute renormalization constants and subtract
  - fermionic definitions use the index theorem to deduce Q from fermionic zero modes
  - smoothing methods use various techniques to smooth gauge fields and recover integer Q

- Sign problem at $\theta \neq 0$
  Taylor expansion from cumulants at $\theta = 0$
  or perform simulations at imaginary values of θ

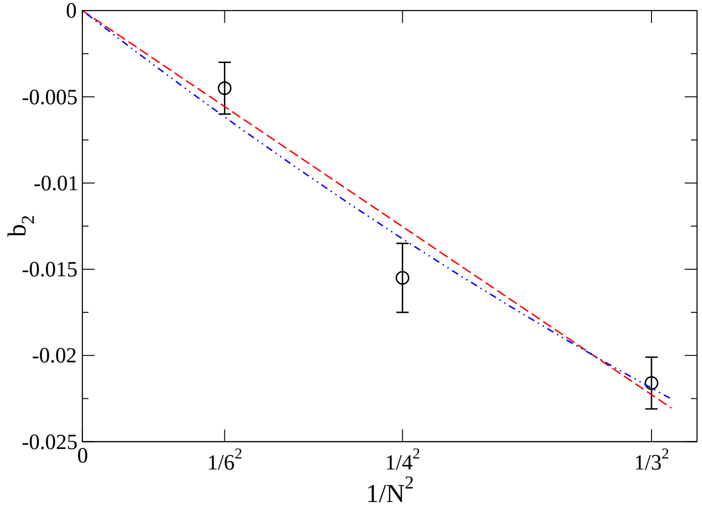
- Freezing of topological modes in the continuum (known algorithms become non-ergodic)

Pure gauge results: $T = 0$ (Yang-Mills vacuum)

Topological susceptibility well known, with increasing refinement, since many years, and compatible with the Witten-Veneziano mechanism for $m_{\eta'}$, $\chi^{1/4} \sim 180$ MeV



Determination of b_2 more difficult. Most recent determination for $SU(3)$ (Bonati, MD, Scapellato, 1512.01544) obtained by introducing an external imaginary θ source to improve signal/noise.



Clear evidence for the predicted large-N_c scaling of b_2:

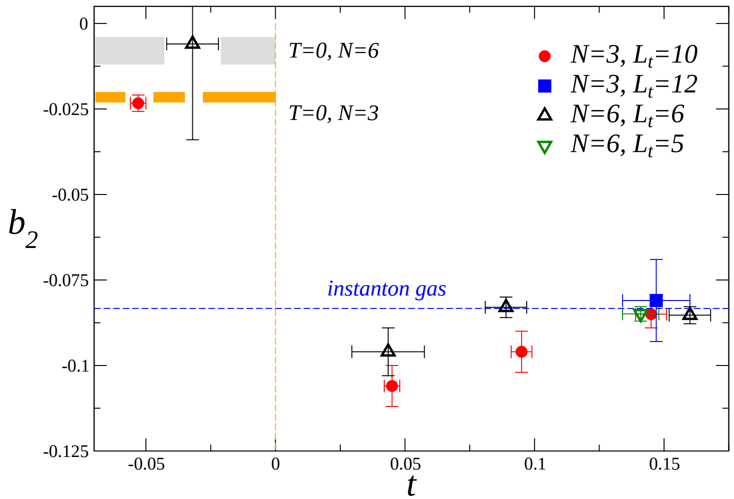
$$b_2 \simeq \frac{\bar{b}_2}{N^2}$$

with $\bar{b}_2 = -0.20(2)$

(Bonati, MD, Rossi, Vicari, 1607.06360)

Pure gauge results: Finite T, across and above T_c

χ **drops suddenly after T_c, known since many years** (B. Alles, MD, A. Di Giacomo, hep-lat/9605013)



from Bonati, MD, Panagopoulos, Vicari 1301.7640

DIGA values for higher cumulants reached quite soon, already for $T \gtrsim 1.1\, T_c$.

