Resurgent analysis and its applications to physics

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Outline

Resurgence: what?

What is it good for

- Resurgence is an old idea
- The first hints of resurgence come from the work of Stokes **[19th century]**
- Recently resurfaced in Écalle's work **[1981]**
- The ideas are now being applied to mathematics and physics

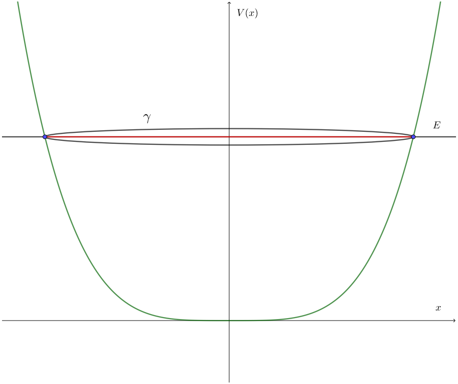
Resurgence and WKB analysis

The WKB method

- In 1D quantum mechanics with potential $V(x)$:

$$\oint_\gamma p(x)\mathrm{d}x = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

- $p(x)$ classical momentum, $\sqrt{2m(E-V(x))}$



Resurgence and WKB analysis

Beyond the first order

- Define the quantum momentum as

$$\psi(x) = \exp\left(\frac{\mathrm{i}}{\hbar}\int_{x_0}^{x} P(t,\hbar)\mathrm{d}t\right)$$

- The momentum solves

$$\mathrm{i}\hbar\partial_x P(x,\hbar) - [P(x,\hbar)]^2 = 2m(E - V(x)) = [p(x)]^2$$

- Series expansion of the momentum in $\hbar$:

$$P(x,\hbar) = \sum_{n=0}^{\infty} \hbar^n P_n(x)$$

- Determines recursion relations for the functions $P_n(x)$

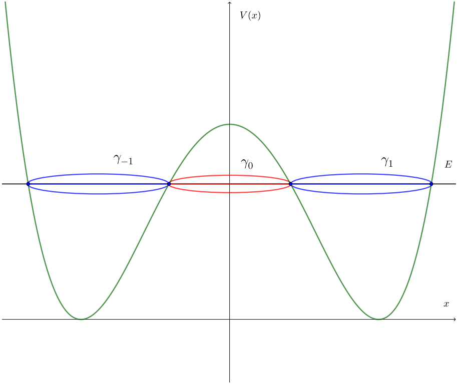
Resurgence and WKB analysis

Integration cycles

- Revelant quantities for non perturbative effects: integrations over cycles

$$S_i(\hbar) = \oint_{\gamma_i} P(t, \hbar)$$

- If γ_i is in a forbidden region, S_i is related to tunneling amplitudes



Resurgence and WKB analysis

Observables and transseries

- Physical quantities are expanded in $\hbar$ in a transseries

$$\mathcal{O}(\hbar) = \sum_{\gamma_i} \exp\left(\frac{\mathrm{i}}{\hbar} \oint_{\gamma_i} p(t)\right) \mathcal{O}_i(\hbar)$$

- Exponential factor: instanton classical action
- $\mathcal{O}_i(\hbar)$ fluctuations around instantons, power series in $\hbar$:

$$\mathcal{O}_i(\hbar) = \hbar^{\beta_i} \sum_{n=0}^{\infty} \hbar^n \mathcal{O}_i^{(n)}$$

- Resurgence: the coefficients $\mathcal{O}_i^{(n)}$ are related!

