The structure and dynamics of multiplex networks

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The Alan Turing Institute

COMPLEXITY SCIENCE HUB VIENNA

``Rich" networks → generalised graphs

Node and links are in space and in time (spatial networks
temporal networks)

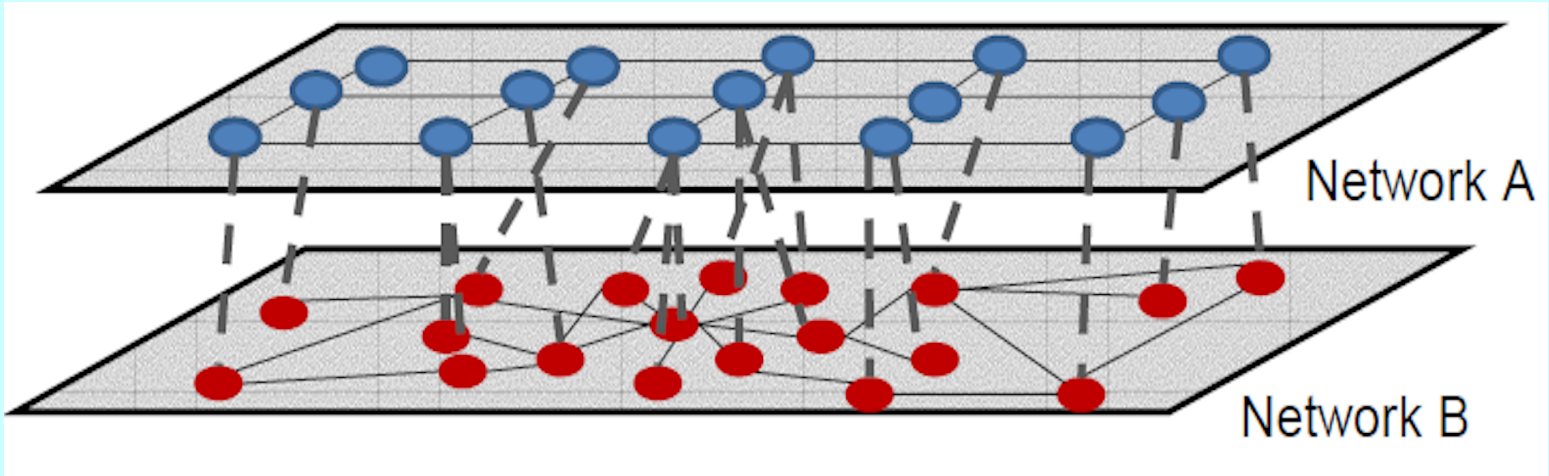
Node and links can be of different types (multiplex,
interconnected networks

The basic units are not only nodes and links (hypergraphs,
simplicial complexes)

Simplicial models of social contagion,
Iacopini, Petri, Barrat, Latora
Nature Communications 10, 2485 (2019)

Interconnected networks

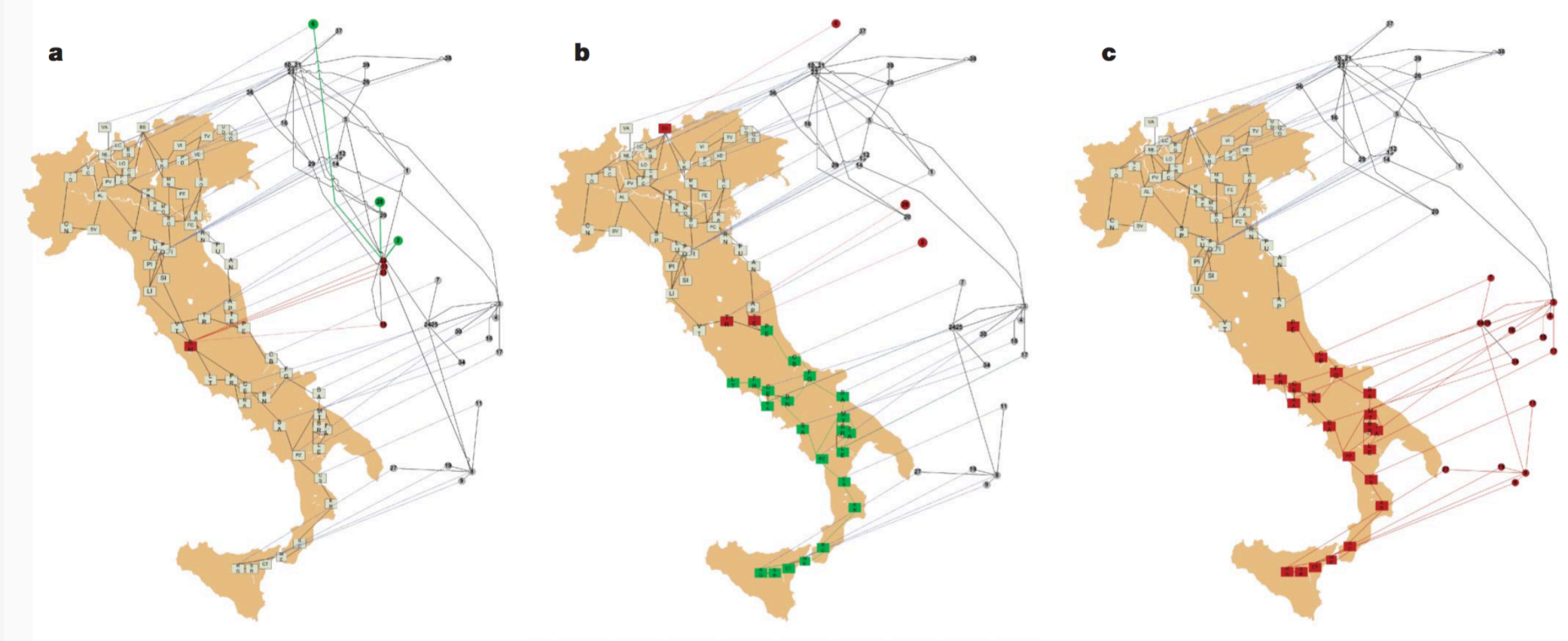
Electric power grid



Communication systems, e.g Internet

Buldyrev et al, Nature 2010

Interconnected networks



Buldyrev et al, Nature 2010

Interconnected networks

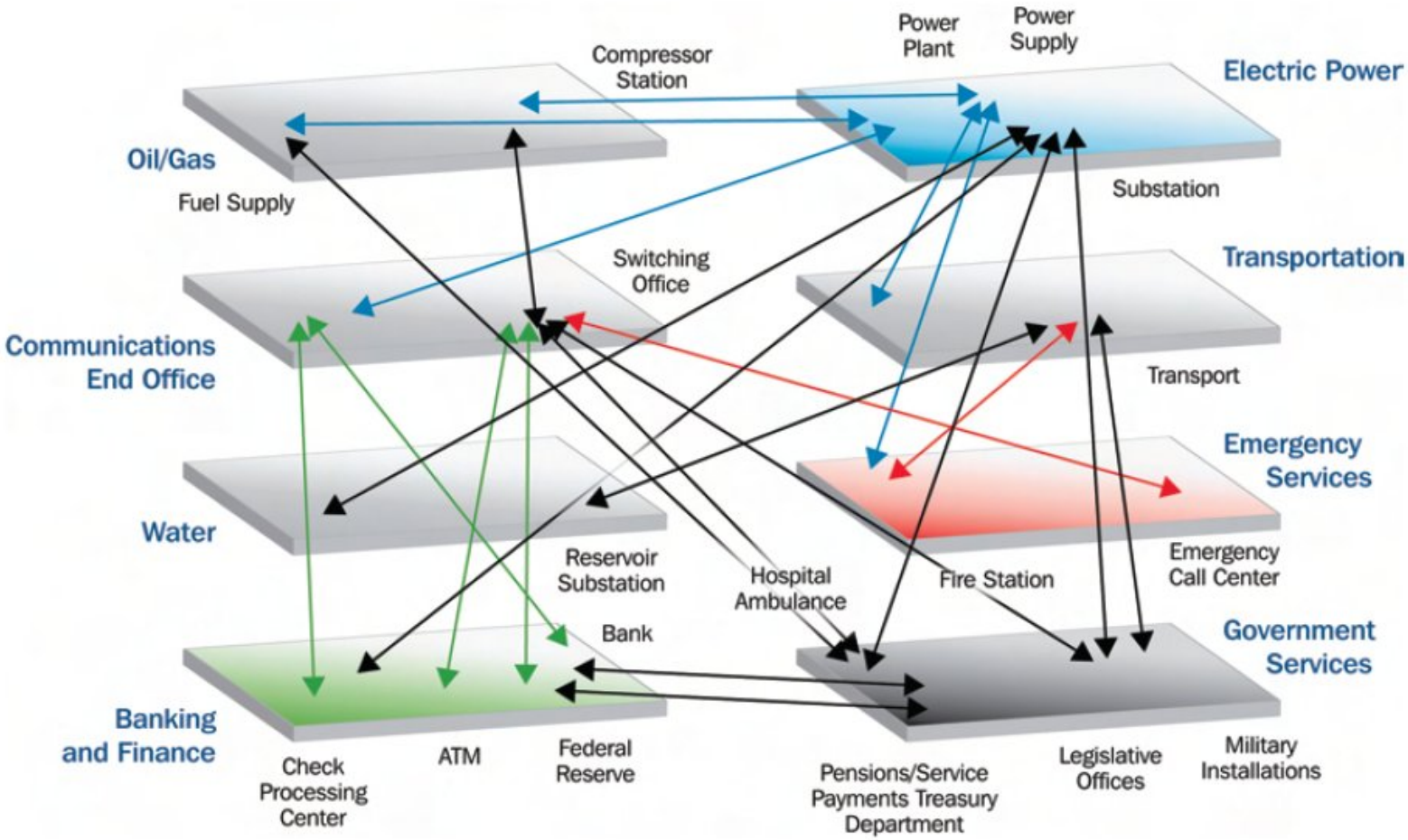
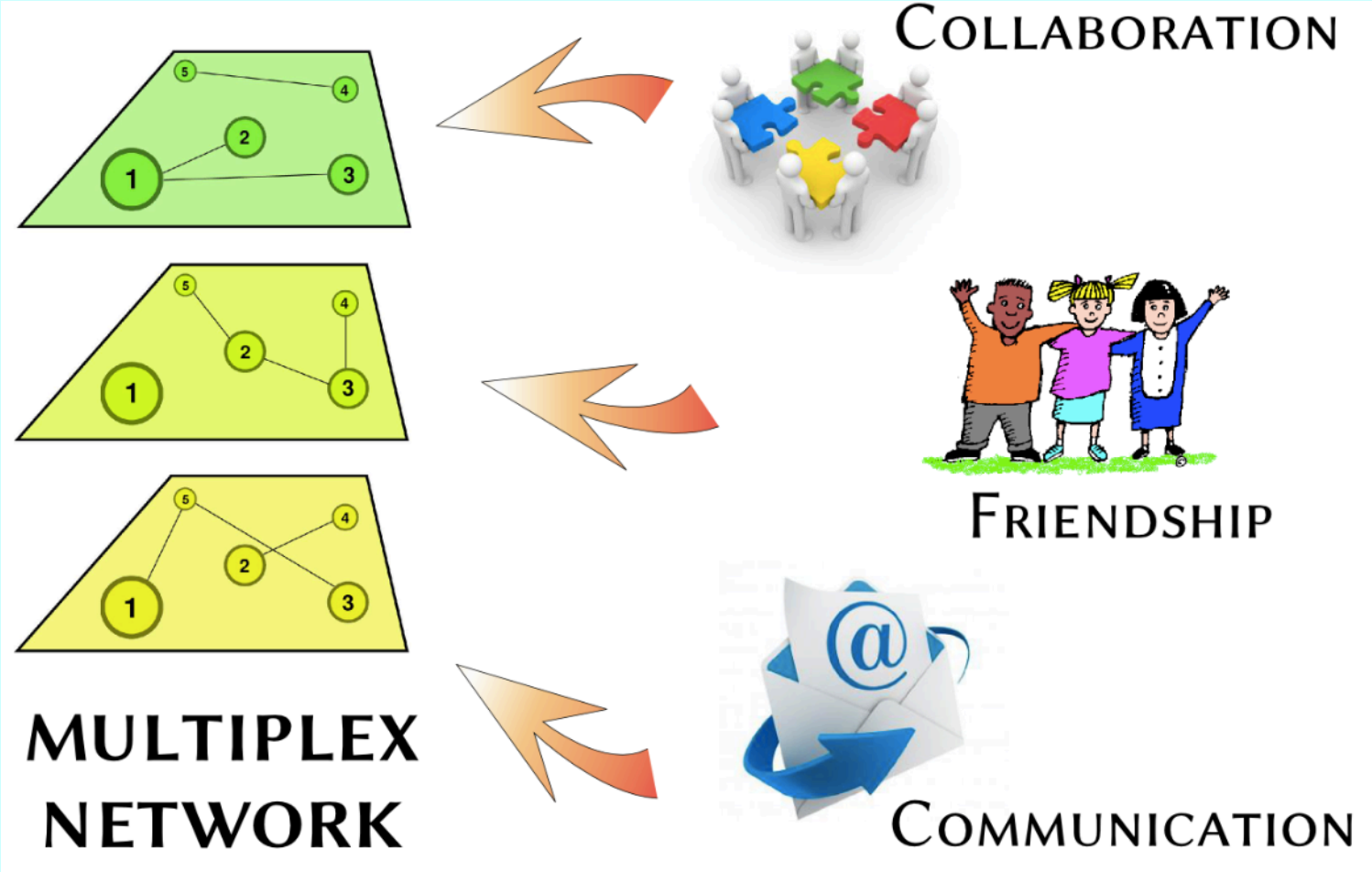


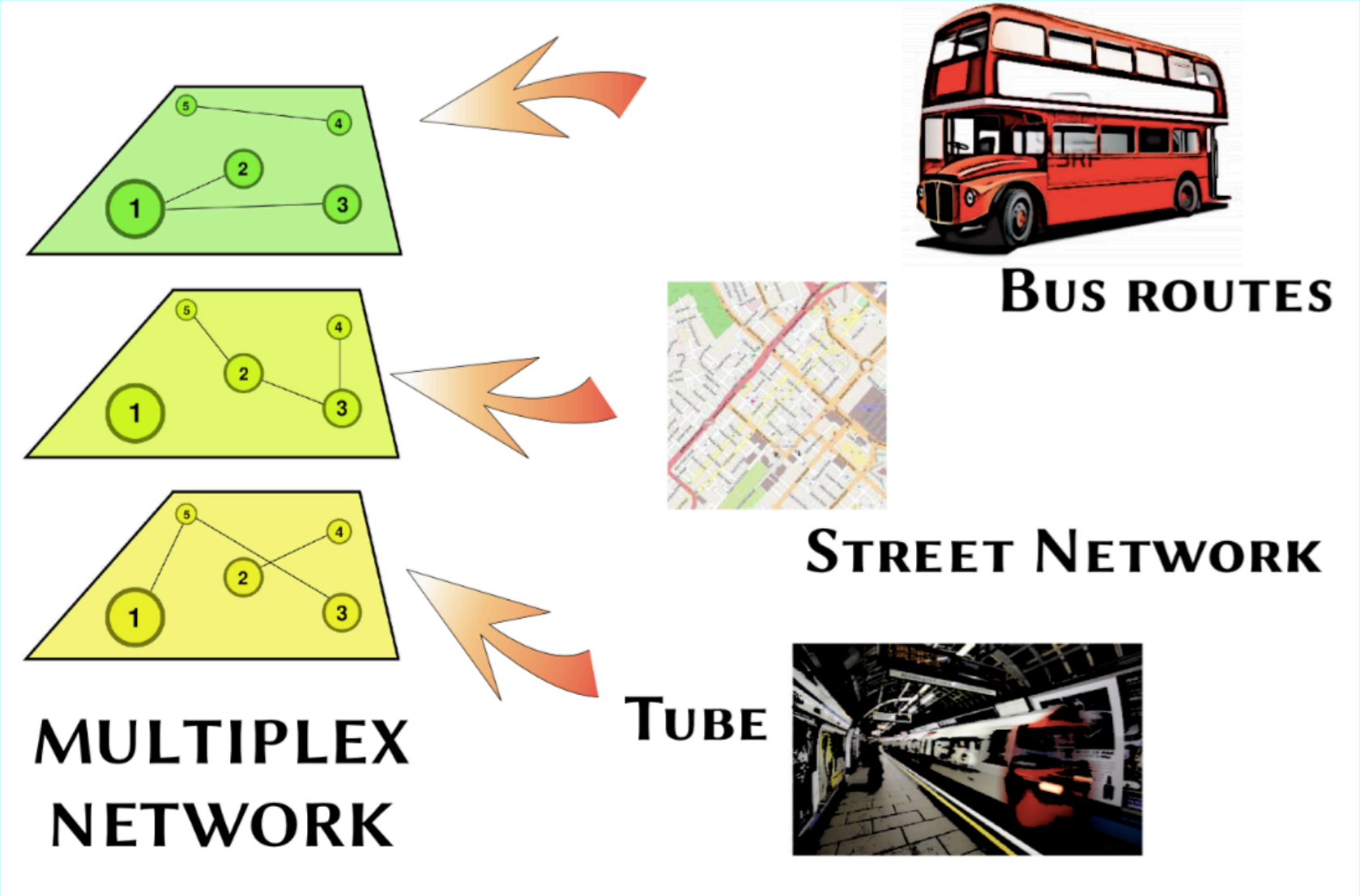
FIGURE 3.1 Connections and interdependencies across the economy. Schematic showing the interconnected infrastructures and their qualitative dependencies and interdependencies. SOURCE: Department of Homeland Security, National Infrastructure Protection Plan, available at http://www.dhs.gov/xprevprot/programs/editorial_0827.shtm.

Multiplex networks



M=3 layers

Multiplex networks



M=3 layers

More is different !!!

4 August 1972, Volume 177, Number 4047 SCIENCE

More Is Different

Broken symmetry and the nature of the hierarchical structure of science.