Emerging picture:

- **shortly after deconfinement (breaking of center symmetry), topological excitations behave as a dilute non-interacting gas, DIGA: $f(\theta) \propto (1 - \cos(\theta))$.**
- **Below T_C, agreement with large-N predictions, $f = f(\theta/N)$.**
  NOT a collection of dilute instanton-quarks with fractional charge $1/N$:

$$f \propto (1{-}\cos(\theta/N)) \implies b_2 = -0.08333/N^2 \;\; \text{instead we find} \;\; b_2 = -0.20(2)/N^2$$

First topic of this talk

A closer look at the relation between center symmetry and θ-dependence

Is it possible to preserve Z_N center symmetry, even with a small compactification radius (high-T, small coupling), by deforming the pure Yang-Mills action?

M. Unsal and L. Yaffe: PRD 78, (2008) 065035

J.C. Myers and C. Ogilvie: PRD 77, (2008) 125030 (first lattice study)

$$S^{def} = S_{YM} + h \sum_{\vec{n}} |TrP(\vec{n})|^2$$

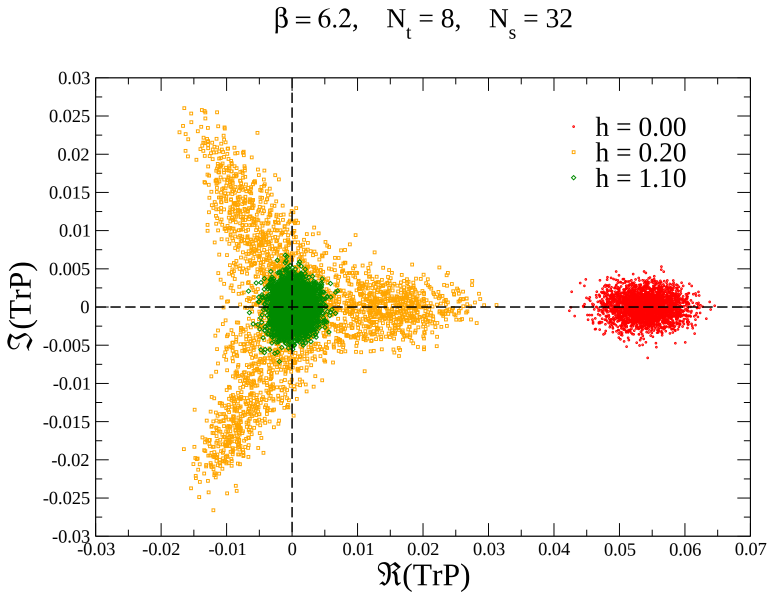
$SU(3)$: **just one deformation, suppresses large values of $|TrP(\vec{n})|$ locally** $\Longrightarrow$
for large enough h, center symmetry is restored even at high-T (small coupling)

QUESTION: what happens to θ dependence?

What is DIGA related to? Small coupling or broken center symmetry?

ANSWER $\Longrightarrow$ C. Bonati, M. Cardinali, MD, Phys. Rev. D 98, 054508 (2018), arXiv:1807.06558

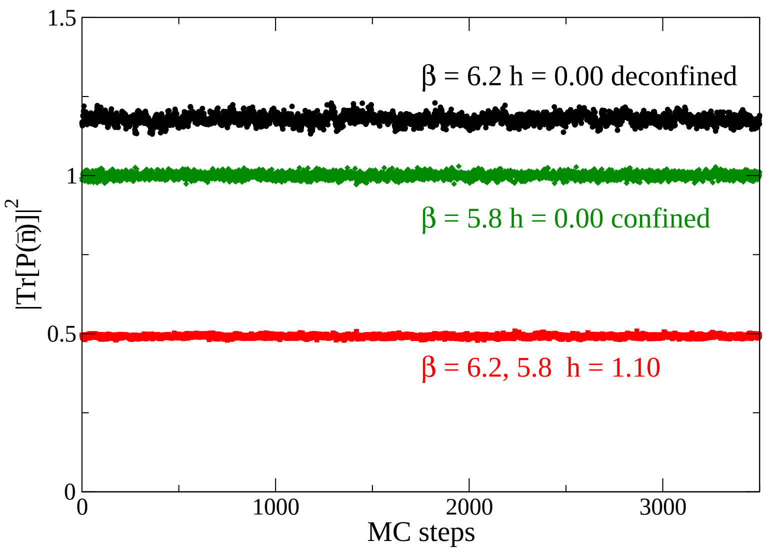
Restoration of Z_3 takes place in a non-trivial way