Resurgence and physics

General considerations

- A transseries collects all possible instanton contributions
- For the partition function, different phases of the system are associated to the different instanton contributions
- Fluctuations around instantons are related
- Conjectured applications:
  - A method to quantize classical orbits with exponential corrections
  - Applications to realistic systems (QFTs, superconductivity, phase transitions...) [Pisani-Smith, 1993; Dunne-Ünsal, 2012; Mariño, 2015; Mariño-Reis 2019]
  - More mathematical: building general solutions to certain differential equations

Asymptotic series

The problem of infinities

- Coupling g: observable $\mathcal{O}(g) \leftrightarrow$ perturbative series

$$\mathcal{O}(g) \sim \sum_{n=1}^{\infty} \mathcal{O}_n g^n$$

- Not an equality: RHS is often asymptotic, zero radius of convergence

$$\mathcal{O}_n \sim A^{-n} n!$$

- Perturbative answer $\rightarrow$ need to resum the series
- Resummations introduce ambiguities
- Ambiguities are solved by non perturbative contributions

Asymptotic series

Taming the infinity

- To resum: remove divergence with Borel transform

$$\mathcal{B}[\mathcal{O}](s) = \sum_{n=1}^{\infty} \frac{\mathcal{O}_n}{n!} s^{n-1}$$

- For many functions, the transform can be analytically continued along almost every ray in the s plane
- Inversion: Borel resummation along a ray

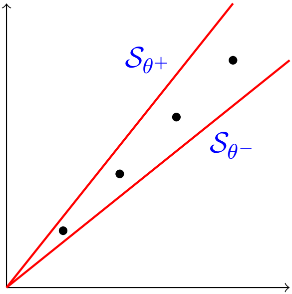
$$\mathcal{S}_\theta[\mathcal{O}](g) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-\frac{s}{g}} \mathcal{B}[\mathcal{O}](s)\mathrm{d}s$$

- Global choice of $\theta \implies$ Global solution in the g plane
- Usual choice: $\theta = \arg g$. Other choices are possible

Asymptotic series

Avoiding singularities

- If there is no singularity of $\mathcal{B}[\mathcal{O}]$ at argument θ, $\mathcal{S}_\theta[\mathcal{O}]$ is well defined and gives unambiguous values
- If there is a singularity, the resummation is undefined
- Deformed summations $\mathcal{S}_{\theta\pm}$: those two do not coincide



The resurgent structure

The transseries

- What is the difference between the summations? The answer will involve non perturbative information
- Perturbative and non perturbative informations are collected in a transseries

$$\Phi(g,\sigma) = \sum_{n=0}^{\infty} \sigma^n \exp\left(-n\frac{A}{g}\right) \Phi_n(g)$$

The resurgent structure

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- σ: transseries parameter

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- A: action (depends on the theory)

The resurgent structure

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$$\Phi(g,\sigma) = \sum_{n=0}^{\infty} \sigma^n \exp\left(-n\frac{A}{g}\right) \Phi_n(g)$$

- $\Phi_n(g)$ usually asymptotic series in g: Borel-Padé resummation is needed!

The resurgent structure

Resurgence!

- What can we extract from the non perturbative ambiguity?
- The structure of the singularities in the Borel plane is dictated by resurgence: as an example

$$\mathcal{B}[\Phi_n](s+kA) = \mathsf{S}_{n\to n+k}\mathcal{B}[\Phi_{n+k}](s)\frac{\log s}{2\pi \mathrm{i}} + \text{hol.}$$

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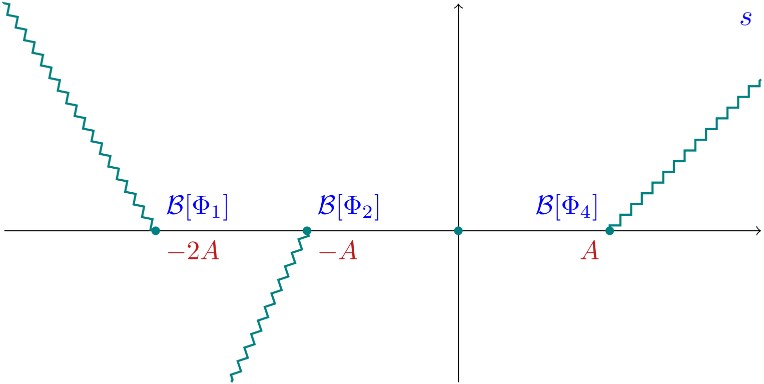
- $\mathsf{S}_{n\to l}$ are complex numbers, Borel residues
- The singularity structure of a sector is determined by the other sectors!
- This also holds for the perturbative series, $n=0$

The resurgent structure

Resurgence!