P. W. Anderson

The reductionist hypothesis may still be a topic for controversy among philosophers, but among the great majority of active scientists I think it is accepted

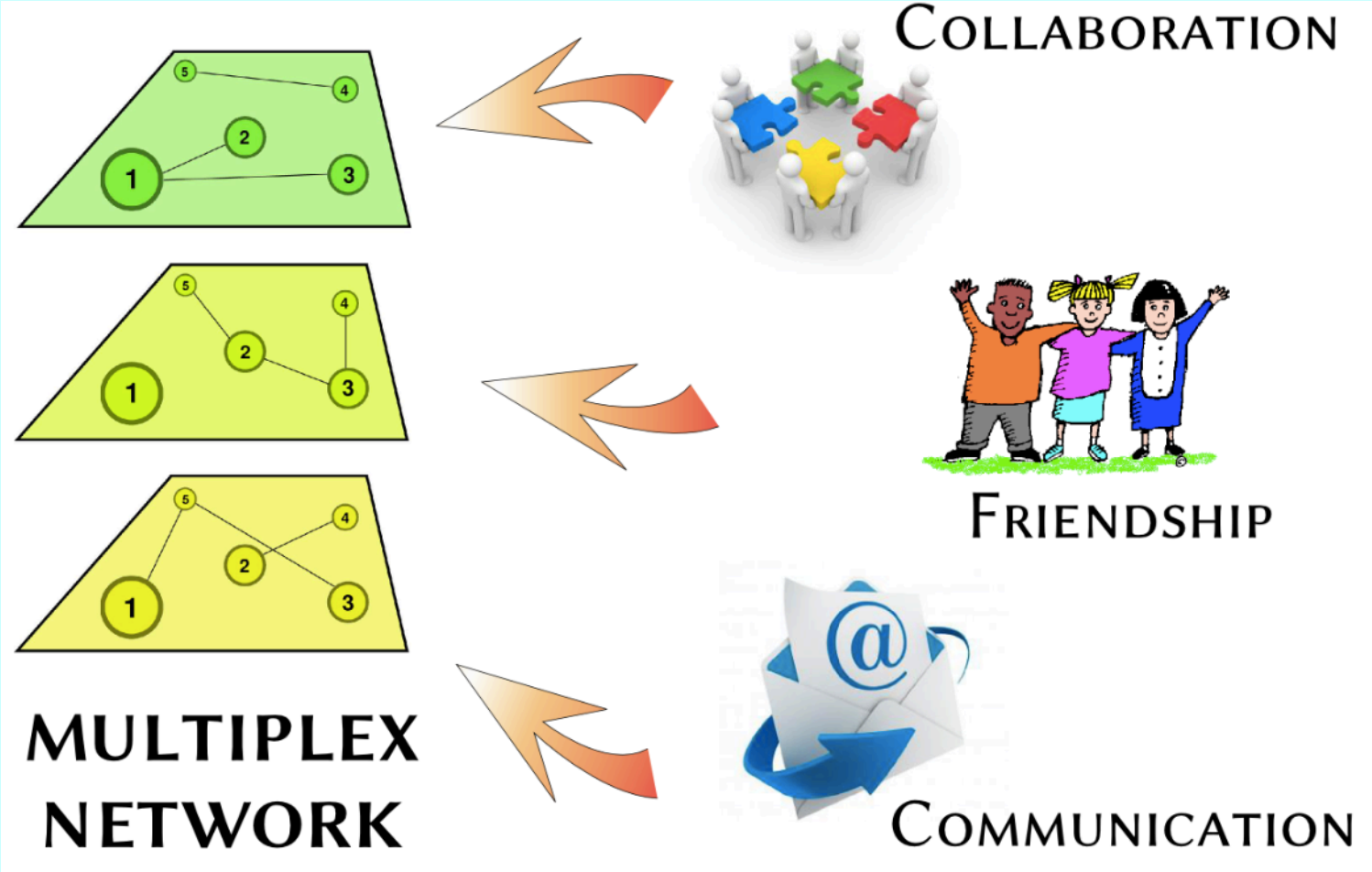
planation of phenomena in terms of known fundamental laws. As always, distinctions of this kind are not unambiguous, but they are clear in most cases. Solid state physics, plasma physics, and perhaps

less relevance they seem to have to the very real problems of the rest of science, much less to those of society.

The constructionist hypothesis breaks down when confronted with the twin difficulties of scale and complexity. The behavior of large and complex aggregates of elementary particles, it turns out, is not to be understood in terms of a simple extrapolation of the properties of a few particles. Instead, at each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other. That is, it seems to me that one may array the sciences roughly linearly in a hierarchy, according to the idea: The elementary entities of science X obey the laws of science Y.

COMPLEX SYSTEMS:
Many units in interaction give rise to emergent behaviors !!

Is multiplex different ??



Q: Is multiplex more than just a sum of many layers?

Q: Emergent ``multiplex behaviors'' ?

Structure and Dynamics of Multilayers
Outline:

How to represent a network with many layers

Extension of standard measures to multiplex networks

Intrinsic multiplex measures

Modeling the growth of a multiplex

Reducibility of a multiplex

Dynamics: Examples of emergent behaviors induced by multiplexity

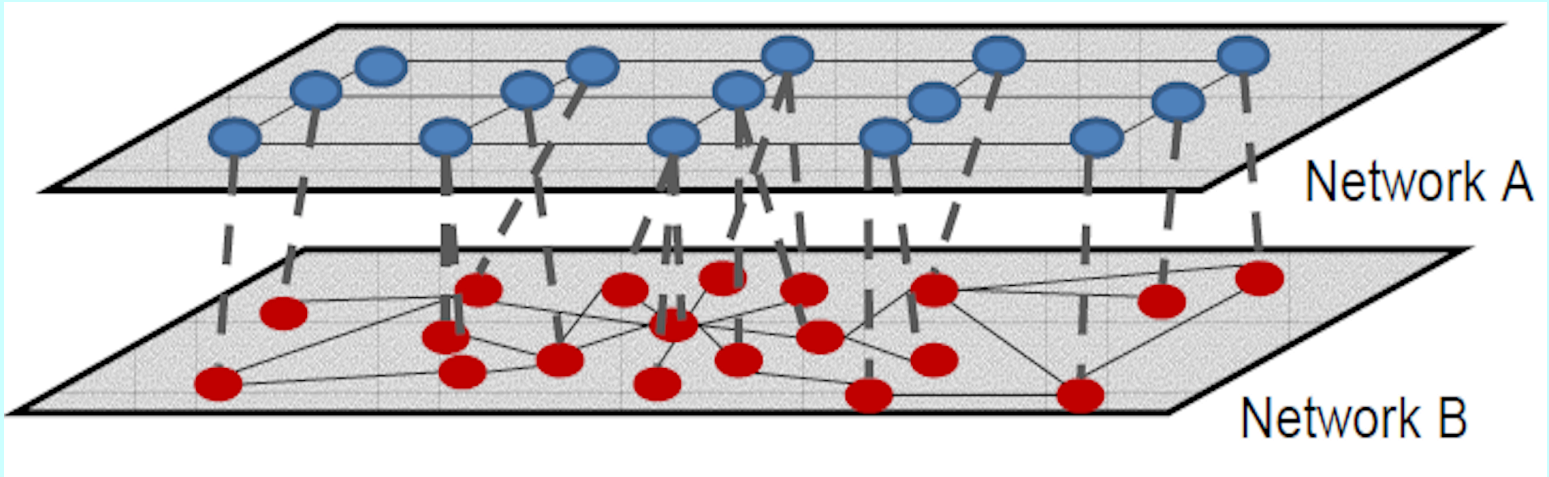
Interconnected networks

Electric power grid

Nodes

N_A

N_B



Internet

Adjacency matrix $A \in \Re^{N_A \cdot N_A}$ $A = \{a_{ij}\}$ Intra-layer

Adjacency matrix $B \in \Re^{N_B \cdot N_B}$ $B = \{b_{ij}\}$ Intra-layer

$M \in \Re^{N_A \cdot N_B}$ $M = \{m_{ij}\}$ Inter-layer

L>2 interconnected networks

Nodes $N_1, N_2, \ldots, N_\alpha, \ldots, N_L$

Intra-layer adjacency matrices $A^\alpha \qquad \alpha = 1, \ldots, L$

Inter-layer adjacency matrices $M^{\alpha\beta} \qquad \alpha, \beta = 1, \ldots, L \quad \alpha \neq \beta$

Supra-adjacency matrix

$$\mathcal{A} = \begin{pmatrix} A^1 & M^{12} & \ldots & M^{1L} \\ M^{21} & A^2 & \ldots & M^{2L} \\ \ldots & \ldots & \ldots & \ldots \\ M^{L1} & M^{L2} & \ldots & A^L \end{pmatrix}$$

$$\mathcal{A} \in \Re^{(N_1+N_2+\ldots+N_L)\cdot(N_1+N_2+\ldots+N_L)}$$

Rank-4 adjacency tensor $T \in \Re^{N \cdot N \cdot L \cdot L} \qquad T = \{t^{i\alpha}_{j\beta}\}$

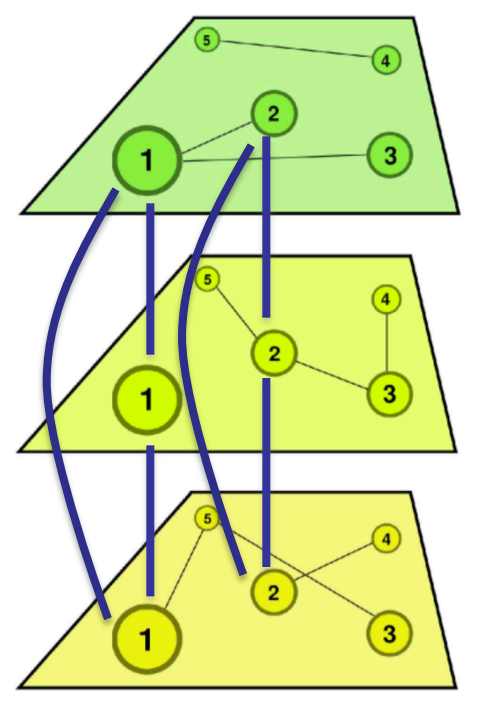
De Domenico at al. PRX 3, 041022 (2013)

Multiplex networks

Layer 1

Layer 2

Layer 3



One-to-one correspondence of nodes

$$\mathcal{A} = \begin{pmatrix} A^1 & I & I & \dots & I \\ I & A^2 & I & \dots & I \\ I & I & A^3 & \dots & I \\ \dots & \dots & \dots & \dots & \dots \\ I & I & I & \dots & A^L \end{pmatrix}$$

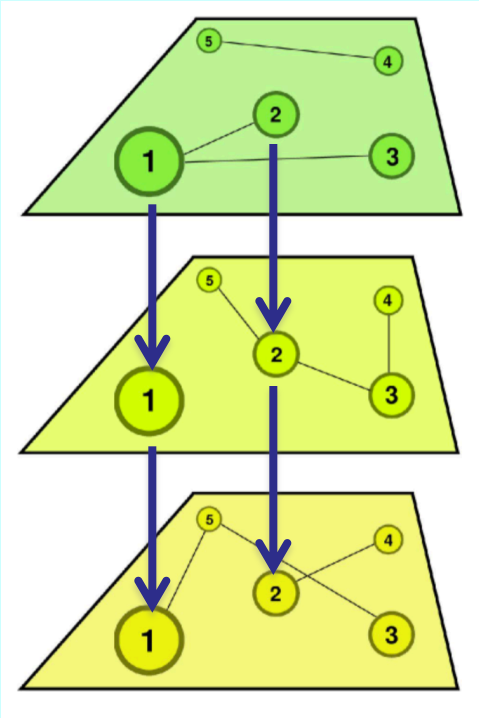
$$\mathbf{A} = \{A^1, A^2, \dots, A^L\}$$

Temporal networks

Time 1

Time 2

Time 3



Temporal causality constraints

$$\mathcal{A} = \begin{pmatrix} A^1 & I & 0 & \dots & 0 \\ 0 & A^2 & I & \dots & 0 \\ 0 & 0 & A^3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A^L \end{pmatrix}$$

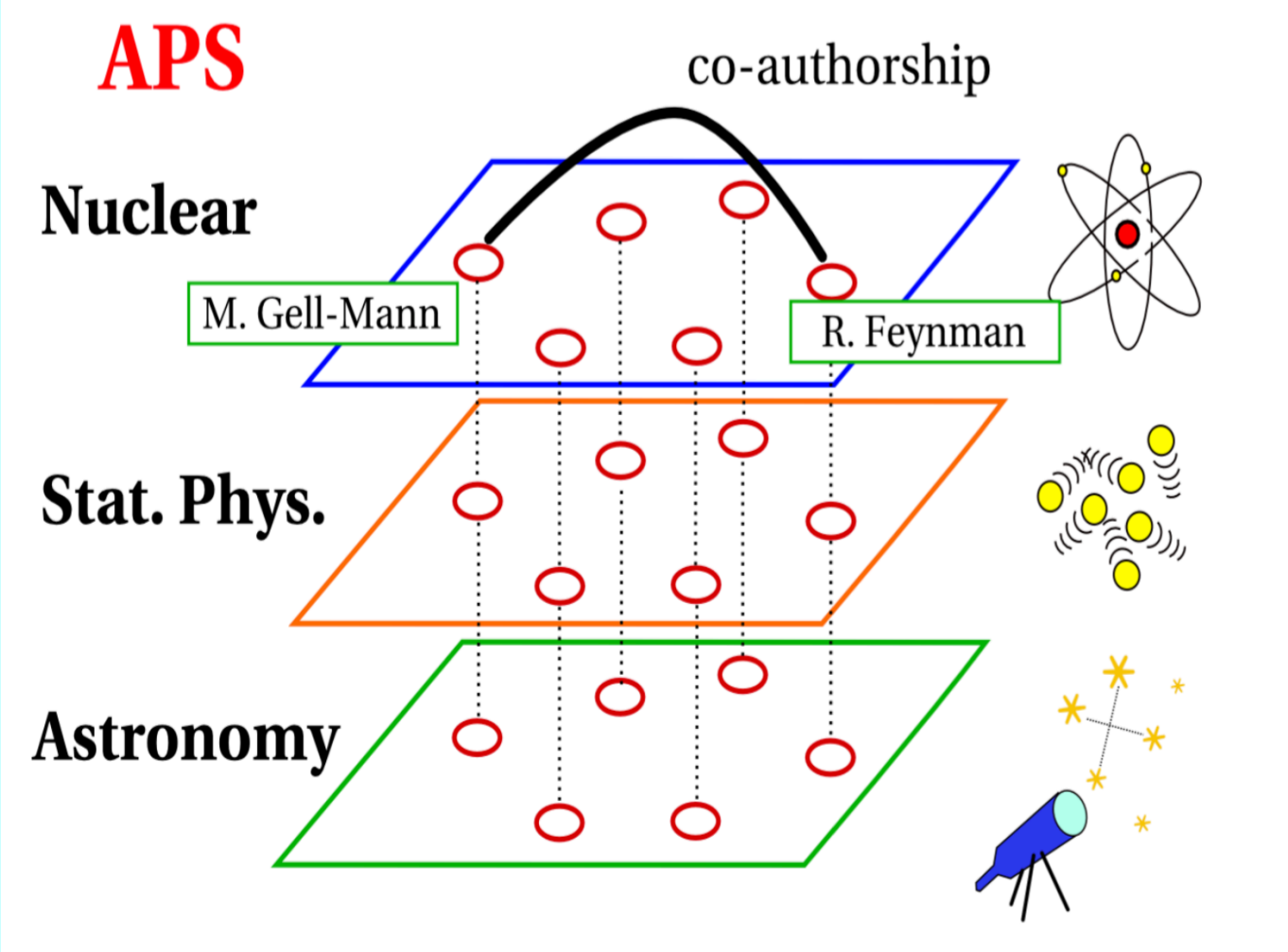
$$\mathbf{A}_{\mathbf{ord}} = \{A^1, A^2, \dots, A^L\}$$

J. Tang, et al, Time-varying graphs, PRE 81, 055101(R) (2010)

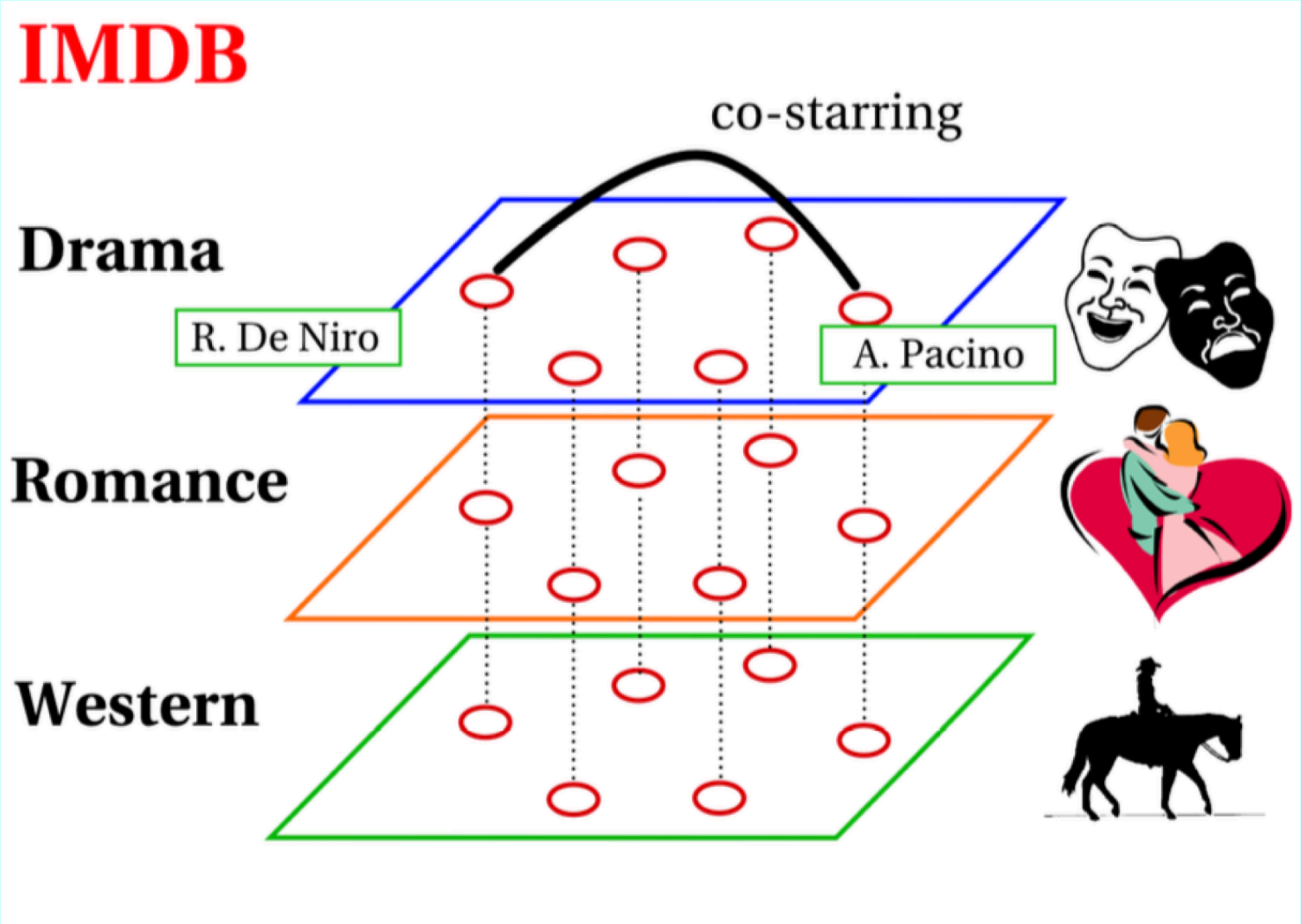
P. Holme, J. Saramaki, Temporal networks, Phys. Rep. 519, 97 (2012)
and Springer book 2013