- $T \simeq 1.4\, T_c$, **broken Z_3 at $h = 0$**

- **Center symmetry recovered by increasing h**

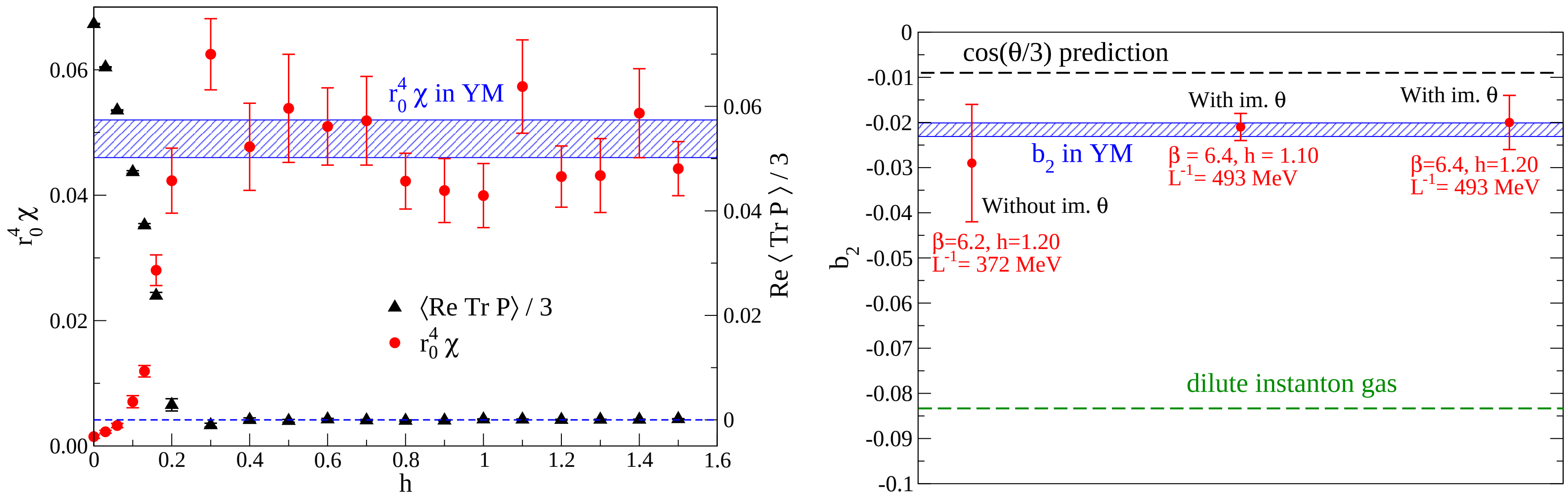
- **Some differences from the standard confined phase emerge looking at the adjoint Polyakov loop**



$$P^{adj} = |TrP|^2 - 1$$

a negative value of P^{adj} means that $|TrP|$ tends to vanish locally (point by point).

For $T < T_c$ it vanishes by long-range disorder



θ-dependence seems to be sensible just to the restoration of center symmetry

(either locally or by long-range disorder)

– **Left:** the topological susceptibility goes back to its $T = 0$ value

– **Right:** the same happens for b_2.