- Example: singularity structure on the Borel plane for Φ_3



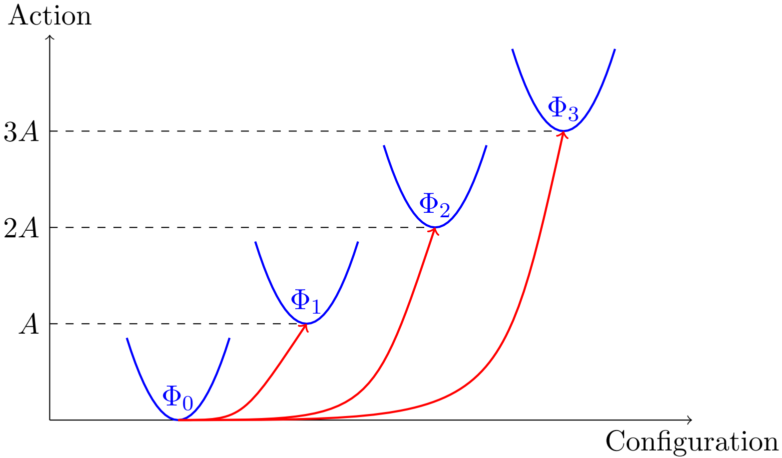
The resurgent structure

Semiclassical decoding

- The transseries provides a way to perform semiclassical decoding
- $\Phi_0 \Longrightarrow$ perturbative sector $\Longrightarrow$ determines the solution near the free point
- $\Phi_n \Longrightarrow$ non perturbative sectors $\Longrightarrow$ invisible to perturbation theory! $\Longrightarrow$ fluctuations around instantons
- Depending on g, perturbative can dominate non perturbative, or vice versa

The resurgent structure

Semiclassical decoding



The resurgent structure

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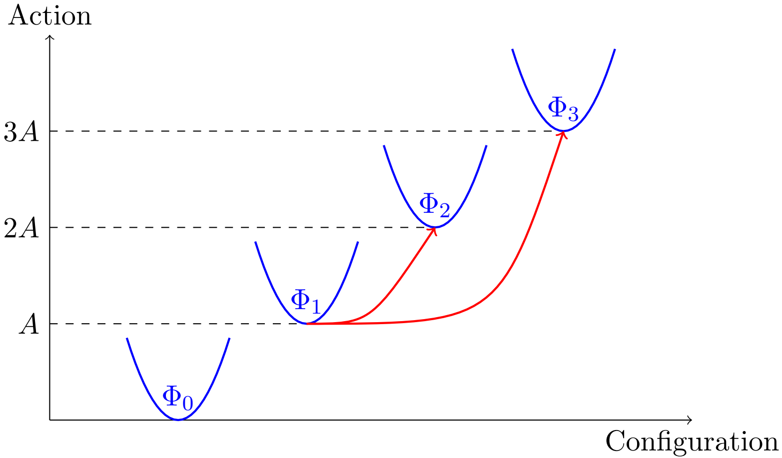