Empirical analysis and modelling of MULTIPLEX NETWORKS

Multiplex networks data sets

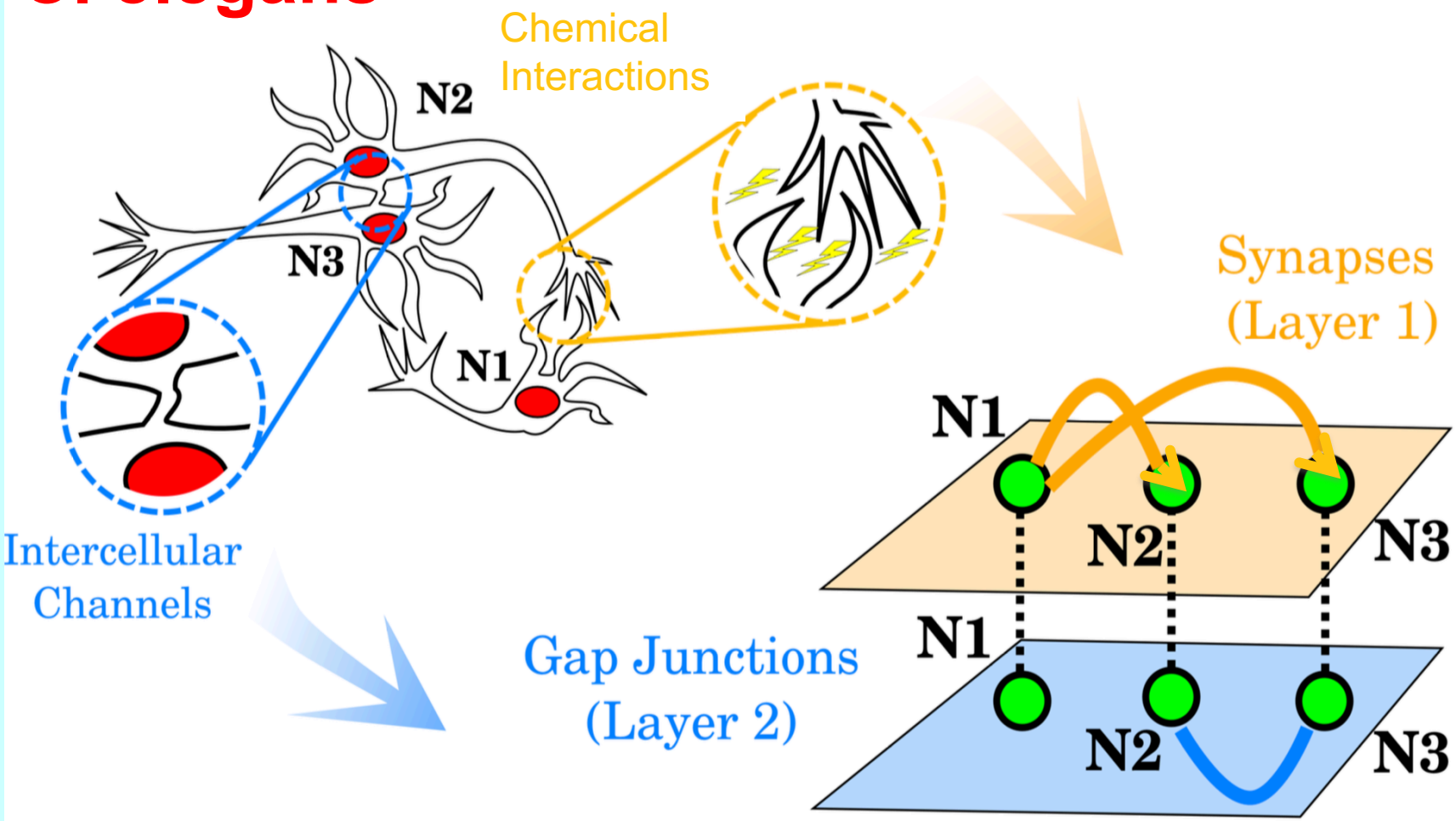


Multiplex networks data sets



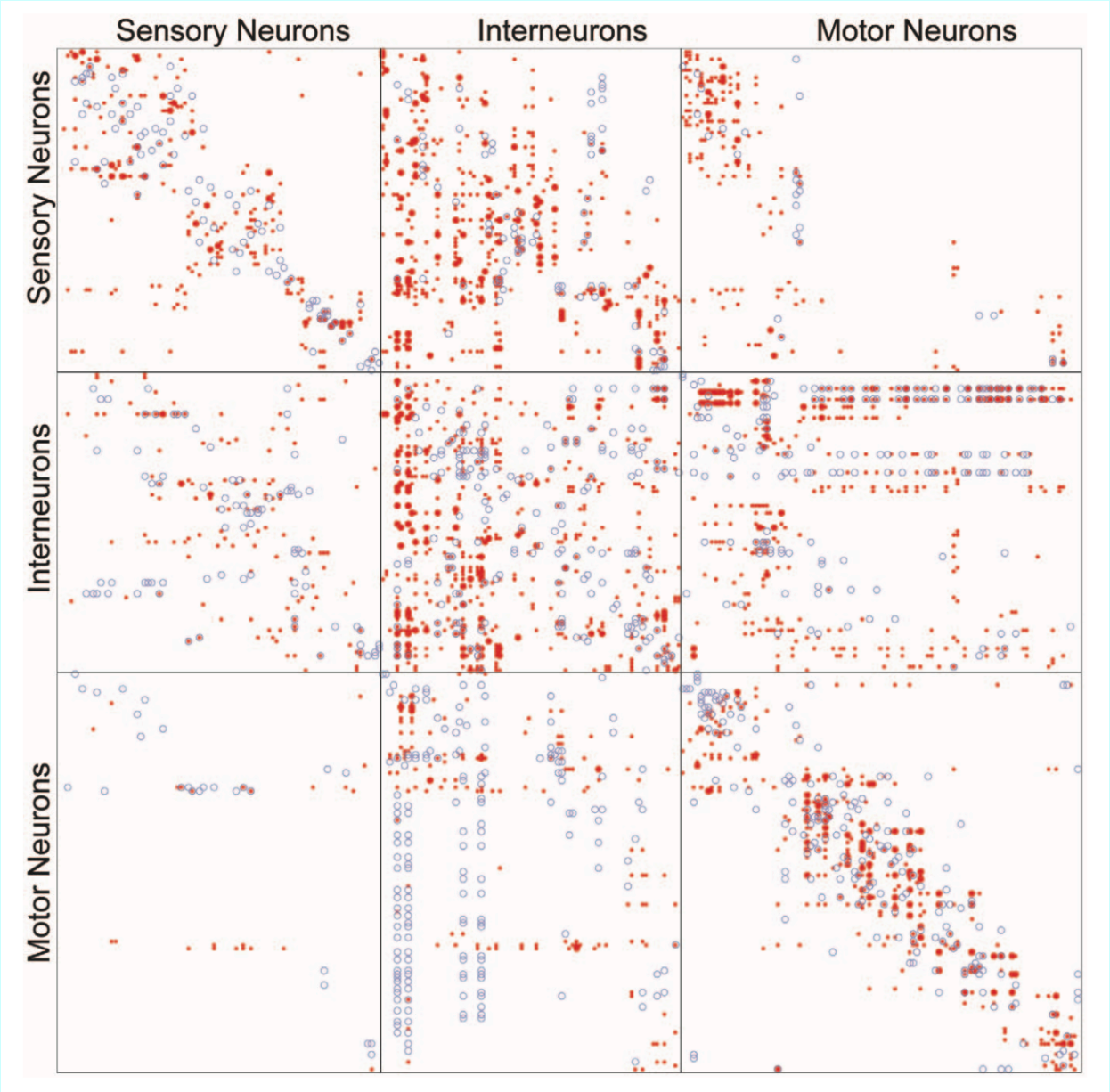
Multiplex networks

C. elegans



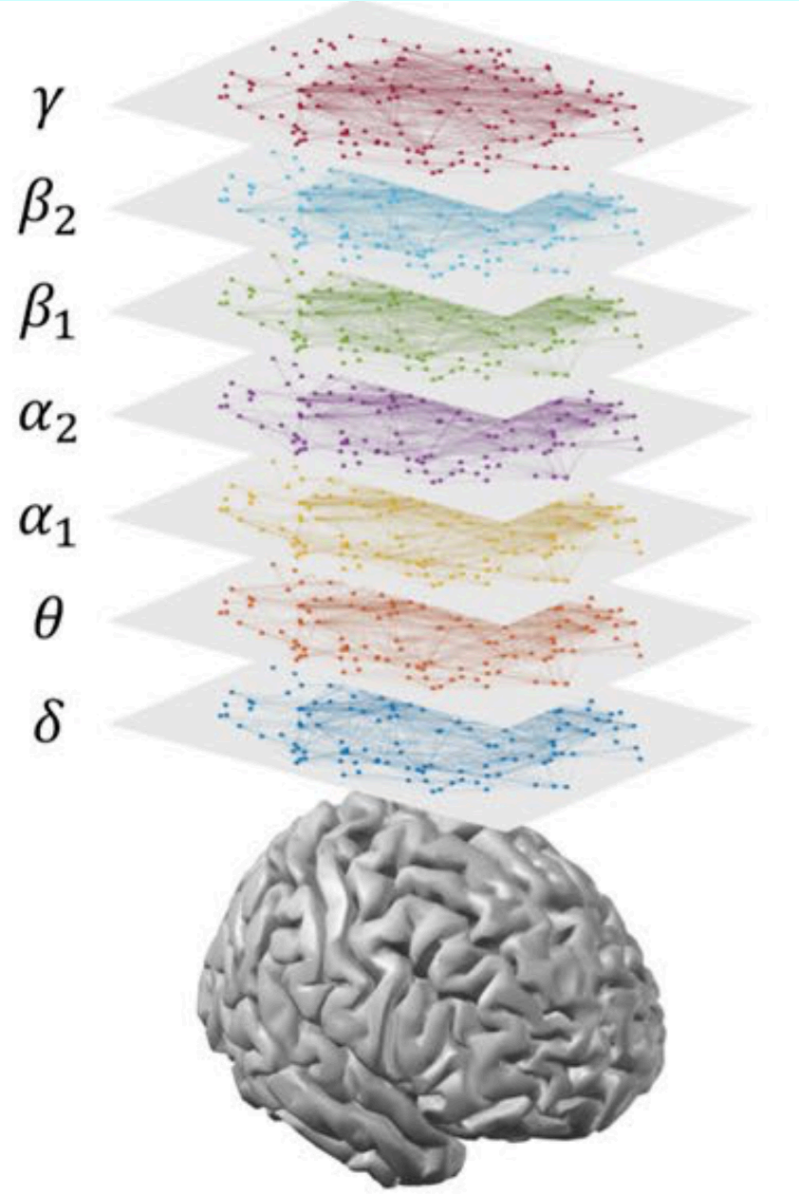
Chemical Synapses

Gap Junctions (asa electrical synapses)



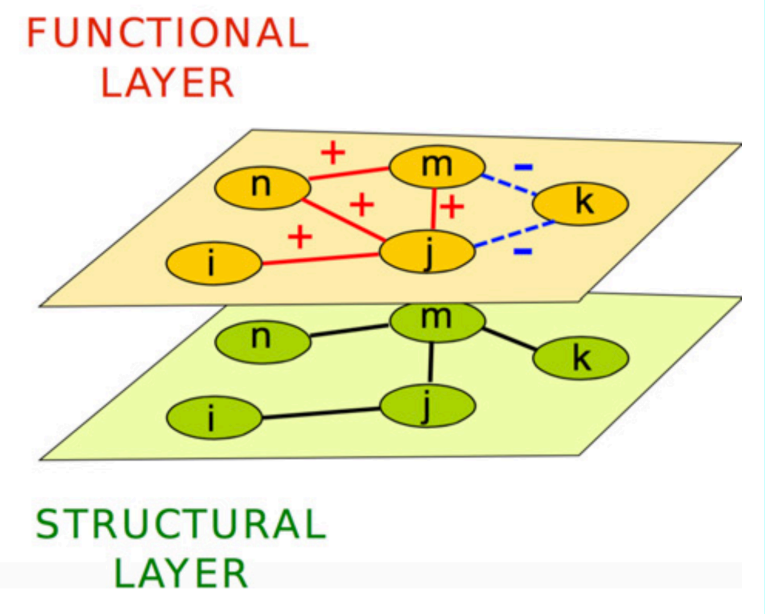
Varshney et al. PLOS Comp. Biology 7, e1001066 (2011)
Bentley et al. PLOS Comp. Biology 12, e1005283 (2016)

EEG - Different frequencies



Guillon et at, Scientific Reports (2017)

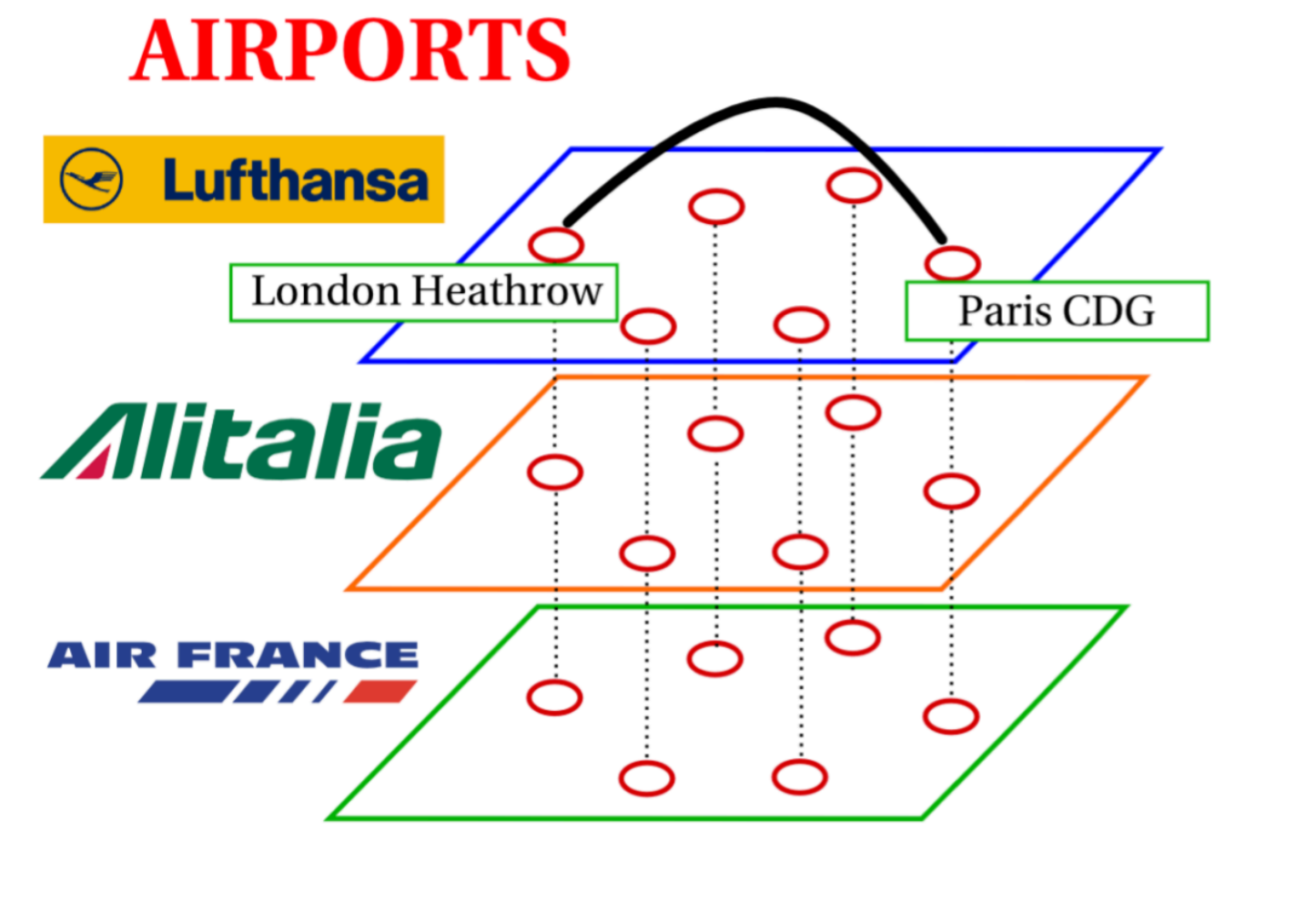
fMRI – Functional magnetic resonance



DTI – Diffusion tensor imaging

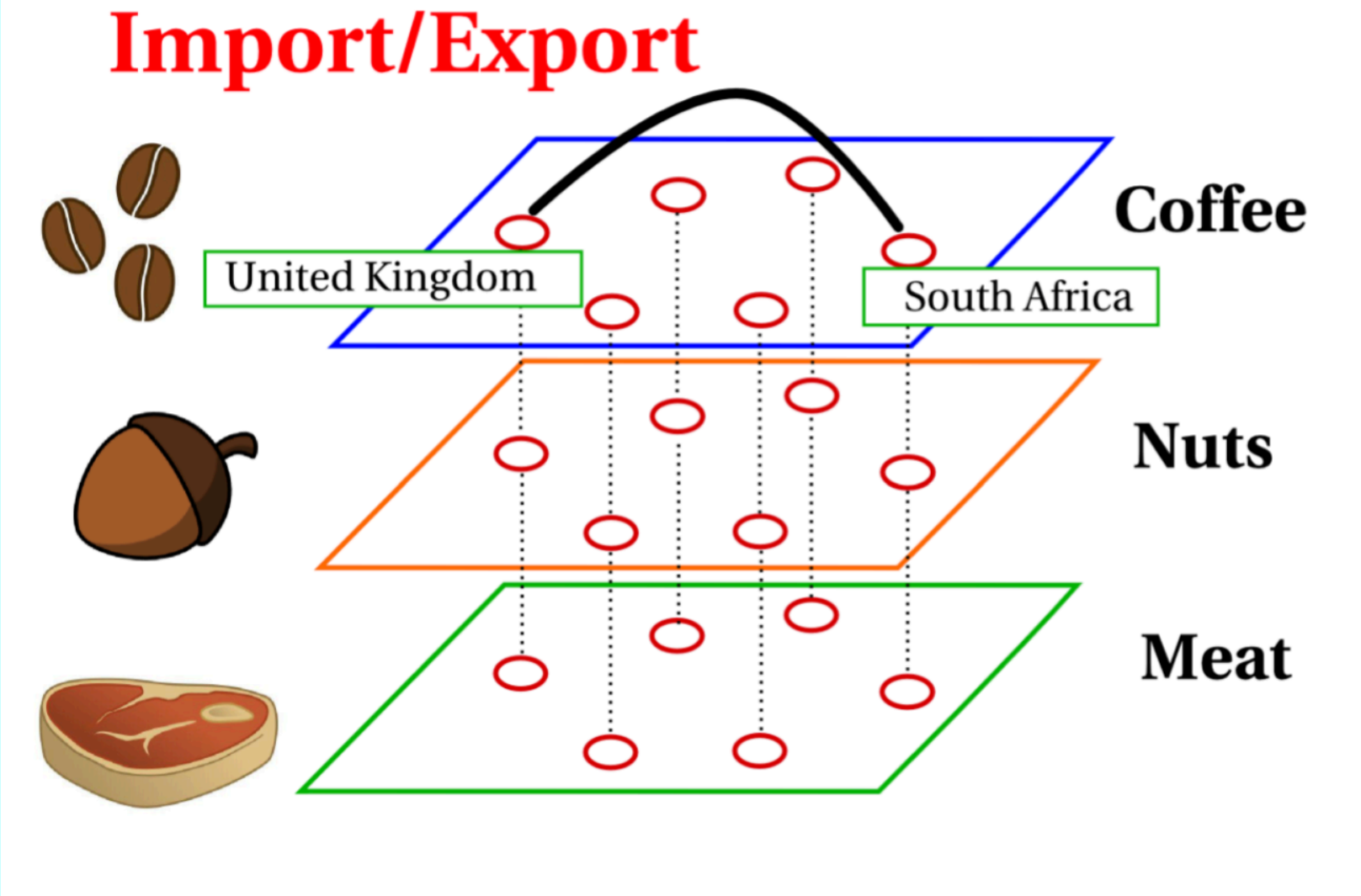
Battiston et at, Chaos (2017)