Notice: semiclassical arguments (Unsal, Yaffe, 2008) predict $b_2 = -1/(12N_c^2)$ (Fractional Instanton Gas Approximation). This is still not observed at the explored T

Better insight by going to $N > 3$

C. Bonati, M. Cardinali, MD, F. Mazziotti, in progress

$SU(4)$: center symmetry has two possible breaking patterns

$$Z_4 \to \mathrm{Id} \ ; \quad Z_4 \to Z_2$$

Complete restoration of Z_4 requires the vanishing of two traces: P and P^2

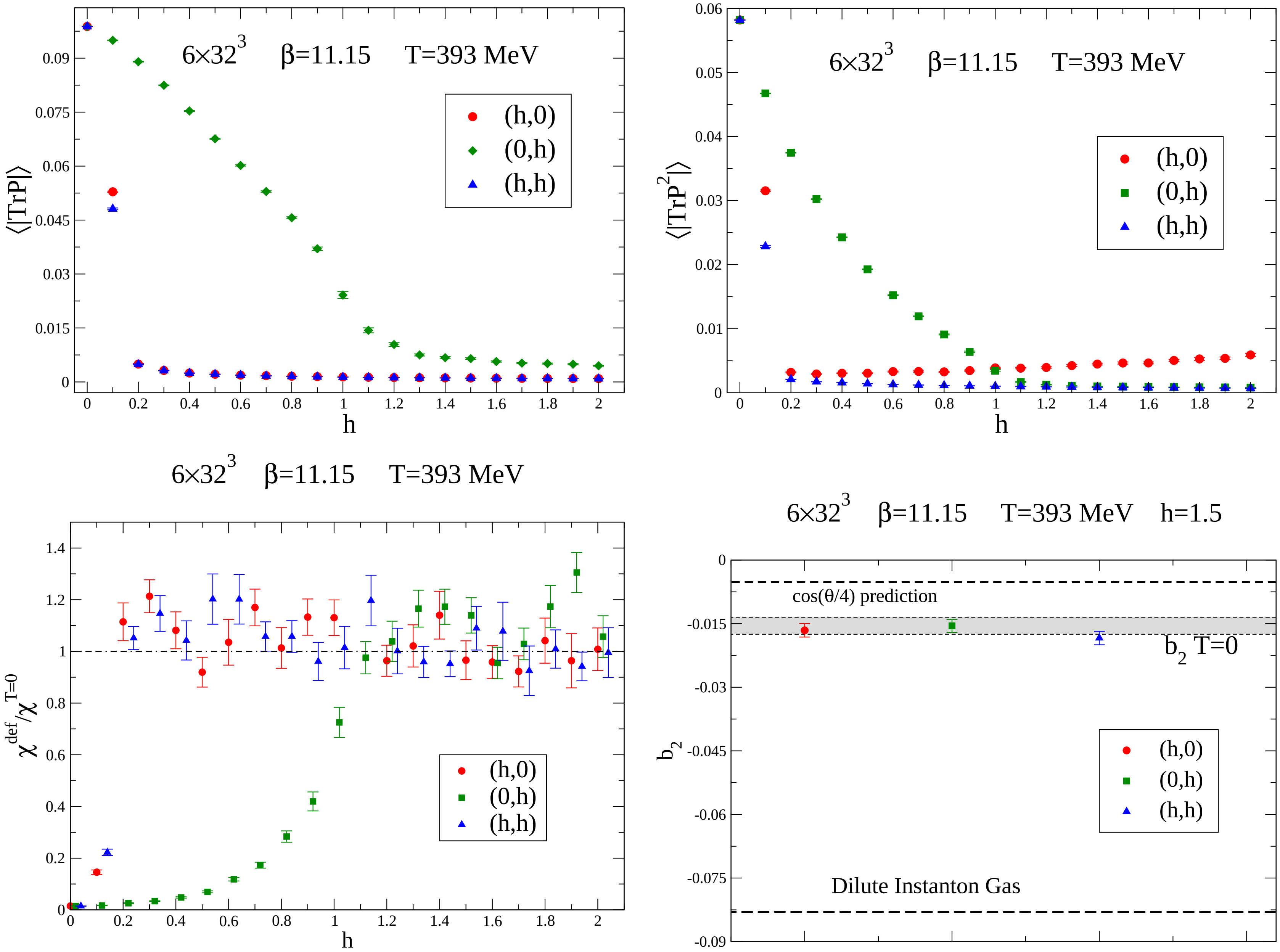
two possible trace deformations to be added to the action

$$S^{def} = S_{YM} + h_1 \sum_{\vec{n}} |Tr P(\vec{n})|^2 + h_2 \sum_{\vec{n}} |Tr P^2(\vec{n})|^2$$

What about θ-dependence?