The resurgent structure

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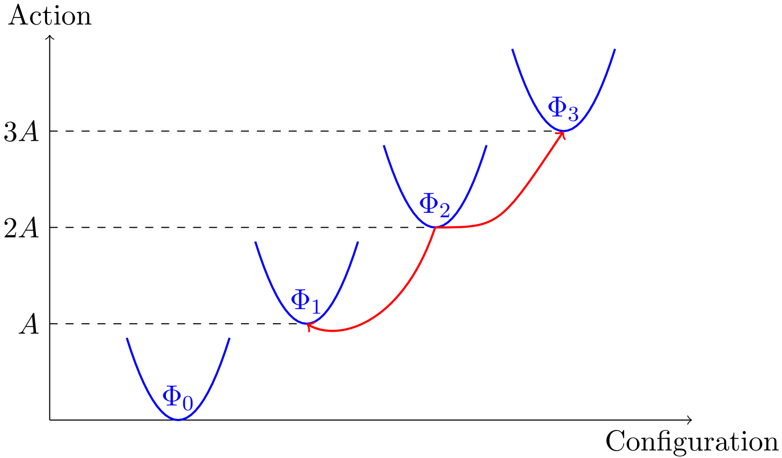
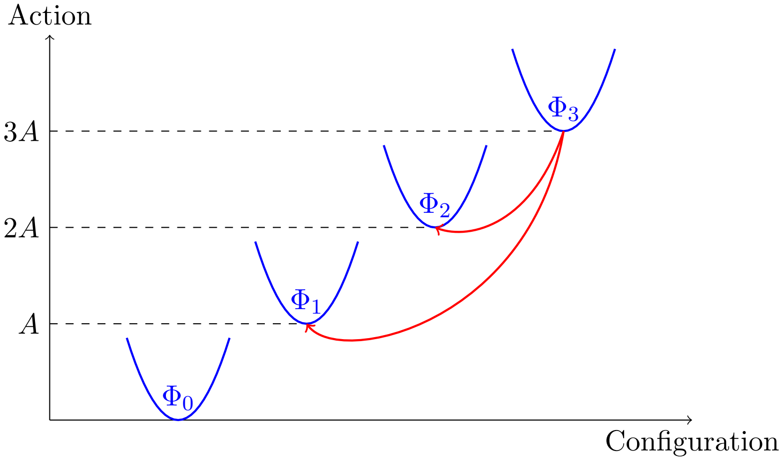


The resurgent structure

Semiclassical decoding



Constructing a global solution

The Stokes automorphism

- At the Stokes lines, the ambiguity can be reabsorbed in a jump of the transseries parameter

$$\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta$$

- $\underline{\mathfrak{S}}_\theta$ is the Stokes automorphism, it allows us to relate the solution on two different sides of a Stokes line
- $\underline{\mathfrak{S}}_\theta$ acts on the transseries parameter:

$$\underline{\mathfrak{S}}_\theta \Phi(x, \sigma) = \Phi(x, f_\theta(\sigma))$$

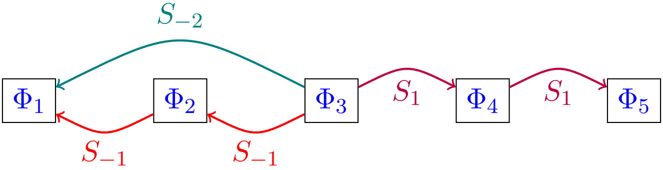
- The Borel residues are determined from the Stokes data, appearing in alien derivatives

$$\Delta_{nA} \Phi_k = S_n (n+k) \Phi_{n+k}$$

Constructing a global solution

The Stokes data

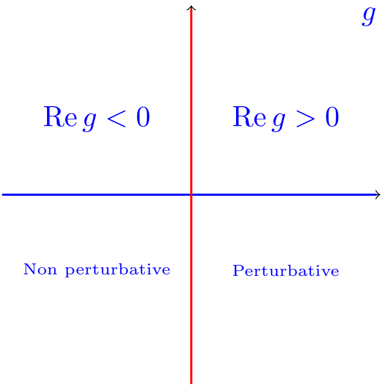
- Stokes data allow us to build the Borel residues
- The action of the Stokes automorphism in terms of the transseries parameters is determined from the Stokes data
- Knowledge of the Stokes data allow us to construct a global solution
- The Stokes automorphism is a precise and analytical way to obtain connection formulae



The final result

What did we get?