Multiplex networks data sets



Cardillo, Gomez-Gardenes et al. Sci Rep 3, 1344 (2013)

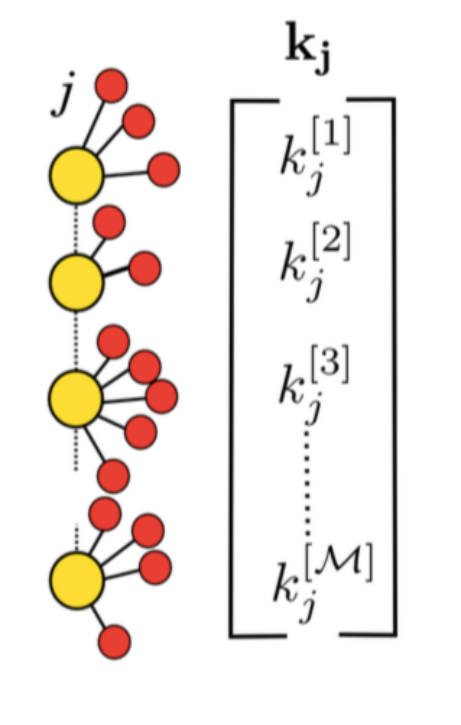
Multiplex networks data sets



Di Domenico et al., Nat Comm. 2015

Basic node properties

degree k_j



degree vector

$$\mathbf{k}_j = (k_j^{[1]}, k_j^{[2]}, \ldots, k_j^{[L]})$$

$$\mathbf{b}_j = (b_j^{[1]}, b_j^{[2]}, \ldots, b_j^{[L]})$$

where $b_j^{[\alpha]} = \begin{cases} 1 & if \quad k_j^{[\alpha]} \neq 0 \\ 0 & if \quad k_j^{[\alpha]} = 0 \end{cases}$

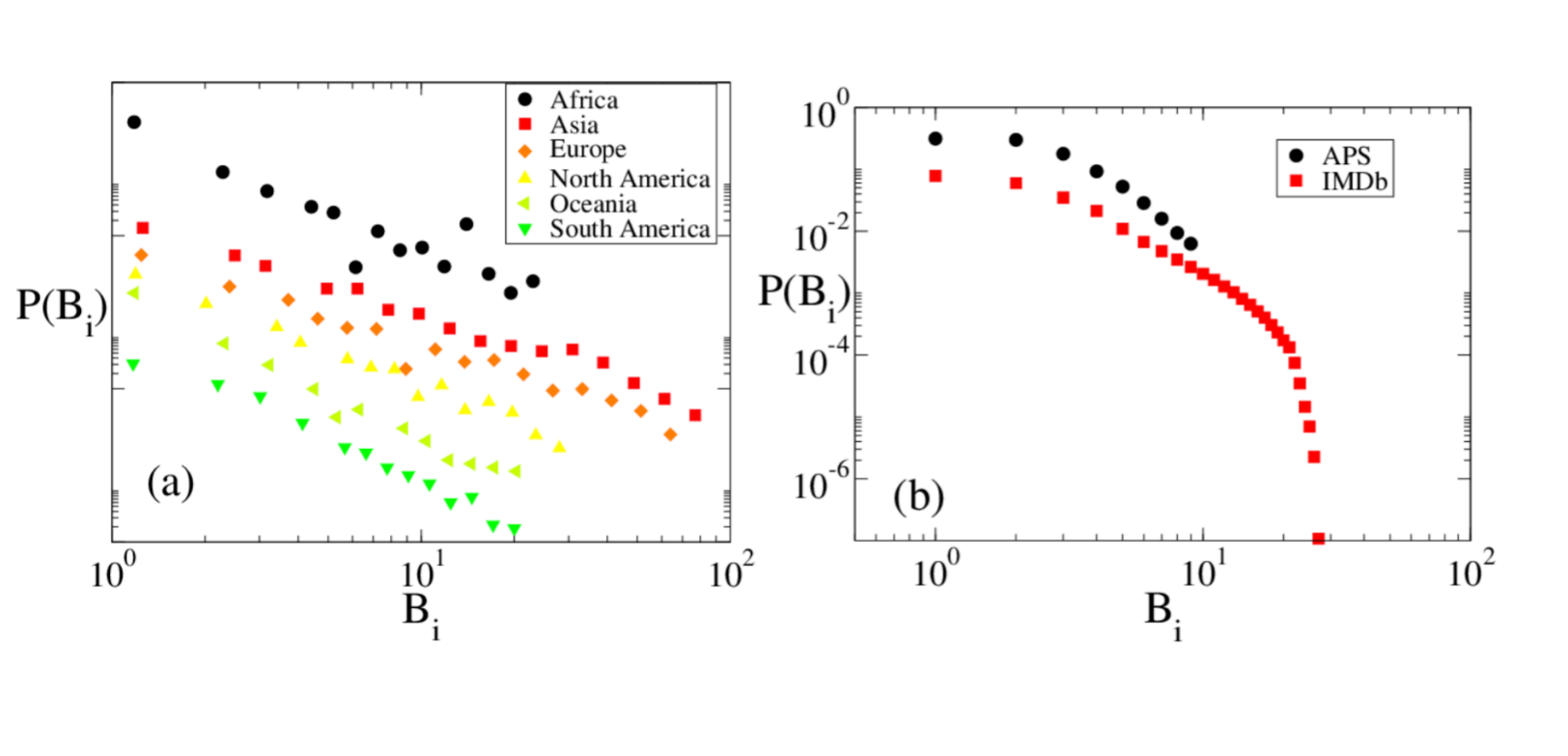
Node-activity vector

$$\mathbf{b}_j = (0,1,0\ldots,0)$$

Basic node properties

$$B_i = \sum_{\alpha} b_i^{[\alpha]}$$

Node activity



Nicosia, Latora, PRE 92, 032805 (2015)

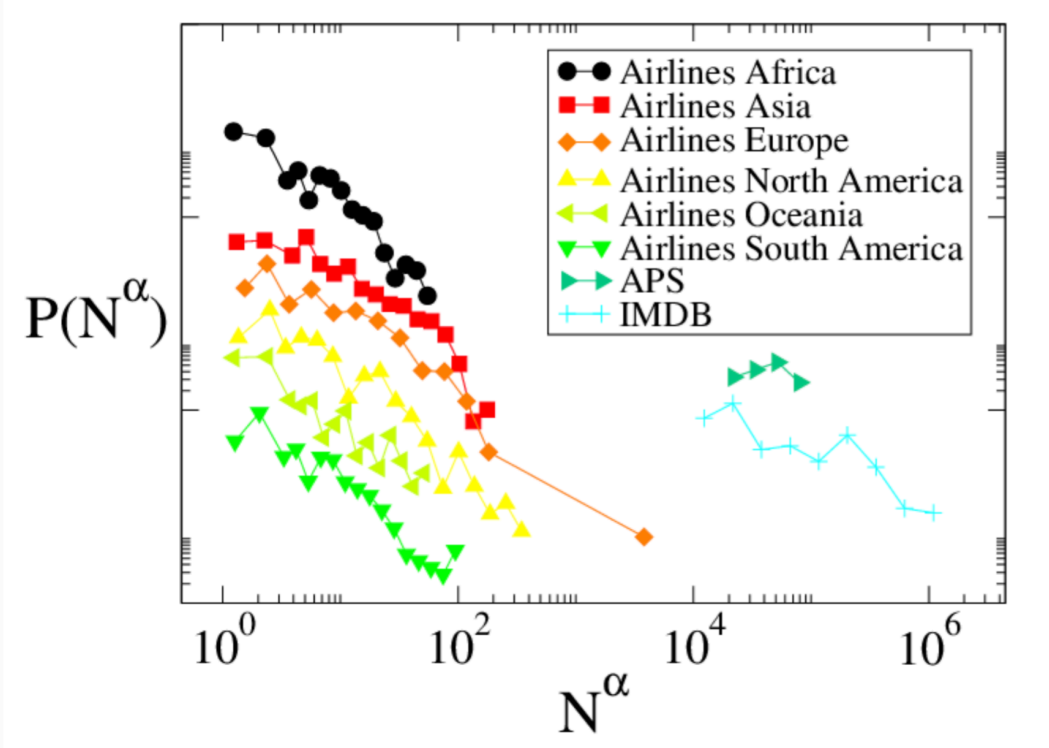
Basic layer properties

Layer-activity vector

$$\mathbf{d}^{[\alpha]} = (b_1^{[\alpha]}, b_2^{[\alpha]}, \ldots, b_N^{[\alpha]}) \quad \text{where} \quad b_j^{[\alpha]} = \begin{cases} 1 & if \quad k_j^{[\alpha]} \neq 0 \\ 0 & if \quad k_j^{[\alpha]} = 0 \end{cases}$$

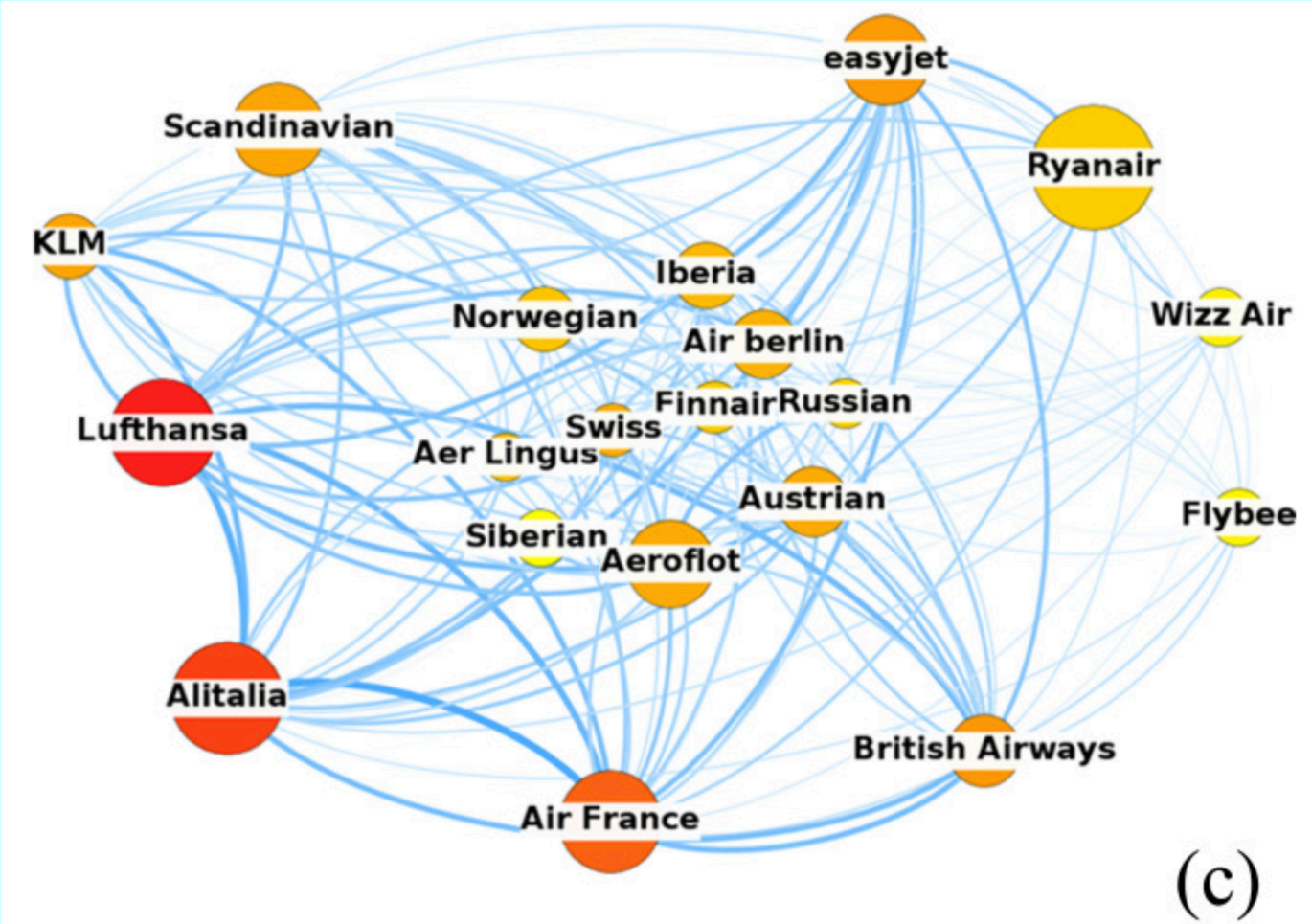
$$N^{[\alpha]} = \sum_i b_i^{[\alpha]}$$

Layer-activity



Basic layer properties

Correlations of layer activity $$Q^{[\alpha],[\beta]} = \frac{1}{N}\sum_i b_i^{[\alpha]} b_i^{[\beta]}$$

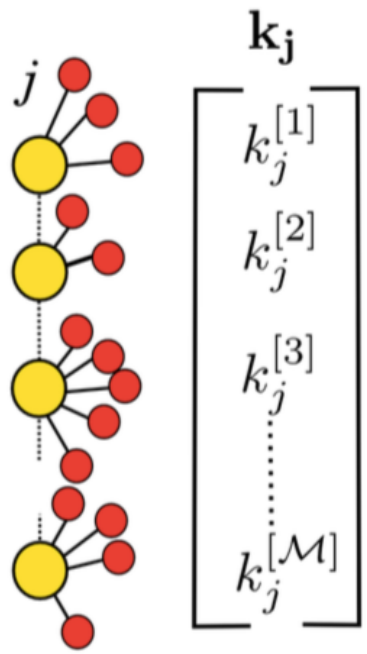


size = $N^{[\alpha]}$

color = strength

Nicosia, Latora, PRE 92, 032805 (2015)

Node cartography

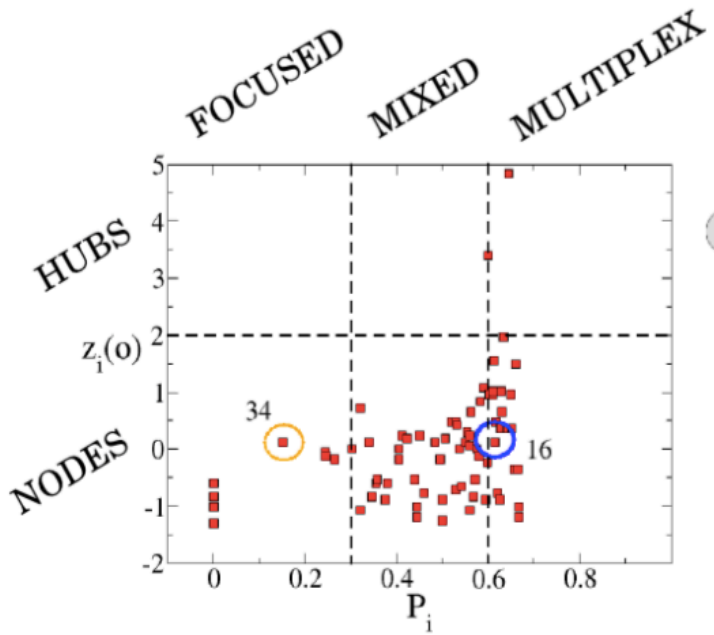


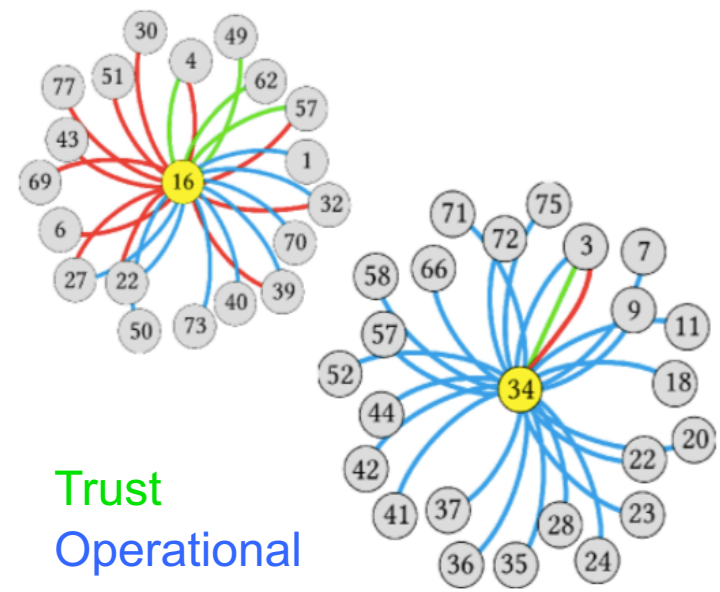
Overlapping degree

$$o_i = \sum_{\alpha=1}^{M} k_i^{[\alpha]}$$

Participation coefficient

$$P_i = \frac{M}{M-1}\left[1 - \sum_{\alpha=1}^{M}\left(\frac{k_i^{[\alpha]}}{o_i}\right)^2\right]$$