Is it sensitive to partial or complete restoration?

ANSWER: θ-dependence back to confined values only for complete restoration!



Second topic of this talk

How well are large-N predictions verified?

Numerical tests of large-N predictions in confined Yang-Mills are semi-quantitative, because the predictions themselves are semi-quantitative.

An interesting playground is represented by $2d\ CP^{N-1}$ models, where large-N predictions are also quantitative!

$$S(\theta) = \int \left[\frac{N}{g} \bar{D}_\mu \bar{z}(x) D_\mu z(x) - i\theta q(x) \right] d^2x \ ; \qquad Q = \int q(x) d^2x = \frac{1}{4\pi} \epsilon_{\mu\nu} \int F_{\mu\nu}(x) d^2x$$

$D_\mu = \partial_\mu + iA_\mu$ $\quad A_\mu$ **is an auxiliary $U(1)$ gauge field**

z **is a normalized complex field with N components**

Large-N predictions for $2d\,CP^{N-1}$ models

$$\chi = \bar{\chi} N^{-1} + O(N^{-2}) \text{ and } b_{2n} = \bar{b}_{2n} N^{-2n} + O(N^{-2n-1}).$$

$$\xi^2\chi = \frac{1}{2\pi N} + \frac{e_2}{N^2} + O\left(\frac{1}{N^3}\right) \;, \quad e_2 = -0.0605 \;; \quad \xi = 2^{nd}\text{ moment corr. length}$$

$$b_2 = -\frac{27}{5}\frac{1}{N^2} + O\left(\frac{1}{N^3}\right) \;, \quad b_4 = -\frac{25338}{175}\frac{1}{N^4} + O\left(\frac{1}{N^5}\right) \;,$$

LO χ: Luscher, PLB 78, 465 (1978) D'Adda, Luscher, Di Vecchia, NPB 146, 63 (1978), Witten, NPB 149, 285 (1979)

NLO χ (e_2): M. Campostrini and P. Rossi, PLB 272, 305 (1991).

LO b_2: L. Del Debbio, G. M. Manca, H. Panagopoulos, A. Skouroupathis, E. Vicari, JHEP 0606, 005 (2006)

LO all b_{2n}: P. Rossi, PRD 94, 045013 (2016) C. Bonati, MD, P. Rossi, E. Vicari, PRD 94, 085017 (2016)

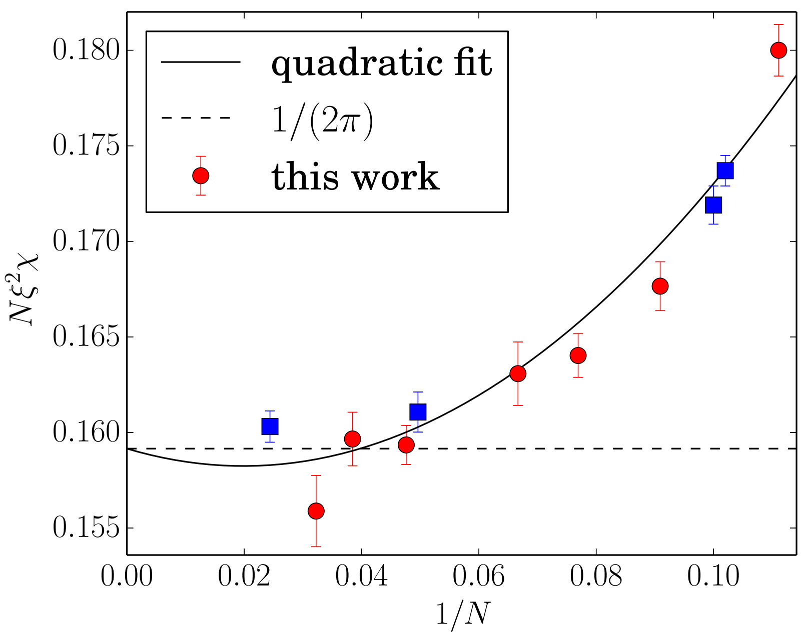
Lattice checks till 2017:

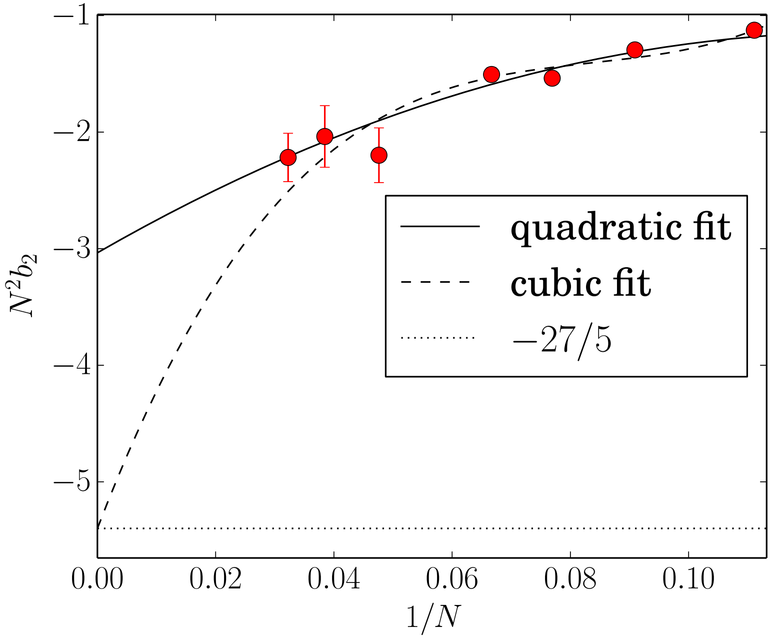
LO χ: OK; NLO χ: disagreement even in sign; LO b_2: never tried

M. Campostrini, P. Rossi and E. Vicari, PRD 46, 2647 (1992) E. Vicari, PLB 309, 139 (1993) L. Del Debbio, G. M. Manca and E. Vicari, PLB 594, 315 (2004) J. Flynn, A. Juttner, A. Lawson and F. Sanfilippo, arXiv:1504.06292 M. Hasenbusch, PRD 96, no. 5, 054504 (2017)

MAIN LIMITATION: critical slowing down of Q for large N

last year update: C. Bonanno, C. Bonati, MD, JHEP 1901, 003 (2019), arXiv:1807.11357