- Borel resummation gives solutions in a Stokes wedge
- Through the Stokes automorphism, we extend the solution globally
- Lines in which g is purely imaginary (anti-Stokes lines) separate regions
- Crossing an anti-Stokes line amounts to a phase transition

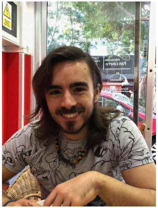


The final result

What's missing?

- Stokes data!
- There is no analytical procedure to determine them
- Numerical methods are available (numerical Borel-Padé approximation, large order analysis, singularities analysis)
- Next goal: find Stokes data analytically for problems of interest
- Ultimate goal: analytical procedure to generically compute Stokes data

The team



Roberto Vega Alvarez

Maximilian Schwick

Ricardo Schiappa

The speaker

The equations of 2D gravity

Painlevé equations

- We are applying resurgence to two dimensional gravity
- By studying matrix models in the appropriate limits, one can obtain equations for the partition function of two dimensional (super)gravity
- The relevant equations are Painlevé I and Painlevé II (a modern review about those topics is [Mariño, 2004])

$$\mathrm{PI} : u(z)^2 - \frac{1}{6}u''(z) = z$$
$$\mathrm{PII} : 2u''(z) - u(z)^3 + zu(z) = 0$$

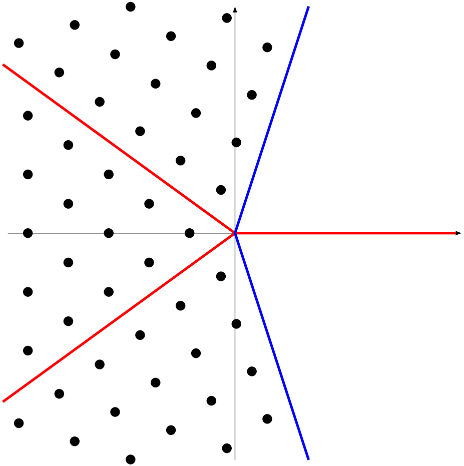
The equations of 2D gravity

Tronquée solution

- There is a one parameter family of solutions to Painlevé I and II: the tronquée solutions
- They can be represented as a standard transseries

$$\sum_{n=0}^{\infty} \sigma^n \mathrm{e}^{-n\frac{A}{x}} \sum_{g=0}^{\infty} u_{2g}^{(n)} x^{g+\frac{n}{2}}$$

- Stokes transitions generate movable poles



The equations of 2D gravity

Transseries ansatz

- Second order differential equations: two integration constants are needed
- Generalized multi-instanton ansatz in the string coupling $g_{\mathrm{s}} = x = z^a$, where the exponent a is $\frac{5}{4}$ for PI, $-\frac{3}{2}$ for PII

$$\Phi(x, \sigma_1, \sigma_2) = \sum_{n,m=0}^{\infty} \sigma_1^n \sigma_2^m \exp\left(-\frac{nA_+ + mA_-}{x}\right) \Phi_{(n,m)}(x)$$

$$\Phi_{(n,m)}(x) = \sum_{k=0}^{k_{(n,m)}} \frac{(\log x)^k}{2^k} \sum_{g=0}^{\infty} u_{2g}^{(n,m)[k]} x^{g+\beta_{(n,m)}^{(k)}}$$

- Actions $A_\pm = \pm\frac{8\sqrt{3}}{5}$ for PI, $A_\pm = \pm\frac{4}{3}$ for PII: resonance
- Coefficients $u_{2g}^{(n,m)[k]}$ can be computed recursively

The equations of 2D gravity

Stokes data

- Stokes data is organized in (Stokes) vectors
- Stokes vectors appear in the alien derivatives

$$\Delta_{sA}\Phi_{(n,m)} = \sum_{p=s-1}^{\min(n+s,m)} \boldsymbol{S}_{(s-p,-p)} \cdot (n+s-p, m-p)\Phi_{(n+s-p,m-p)}$$

$$\Delta_{-sA}\Phi_{(n,m)} = \sum_{p=s-1}^{\min(n,m+s)} \boldsymbol{S}_{(-p,s-p)} \cdot (n-p, m+s-p)\Phi_{(n-p,m+s-p)}$$