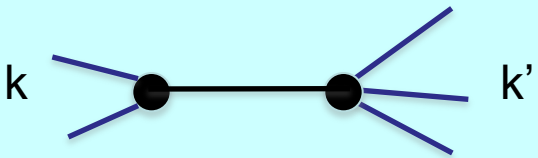




Trust
Operational
Communication

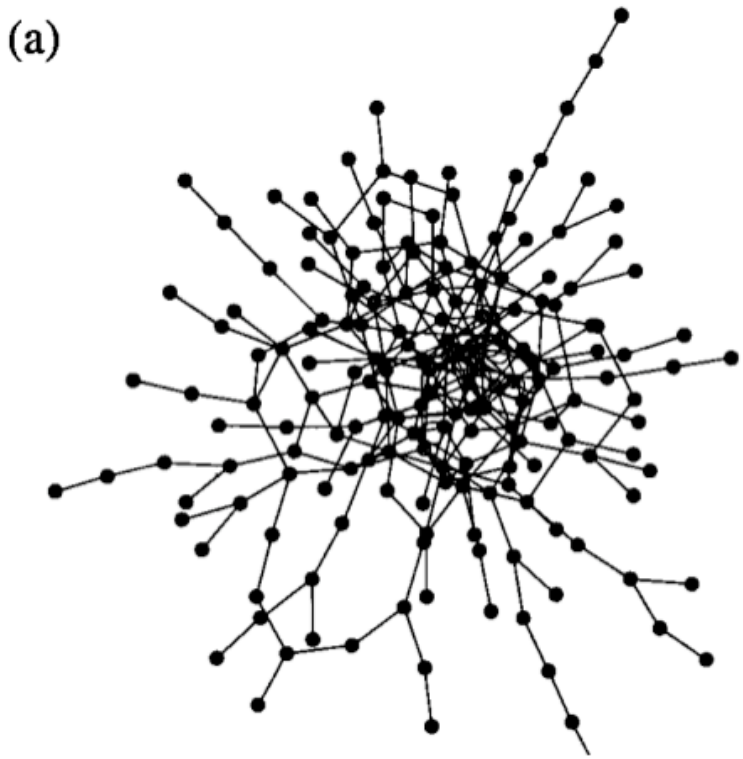
Battiston, Nicosia, Latora,
Structural measures for multiplex, PRE 89, 032804 (2014)

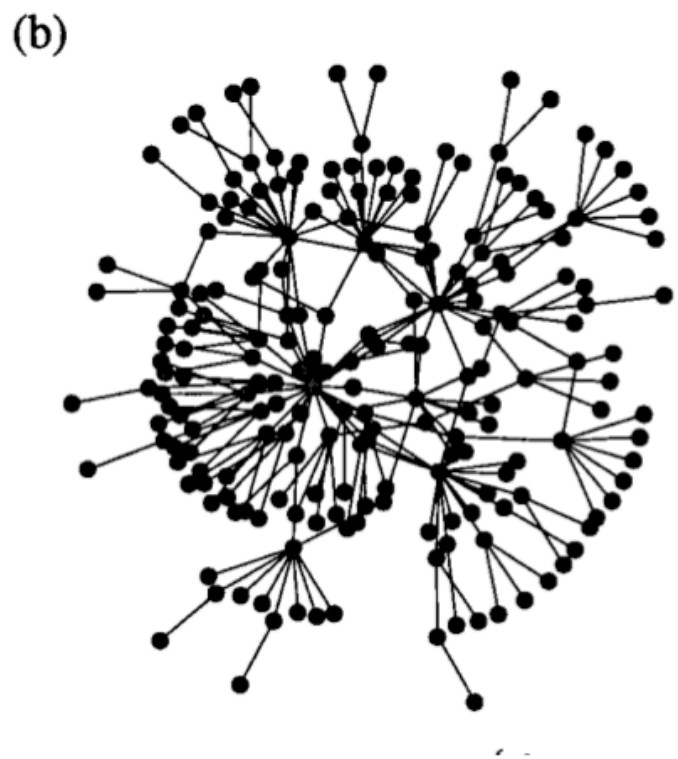
Degree-degree correlations



$$P_{k'|k}$$

Conditional probability that a link from a node of degree k goes to a node of degree k'

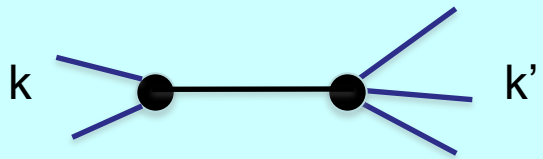




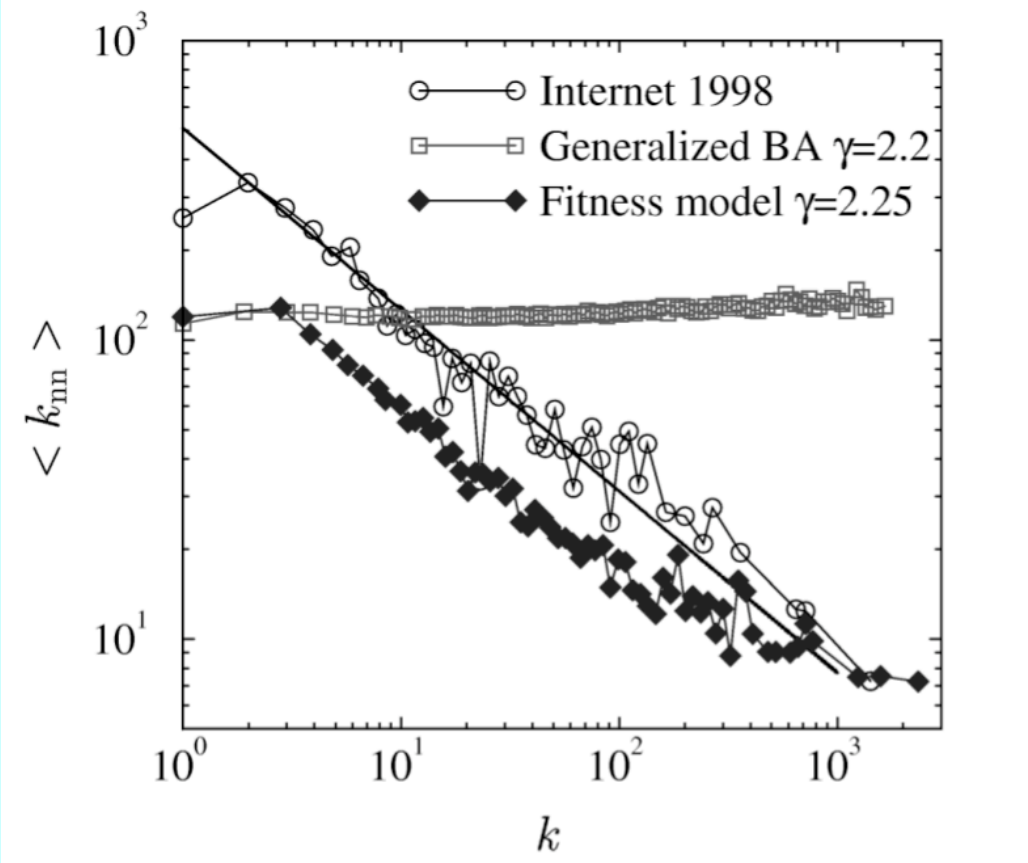
Assortative:
positive correlations

Disassortative:
negative correlations

Degree-degree correlations: assortative vs disassortative

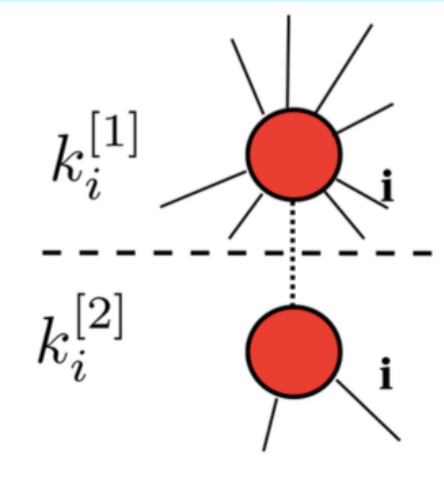


$$\langle k_{nn} \rangle (k) = \sum_{k'} k' P_{k'|k}$$



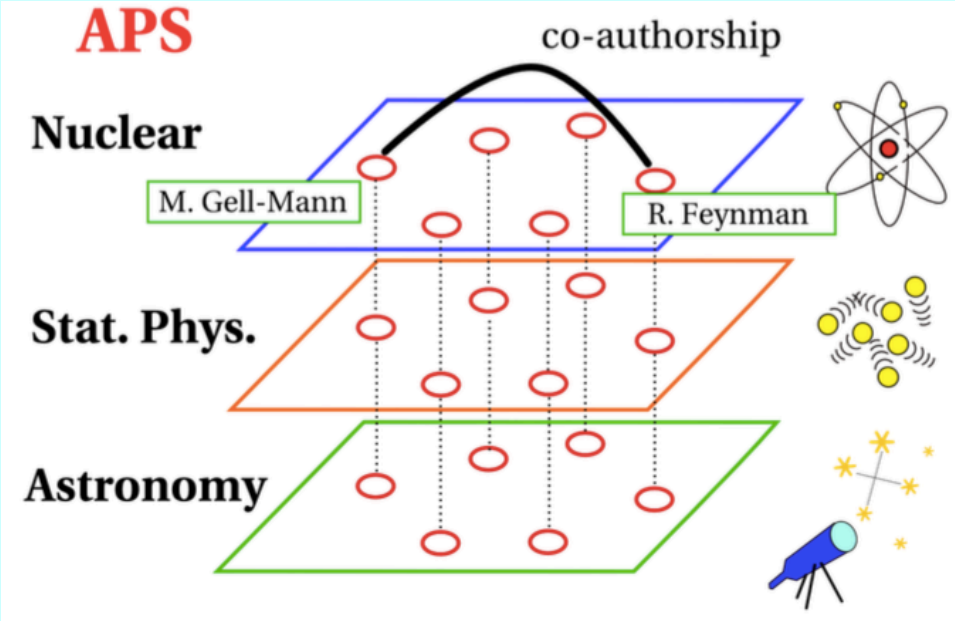
Pastor-Satorras, Vespignani PRL 01

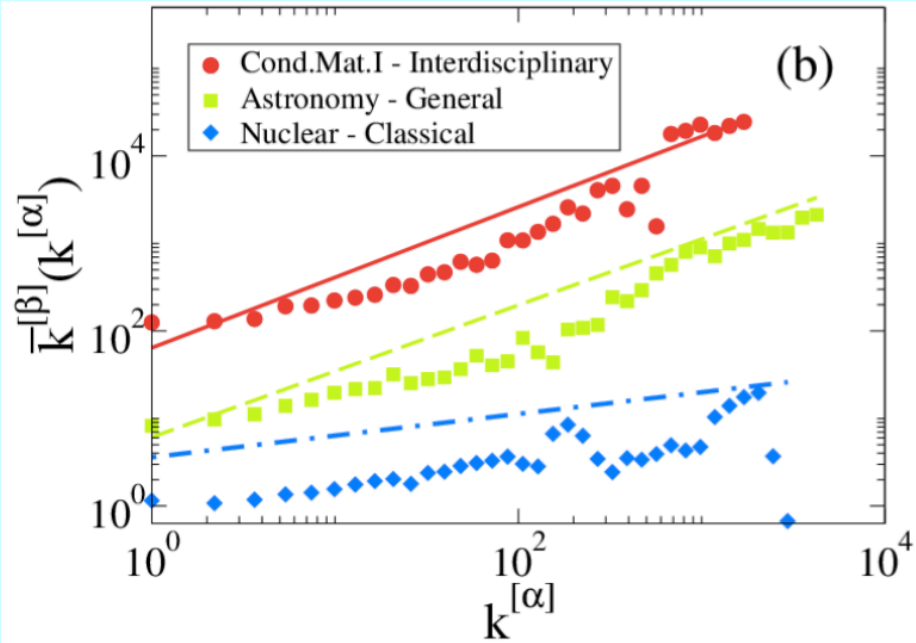
Inter-layer degree correlations



$$P(k^{[\beta]}|k^{[\alpha]})$$

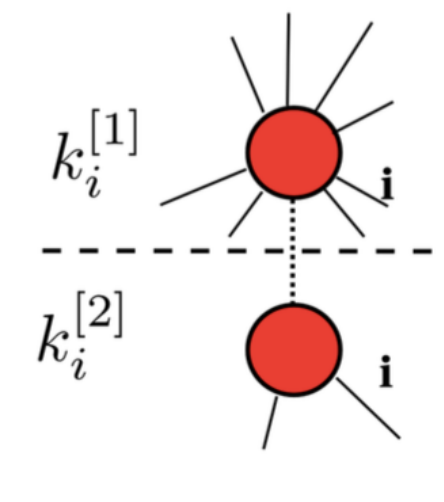
$$\overline{k^{[\beta]}}(k^{[\alpha]}) = \sum_{k^{[\beta]}} k^{[\beta]} P(k^{[\beta]}|k^{[\alpha]})$$





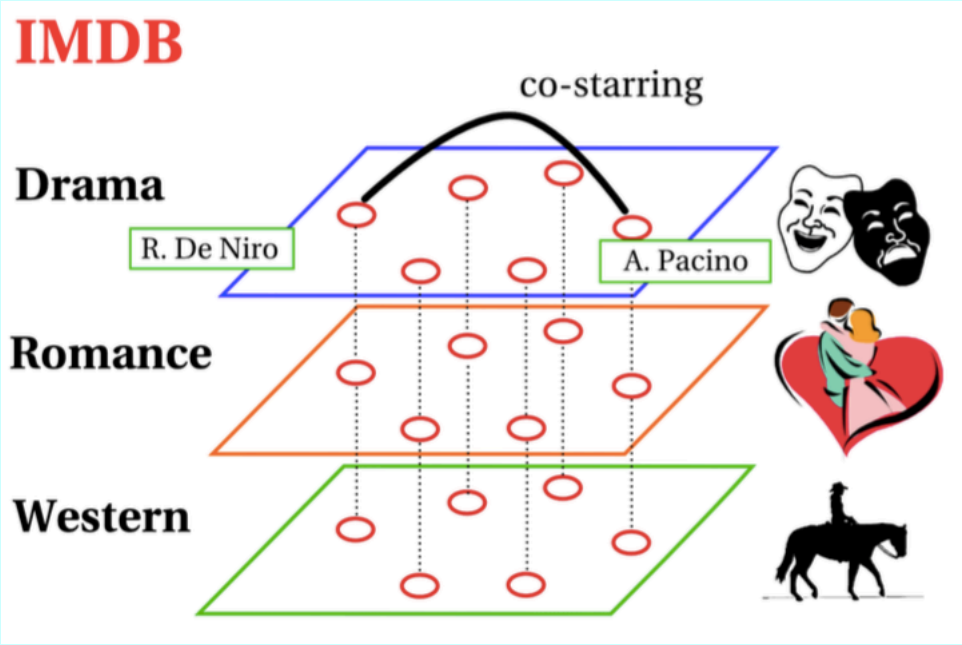
Nicosia, Latora, Measuring and modelling correlations in multiplex networks
PRE 92, 032805 (2015)

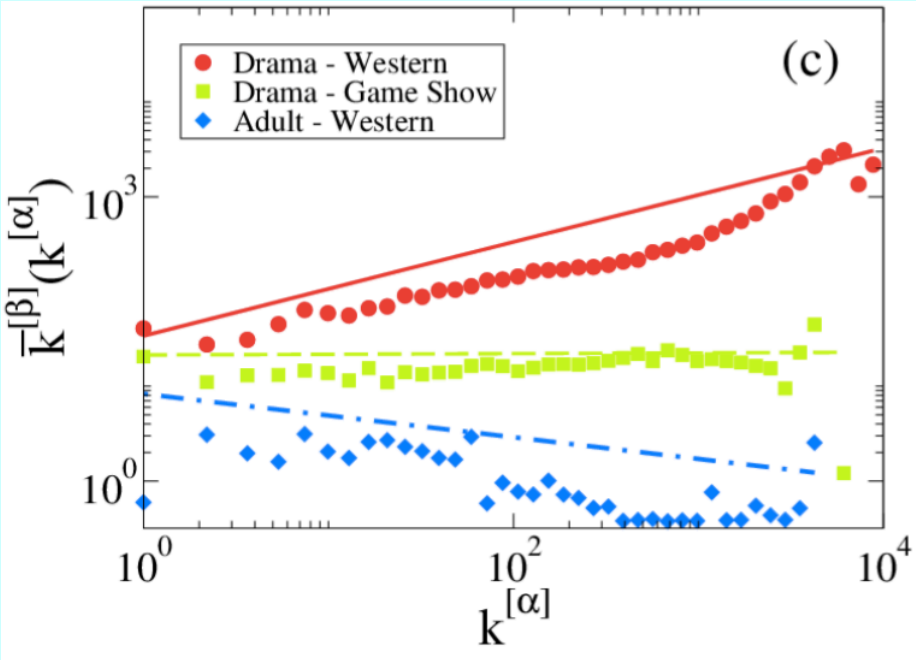
Inter-layer degree correlations



$$P(k^{[\beta]}|k^{[\alpha]})$$

$$\overline{k^{[\beta]}}(k^{[\alpha]}) = \sum_{k^{[\beta]}} k^{[\beta]} P(k^{[\beta]}|k^{[\alpha]})$$

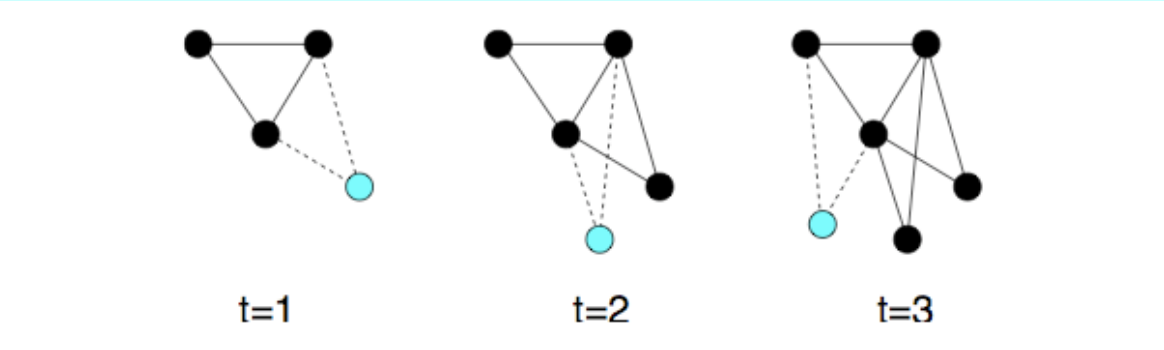




Nicosia, Latora, Measuring and modelling correlations in multiplex networks
PRE 92, 032805 (2015)