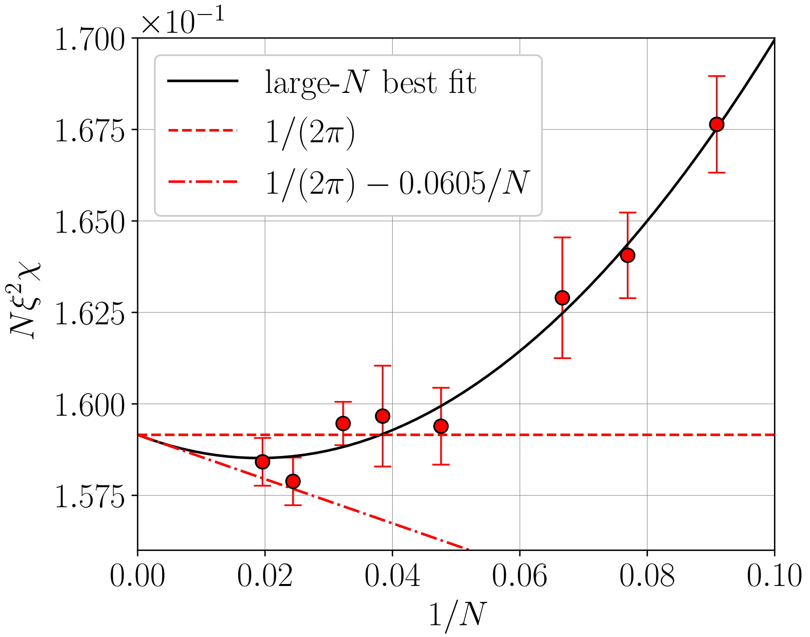


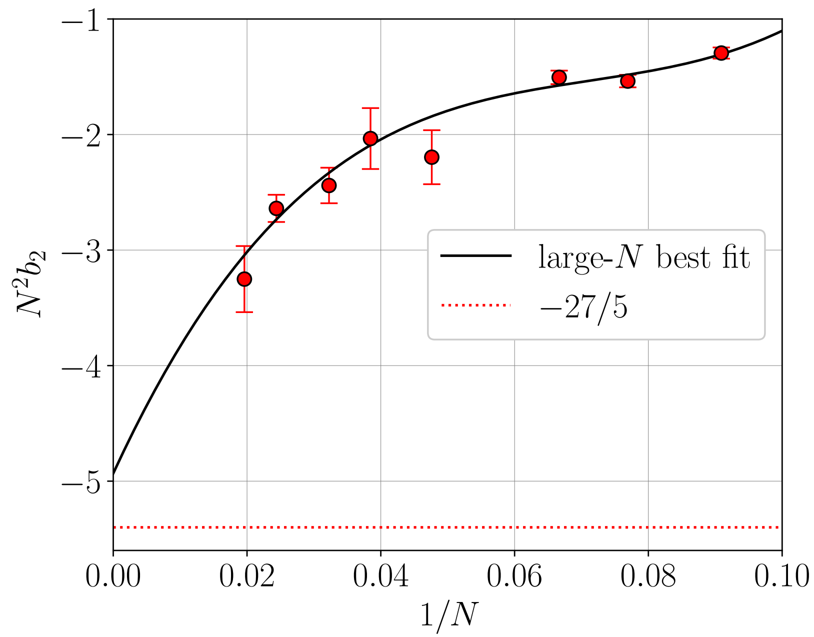
– **Progress thanks to: simulated tempering plus imaginary θ, up to $N = 31$**

– **results for χ (left): deviations from LO consistently positive, but a fit including NNLO corrections is barely consistent with NLO $e_2 = -0.0605$**

– **results for b_2 (left): inconsistency with LO if NLO and NNLO included in the fit, consistency forced including also NNNLO, but looks like wishful thinking ...**

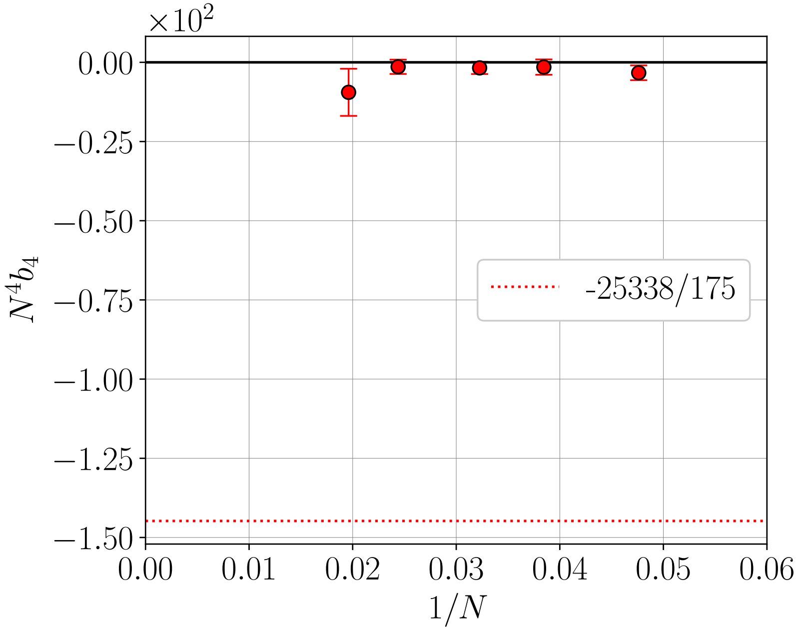
this year update: **M. Berni (Master Thesis), C. Bonanno, MD, preliminary, in progress ...**