- What we have done: check resurgent structure and compute numerically the Stokes vectors for $s = \pm 1, 2$ and various values of p
- What we want: analytical data

Deciphering the Stokes data

Backward-forward relation

- We have discovered an analytical relation between the forward Stokes data $\{\boldsymbol{S}_{(s-p,-p)}\}$ and the backward Stokes data $\{\boldsymbol{S}_{(-p,s-p)}\}$

$$\mathsf{S}_{(n,n)\to(p,p+s)} = \mathrm{i}^s (-1)^{n+p+\frac{s}{2}} \sum_{q=0}^{n-p} \left(-\mathrm{i}\frac{\alpha\pi}{2} s\right)^q \frac{1}{q!} \mathsf{S}_{(n,n)\to(p+q+s,p+q)}$$

- $\alpha = \frac{4}{\sqrt{3}}$ for PI, $\alpha = 8$ for PII
- We can use those relations to determine backward data from forward data: focus on forward data

Deciphering the Stokes data

The vector structure

- For the vectors $\boldsymbol{S}_{(1-p,-p)}$ and $\boldsymbol{S}_{(2-p,-p)}$, we observed the following vector structure

$$\boldsymbol{S}_{(s-p,-p)} = N^{(s)}_{s-p} \begin{pmatrix} p+1 \\ s-p-1 \end{pmatrix}$$

- This structure is hinted by the algebra closure relation valid for non resonant transseries

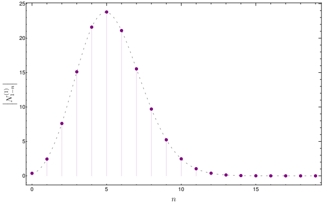
$$(\boldsymbol{S_n} \cdot \boldsymbol{m})\boldsymbol{S_m} - (\boldsymbol{S_m} \cdot \boldsymbol{n})\boldsymbol{S_n} \propto \boldsymbol{S_{n+m}}$$

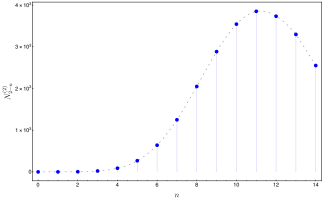
- Open question: in the context of resonant transseries, does this relation imply algebra closure? If not, from where does this structure come from?

Deciphering the Stokes data

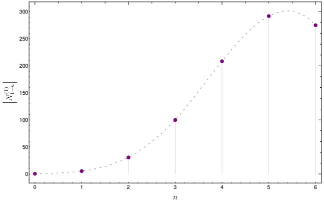
The data

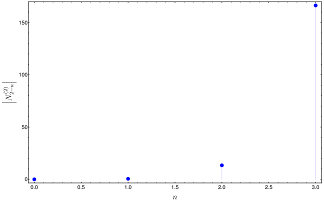
- Painlevé I: a large set of data is available





- Painlevé II: same structure, computationally harder





Deciphering the Stokes data

The future

- Main objective: use the numerical insight to get analytical Stokes data
- Brute-force approach: guess a closed form of the computed numbers
- Other approaches are being explored (asymptotics, transasymptotics...)
- If we get Stokes data: a significant step forward in mathematics (towards full solutions of PI and PII) and physics (towards full partition function of 2D (super)gravity)

Conclusions

Resurgence and physics

- Resurgence extends asymptotic series in full transseries that consider instanton contributions
- The main statement: non perturbative contributions can be (in principle) deciphered from perturbative contributions
- Resurgence is a very natural setting to describe phase diagrams and understand phase transitions
- Resurgence is being applied to 2D gravity with good numerical results
- Resurgence is a very promising tool!

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