Modelling
of
MULTIPLEX NETWORKS

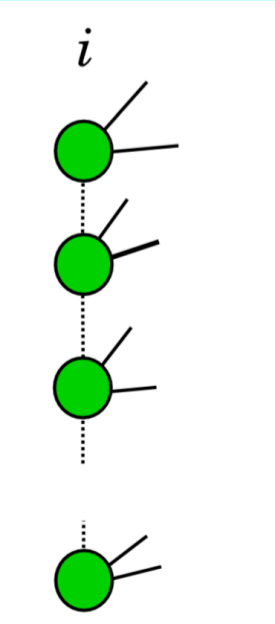
Growing multiplex graphs

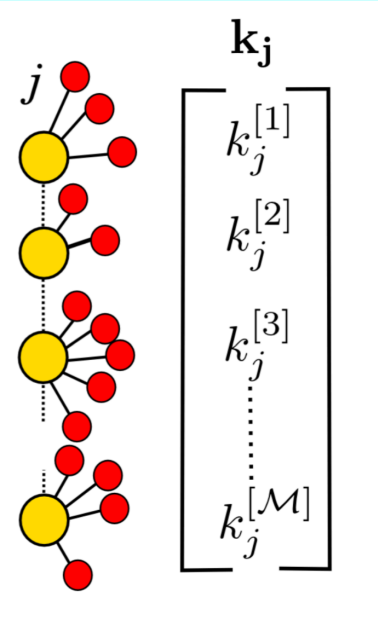


New nodes with m=2 stubs

Linear attachment

$$\Pi_{i \to j} \propto k_j$$





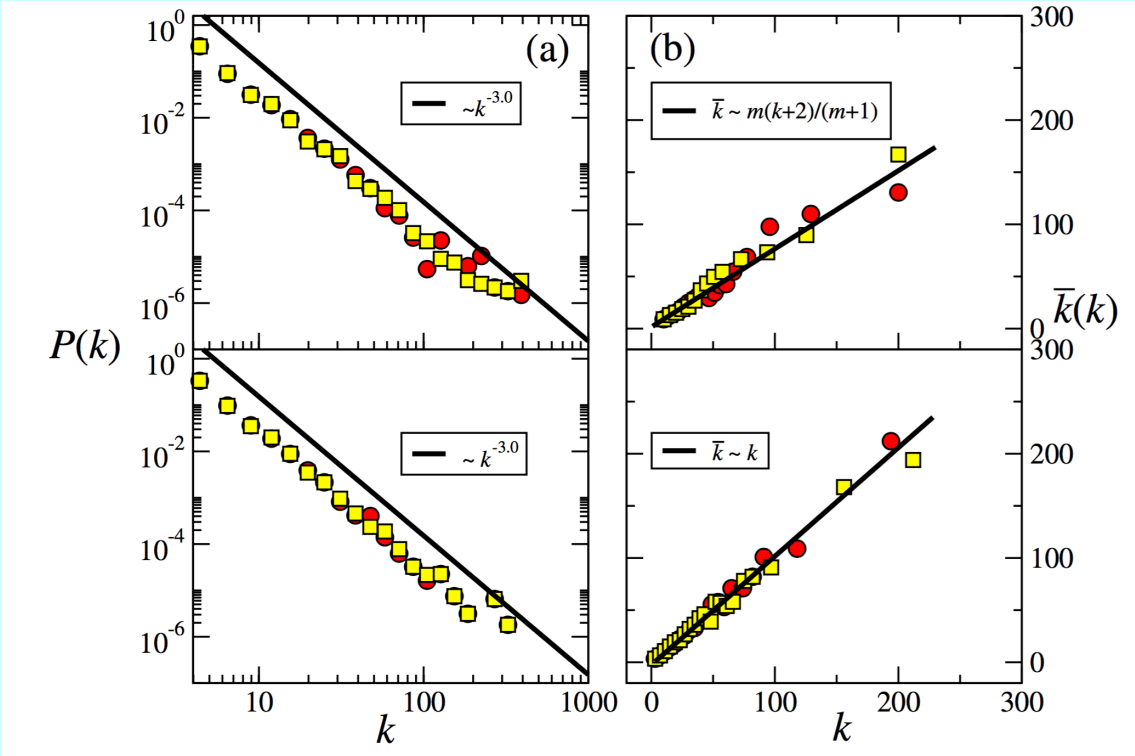
$$\Pi_{i \to j}^{[\alpha]} \propto F^{[\alpha]}(\mathbf{k_j})$$

m=2 stubs at each layer

Linear-attachment model

$$\begin{bmatrix} F^{[1]}[k,q] \\ F^{[2]}[k,q] \end{bmatrix} = C \begin{bmatrix} k \\ q \end{bmatrix} = \begin{bmatrix} c^{[1,1]} & c^{[1,2]} \\ c^{[2,1]} & c^{[2,2]} \end{bmatrix} \begin{bmatrix} k \\ q \end{bmatrix}$$

+ simultaneous or power-law delayed arrival



Assortative correlations (only)

Nicosia et al, PRL 111, 058701 (2013)

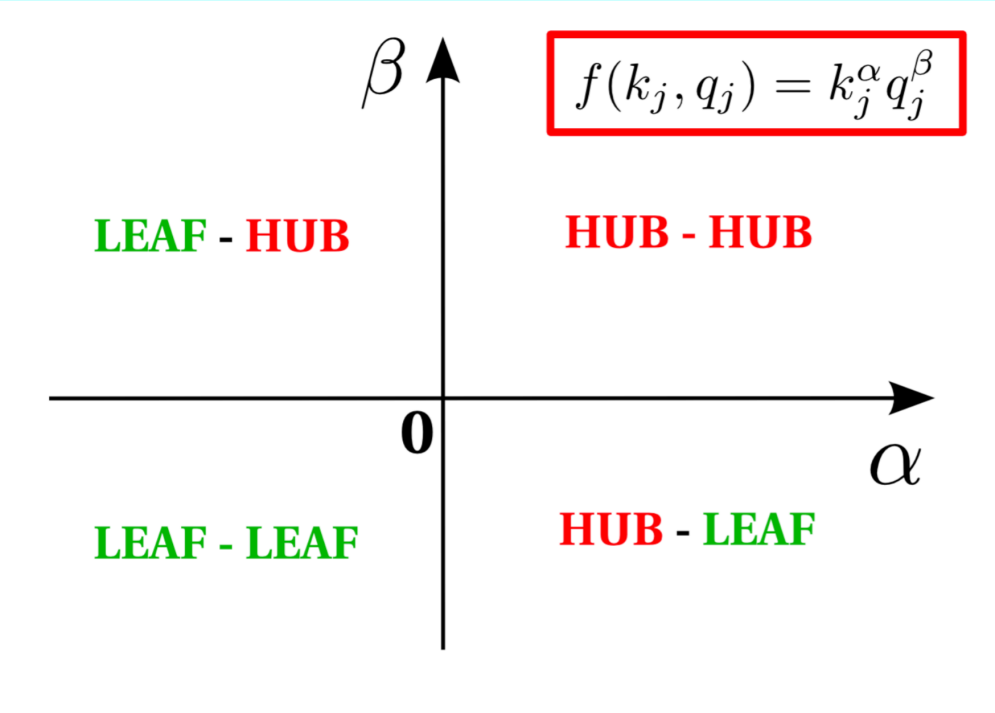
Nonlinear attachment model

$$\Pi^{[1]}_{i\to j} \propto f(k_j, q_j)$$

$$\Pi^{[2]}_{i\to j} \propto f(q_j, k_j)$$

Nonlinear attachment

$$f(k,q) = k^{\alpha} q^{\beta}$$

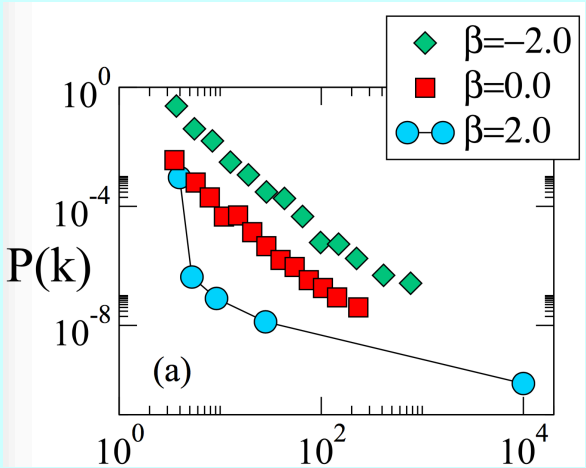


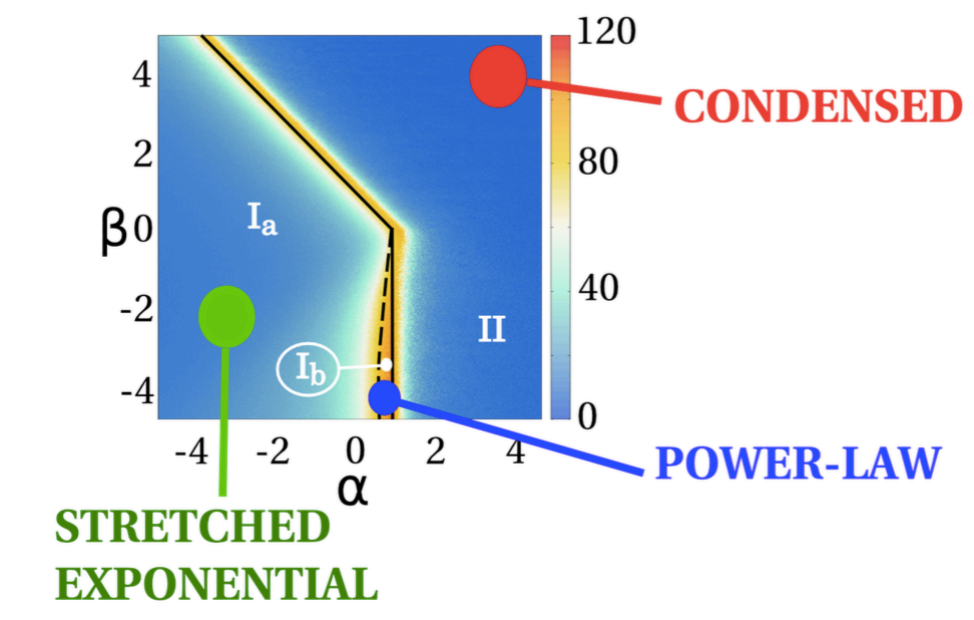
Nicosia et al, PRE 90, 042807 (2014)

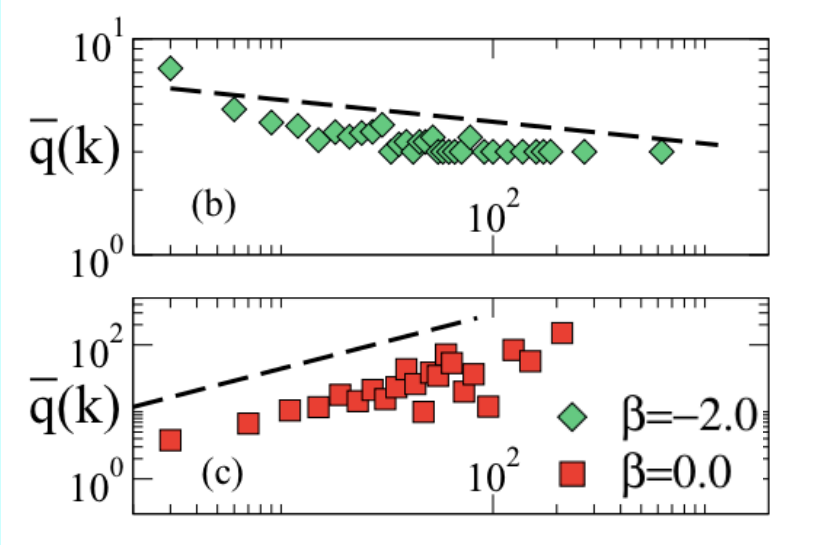
Nonlinear attachment model

$$f(k,q) = k^{\alpha} q^{\beta}$$

$\alpha = 1$





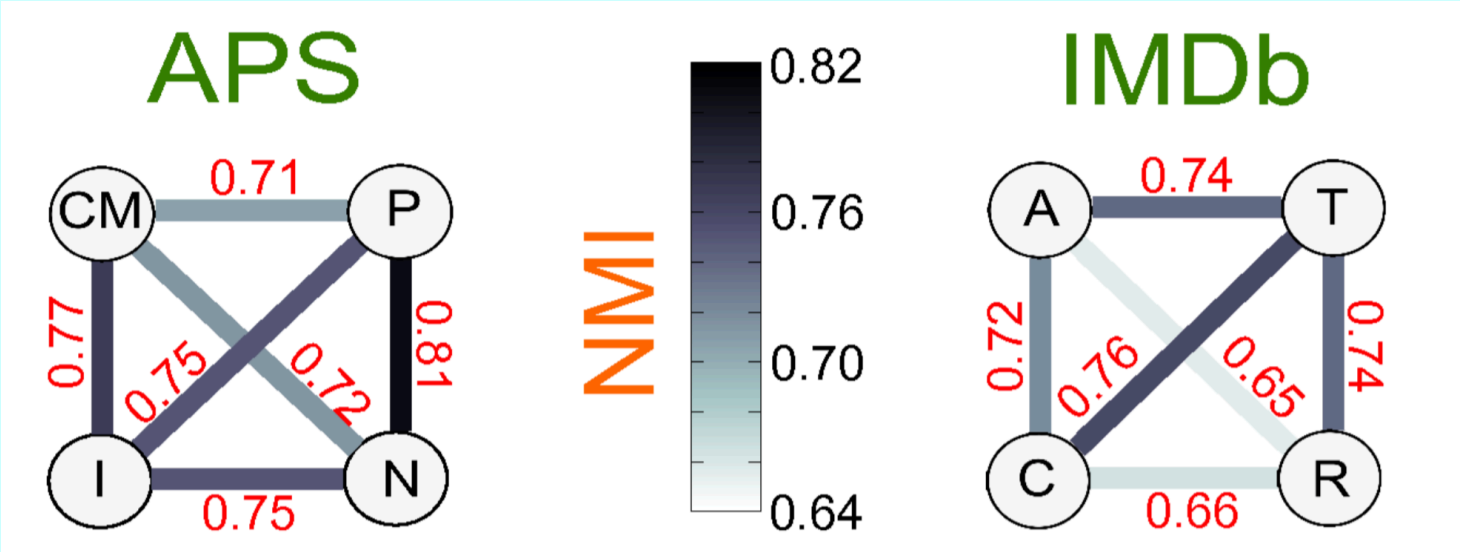


Nicosia et al, PRE 90, 042807 (2014)

Assortative and disassoartative correlations

Communities

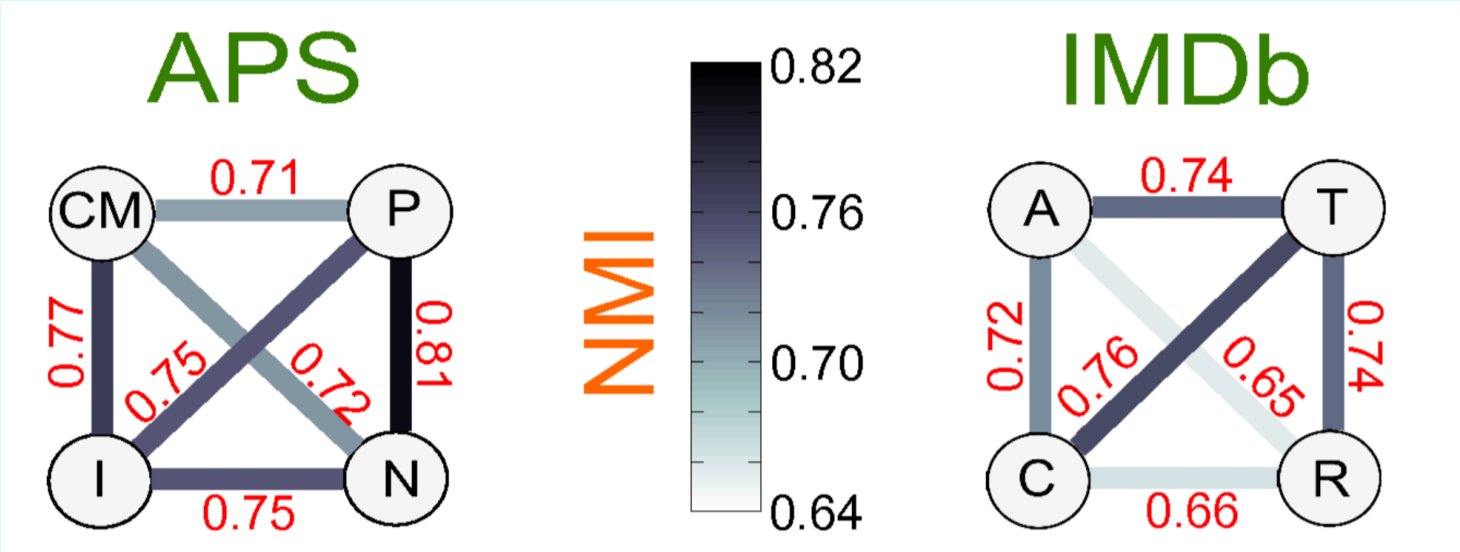
APS	N	$\langle k \rangle$	C
Nuclear (N)	1238	4.75	0.27
Particle (P)	1238	4.66	0.30
Cond. Matt. I (CM)	1238	10.29	0.24
Interdisciplinary (I)	1238	7.37	0.26
IMDb	N	$\langle k \rangle$	C
Action (A)	55797	83.56	0.61
Crime (C)	55797	82.30	0.58
Romance (R)	55797	86.00	0.59
Thriller (T)	55797	77.75	0.56