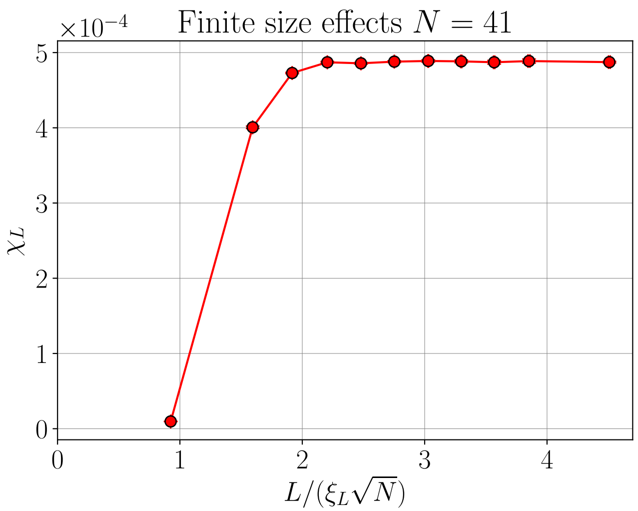


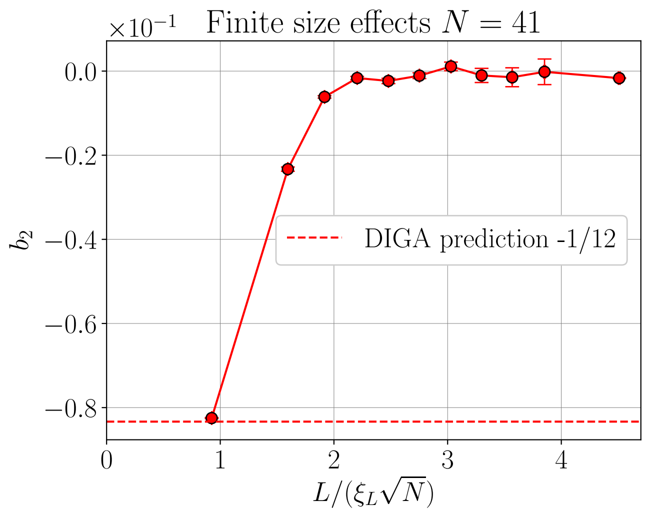
- **Progress thanks to:** **new algorithm proposed by M. Hasenbusch,** **(PRD 96, no. 5, 054504 (2017), arXiv:1706.04443)** **We manage to reach up do $N = 51$.**
- **results for χ (left):** $\xi^2\chi = 1/(2\pi N) + e_2/N^2 + e_3/N^3$
  $e_2 = -0.066(13)\ ;\ \ e_3 = 1.75(20)\ ;\ \ \tilde{\chi}^2 = 0.5$
- **results for b_2 (right):** $b_2 = p_2/N^2 + p_3/N^3 + p_4/N^4 + p_5/N^5$
  $p_2 = -4.9(1.1)\ ;\ \ p_3 = 125(67)\ ;\ \ p_4 = -1600(1000)\ ;\ \ p_5 = -7700(6000)\ ;\ \ \tilde{\chi}^2 = 1.6$
- **Conclusions:** **NLO for χ and LO for b_2 successfully checked; NNLO for χ and NLO for b_2 predicted; slow $1/N$ convergence, due to singularity at $N = 2$??**

What about higher order terms in θ?



- for b_4 only upper bounds, and very far from LO
- confirming LO soon is really wishful thinking
- likely much larger N needed
- difficult because of freezing, but also for finite size effects, need $(L^2/(\xi^2 N) \gg 1$ (M. Aguado and M. Asorey, Nucl.Phys. B844 (2011) 243-265, arXiv:1009.2629)





SUMMARY

– θ-dependence in 4D $SU(N)$ Yang-Mills theories matches large-N predictions in the confined phase and DIGA in the deconfined phase

– analysis of trace deformed $SU(N)$ theories confirms the strict link between θ-dependence and the realization of Z_N center symmetry

– lattice results for 2D CP^{N-1} finally confirm analytical large-N predictions: NLO for χ and LO for b_2. The $1/N$ convergence is however quite slow, much slower than for 4D $SU(N)$