Communities

CM condensed matter
P particle
N nuclear
I interdisciplinary

A Action
T Thriller
R Romance
C Crime

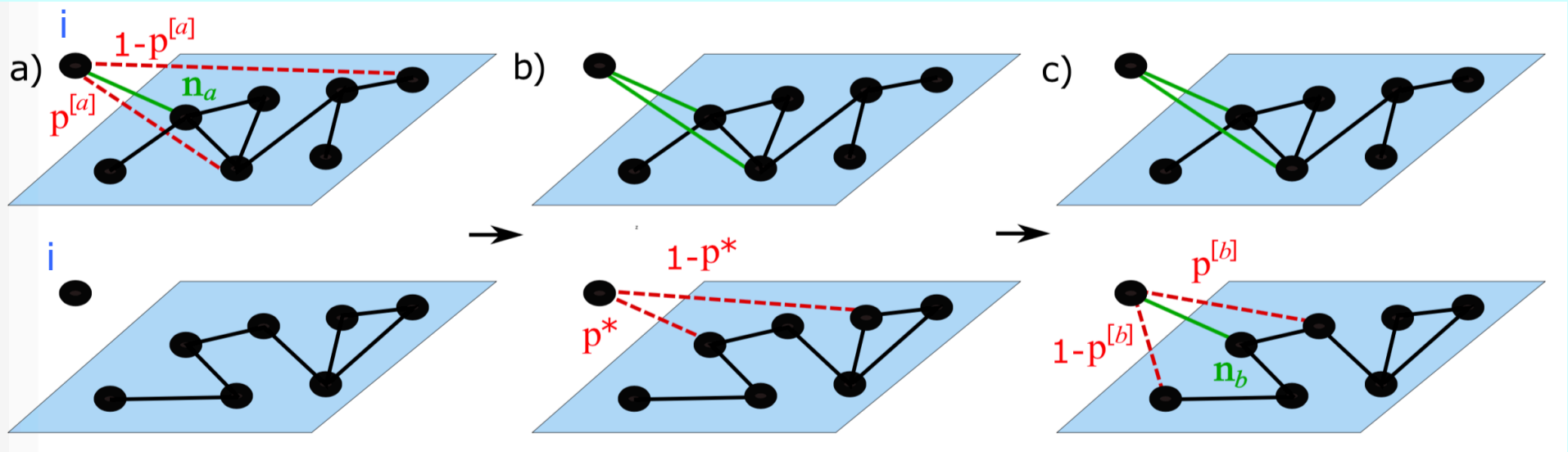


$$NMI(\mathcal{P}_\alpha, \mathcal{P}_\beta) = \frac{-2\sum_{m=1}^{M_\alpha}\sum_{m'=1}^{M_\beta} N_{mm'} \log\left(\frac{N_{mm'}N}{N_m N_{m'}}\right)}{\sum_{m=1}^{M_\alpha} N_m \log\left(\frac{N_m}{N}\right) + \sum_{m'=1}^{M_\beta} N_{m'} \log\left(\frac{N_{m'}}{N}\right)}$$

Danon, Diaz-Guilera, Duch, Arenas, JSM 2005

A model to grow communities

Five parameters $m^{[\alpha]}$ $m^{[\beta]}$ $p^{[\alpha]}$ $p^{[\beta]}$ $p*$



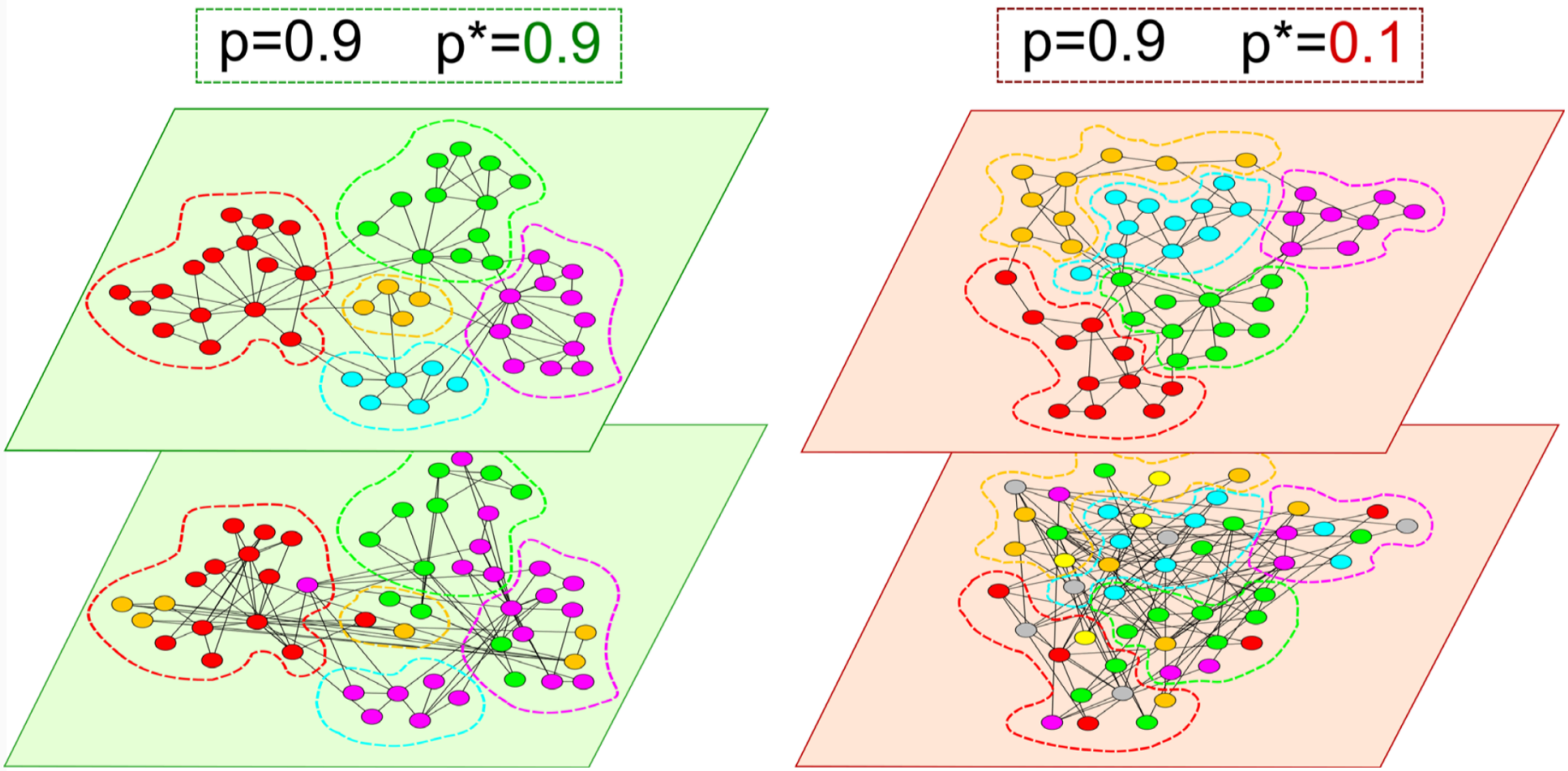
Intra-layer triadic closure $p^{[\alpha]}$ $p^{[\beta]}$

Overlap of neighbours of new node *i* $p*$

Battiston, Iacovacci, Nicosia, Bianconi, Latora, PLOS ONE 2016

A model to grow communities

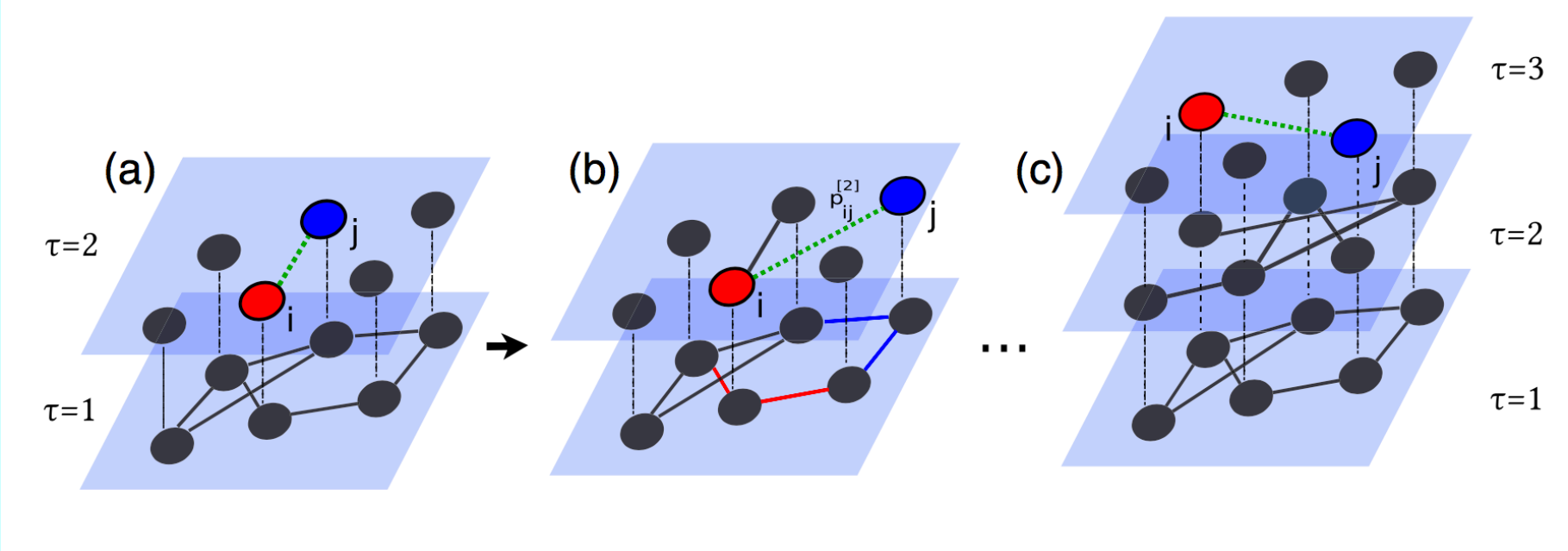
$$m^{[\alpha]} \quad m^{[\beta]} \quad p^{[\alpha]} \quad p^{[\beta]} \quad p*$$



Battiston, Iacovacci, Nicosia, Bianconi, Latora, PLOS ONE 2016

A new class of models

Multi-objective optimization (MOO) model

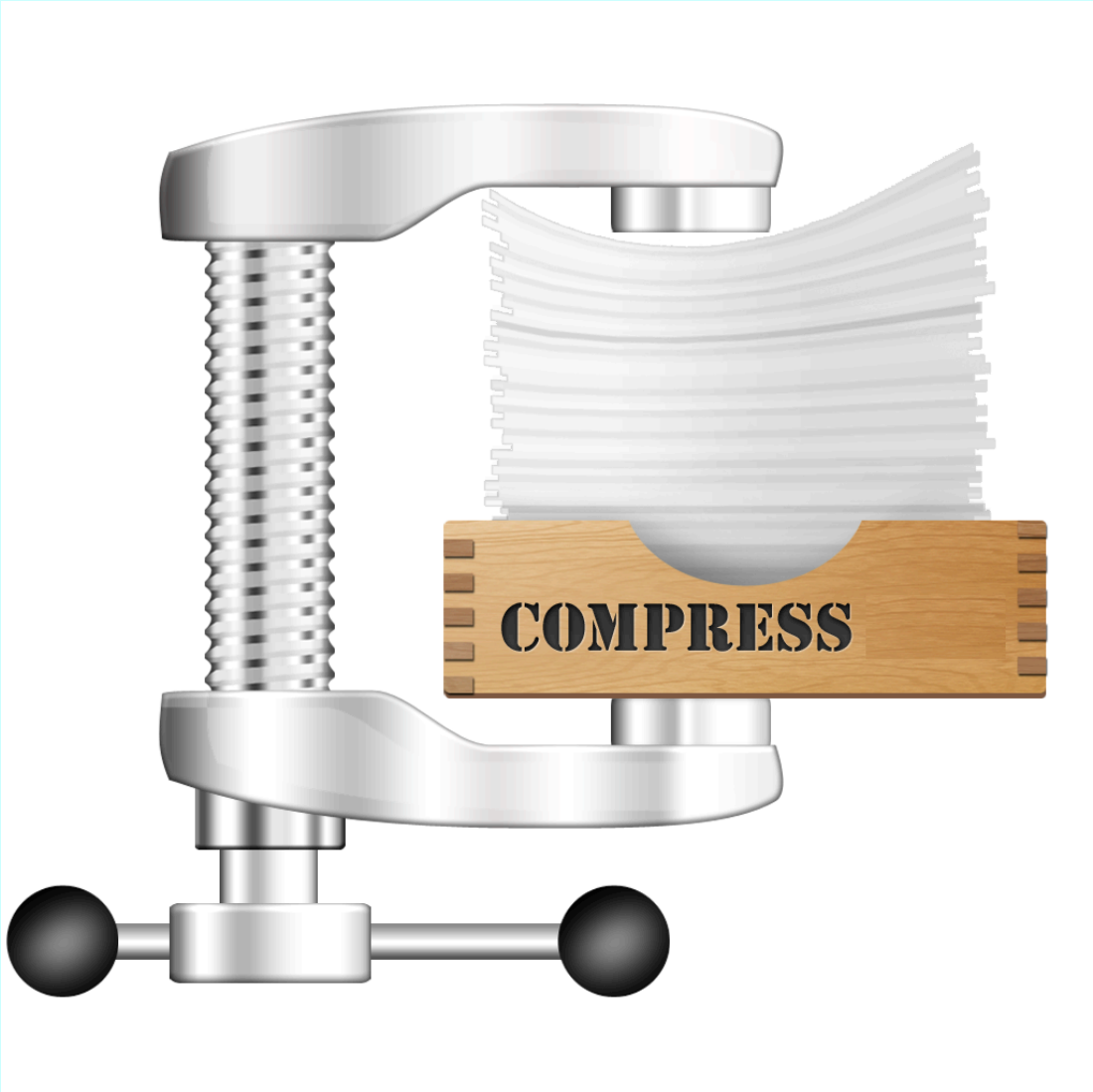


$$p_{ij}^{[\tau]} \propto \frac{o_i^{[\tau-1]} o_j^{[\tau-1]} + c_1}{o_{ij}^{[\tau-1]} + c_2}$$

$$\begin{matrix} \max \mathbf{F} \\ \min \mathbf{G} \end{matrix} = \begin{cases} F^{[\tau]} = \sum_{i,j:a_{ij}^{[\tau]}=1} (o_i o_j + c_1) \\ G^{[\tau]} = \sum_{i,j:a_{ij}^{[\tau]}=1} (o_{ij} + c_2) \end{cases}$$

Pareto optimality in multilayer network growth
Santoro, Latora, Nicosia, Nicosia, PRL 121, 128302 (2018)

Reducibility of a multiplex



Do we need to keep all the layers ?

Analogy between:

QUANTUM SYSTEMS

Density operator

$$\rho = \sum_s p_s |\psi_s\rangle\langle\psi_s|$$

$$Tr(\rho) = 1$$

Von Neumann (entanglement) entropy:

$$h_{VN} = -Tr[\rho \log \rho]$$

measures the mixedness of a system

$h_{VN} = 0$ for pure states

GRAPHS

Combinatorial Laplacian

$$L = \frac{1}{2K}[D - A]$$

$$Tr(L) = 1$$

Von Neumann entropy of a graph:

$$h_G = -Tr[L \log L]$$

measures the complexity of a graph

Braunstein, Ghosh, Severini, Ann. Comb. 10, 291 (2006)

The Von Neumann entropy of a multiplex

Original Multilayer

$$\mathcal{A} = \{A^{[1]}, \ldots, A^{[M]}\}$$

Reduced multilayer

$$C = \{C^{[1]}, C^{[2]}, \ldots, C^{[X]}\}$$

$$X \leq M$$

$$\overline{H}(C) = \frac{1}{X}\sum_{\alpha=1}^{X} h_{C^{[\alpha]}}$$

Where h_G is the Von Neumann entropy of graph G

$$h_G = -Tr[L_G \log_2 L_G] = -\sum_{i=1}^{N} \lambda_i \log_2 \lambda_i$$

THE IDEA: look for reduced multilayer networks which **minimize** the **Von Neumann entropy** (complexity)

Distinguishibility from the aggregate

Reduced multilayer

$$C = \{C^{[1]}, C^{[2]}, \ldots, C^{[X]}\} \qquad X \leq M$$

$$\overline{H}(C) = \frac{1}{X} \sum_{\alpha=1}^{X} h_{C^{[\alpha]}}$$

Aggregate

$$A = A^{[1]} + A^{[2]} + \ldots + A^{[M]} \qquad X = 1$$

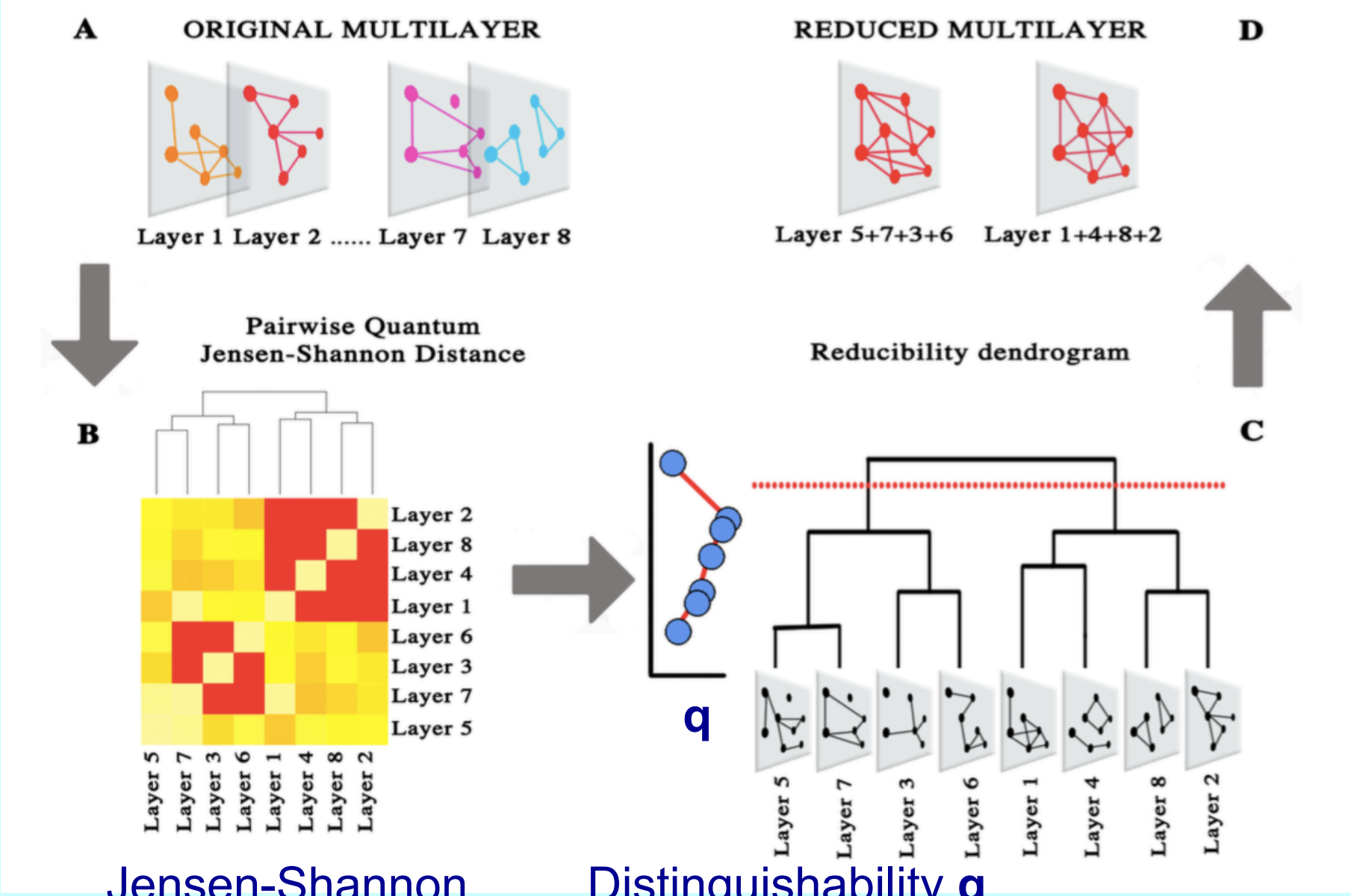
Distinguishability between C and A

$$q(C) = 1 - \frac{\overline{H}(C)}{h_A}$$

We look for reduced multilayer graphs which **maximize** the **distinguishability** from the aggregate.

Exhaustive search: number of partitions of M objects grows exponentially with M

A greedy hierarchical algorithm



Jensen-Shannon divergence

Distinguishability **q** (based on Von Neumann entropy)

Jensen-Shannon Divergence

$$\mathcal{D}_{KL}(\boldsymbol{\rho}||\boldsymbol{\sigma}) = \mathrm{Tr}\left[\boldsymbol{\rho}\left(\log_2(\boldsymbol{\rho}) - \log_2(\boldsymbol{\sigma})\right)\right]$$

Kullback-Liebler divergence:
the information gained about σ when the expectation is based only on ρ

$$\mathcal{D}_{KL}(\boldsymbol{\rho}||\boldsymbol{\sigma}) \neq \mathcal{D}_{KL}(\boldsymbol{\sigma}||\boldsymbol{\rho})$$

is not a metric

$$\mathcal{D}_{\mathrm{JS}}(\boldsymbol{\rho}||\boldsymbol{\sigma}) = \frac{1}{2}\mathcal{D}_{\mathrm{KL}}(\boldsymbol{\rho}||\boldsymbol{\mu}) + \frac{1}{2}\mathcal{D}_{\mathrm{KL}}(\boldsymbol{\sigma}||\boldsymbol{\mu}) =$$

$$= h(\boldsymbol{\mu}) - \frac{1}{2}[h(\boldsymbol{\rho}) + h(\boldsymbol{\sigma})]$$

$$\boldsymbol{\mu} = \tfrac{1}{2}(\boldsymbol{\rho} + \boldsymbol{\sigma})$$

Reducibility?

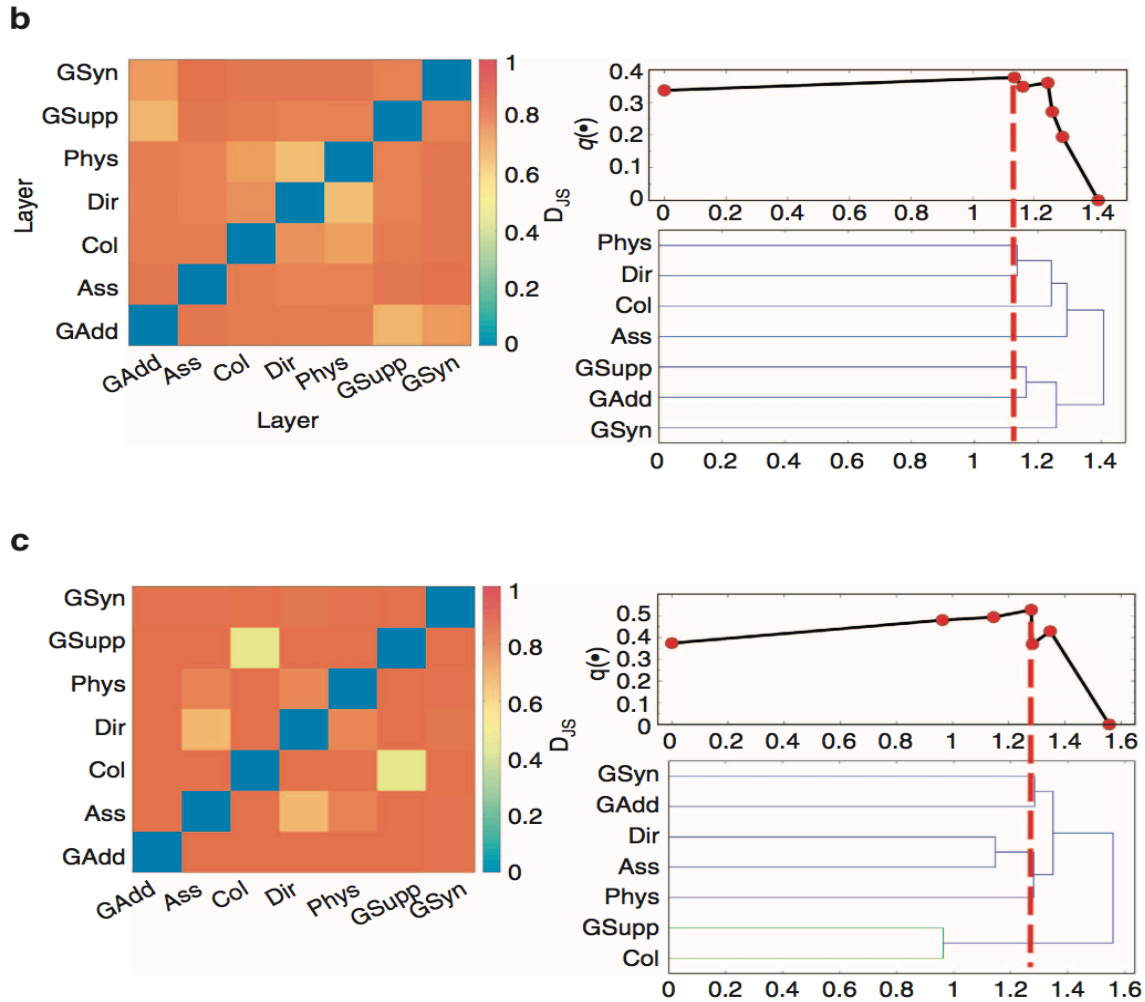


Figure 3 | Layer aggregation of protein–genetic interaction networks. The multilayer protein–genetic networks of different species have

Structural reducibility of multilayer networks

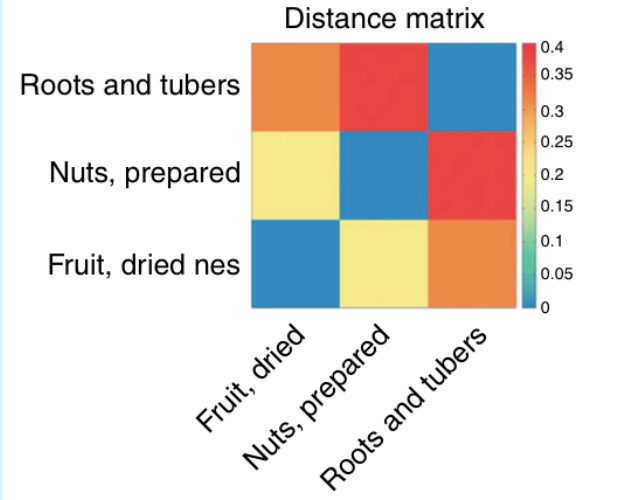
De Domenico, Nicosia, Arenas, Latora, Nature Communications 2015

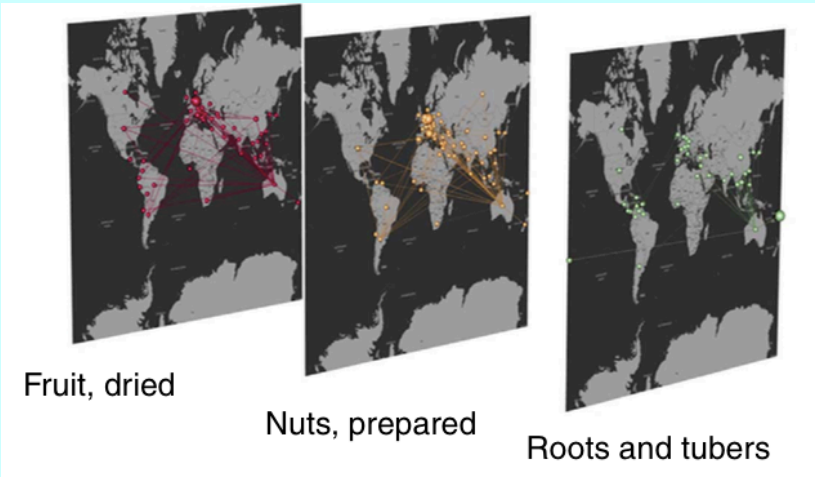
Reducibility?

Table 1 | Reducibility of empirical multilayer networks.

Network	N	M	M_{opt}	max[$q(\bullet)$]	χ
Arabidopsis	6981	7	5	0.436	0.33
Bos	326	4	3	0.494	0.33
Candida	368	7	4	0.527	0.50
C. elegans	3880	6	4	0.390	0.40
Drosophila	8216	7	5	0.426	0.33
Gallus	314	6	4	0.505	0.40
Human HIV-1	1006	5	2	0.499	0.75
Mus	7748	7	6	0.376	0.17
Plasmodium	1204	3	2	0.500	0.50
Rattus	2641	6	4	0.504	0.40
S. cerevisiae	6571	7	4	0.115	0.50
S. pombe	4093	7	4	0.197	0.50
Xenopus	462	5	3	0.424	0.50
Arxiv coauthorship	14065	13	11	0.231	0.17
Terrorist network	78	4	2	0.239	0.67
FAO Trade network	184	340	182	0.354	0.47
London Tube	369	13	12	0.441	0.08
Airports Europe	1064	175	165	0.667	0.06
Airports Asia	1130	213	202	0.653	0.05
Airports North America	2040	143	136	0.686	0.05

Worldwide food import/export





Structural reducibility of multilayer networks

De Domenico, Nicosia, Arenas, Latora, Nature Communications 2015

Reducibility?

Worldwide food import/export

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Airports Europe	1064	175	165	0.667	0.06
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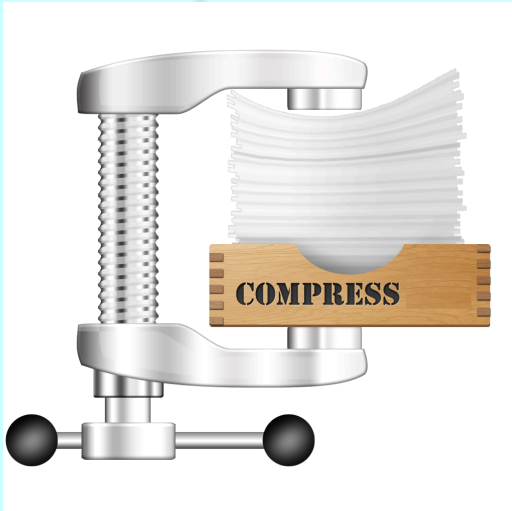
340 layers

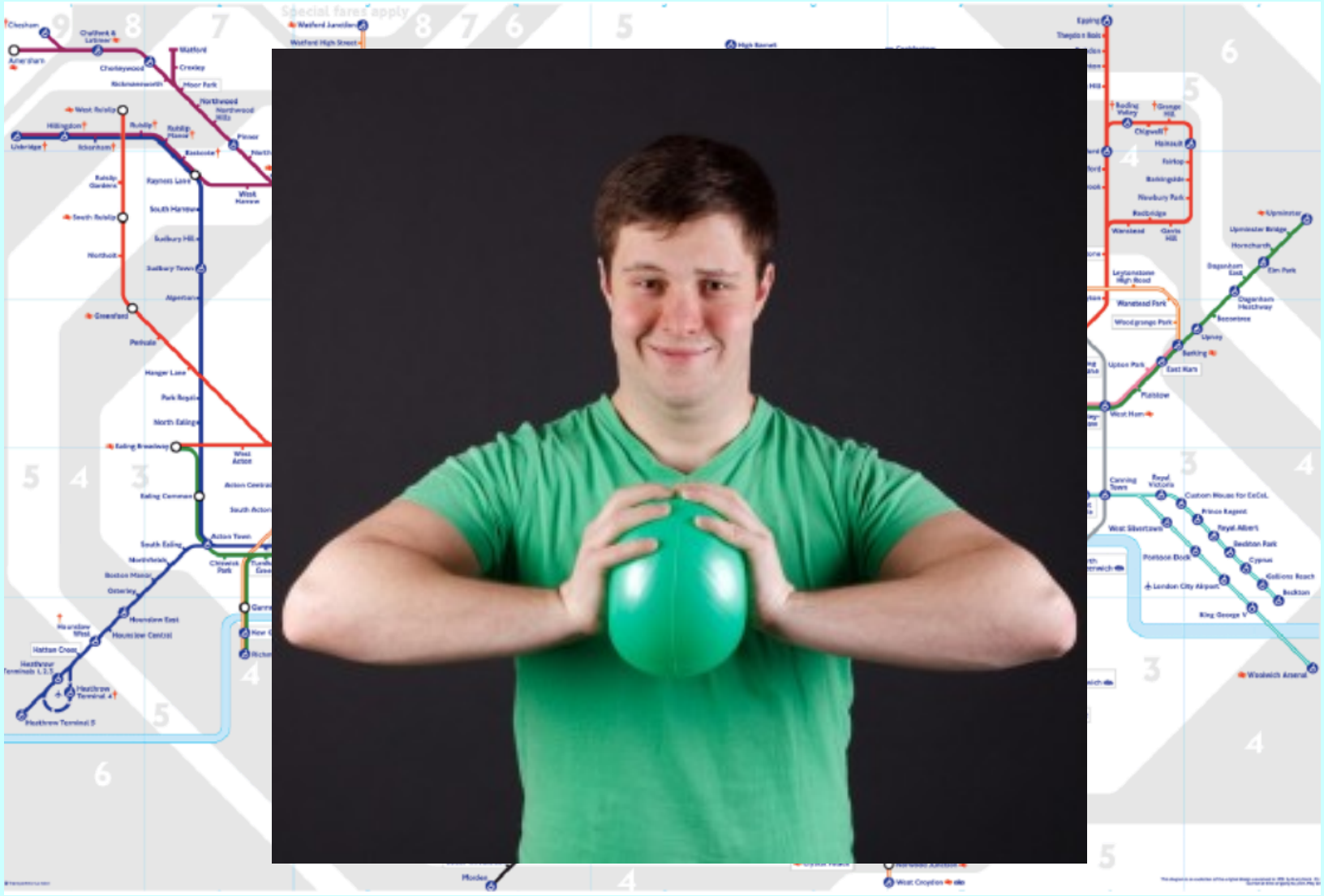


182 layers

Structural reducibility of multilayer networks

De Domenico, Nicosia, Arenas, Latora, Nature Communications 2015

Reducibility?



From 13 layers → 12 layers

Summing up

How to represent the structure of a multiplex network

Extension of basic measures

Intrinsic multiplex measures

Modeling the growth of a multiplex

Reducibility of a multiplex

Dynamics on multiplex networks

Kivela et al., Multilayer networks, Journal of Complex Networks 2014

Boccaletti et al, Structure and dynamics of multilayer networks, Phys. Rep 2014

De Domenico et al, Physics of spreading processes in multilayer, Nat Phys 2016

Battiston et al, When multiplex really matters: the emergent dynamics of multiplex networks, soon on ArXiv

DYNAMICS

Irreducibility of the voter model

Dynamical inequivalence of multiplex and aggregate

Diakonova, Nicosia, Latora, San Miguel, New J. Phys. 18 (2016) 023010

Axelrod model on multiplex nets

The emergence of robust multiculturality

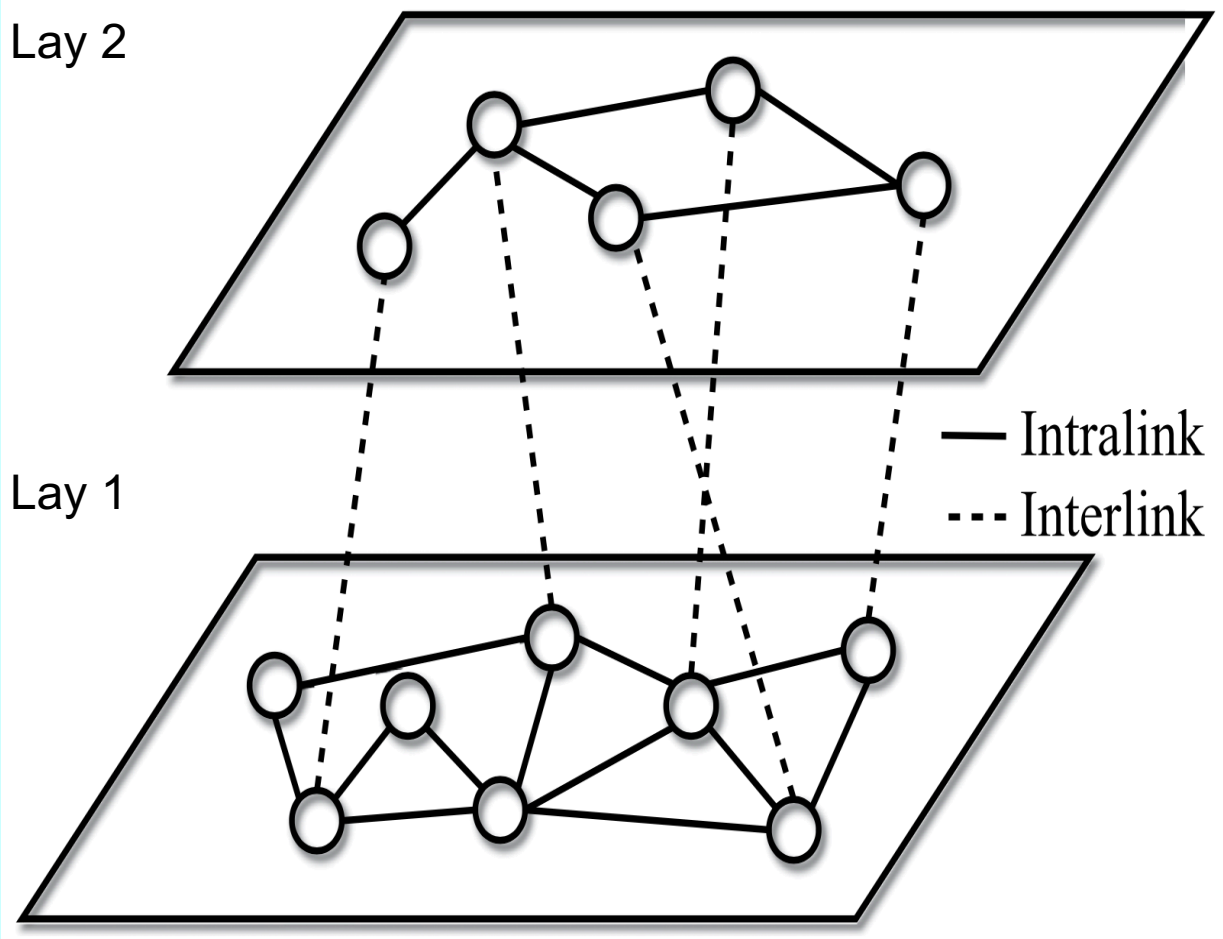
Battiston, Nicosia, Latora, San Miguel, Scientific Reports 7, 1809 (2017)

Multiplex mobility networks

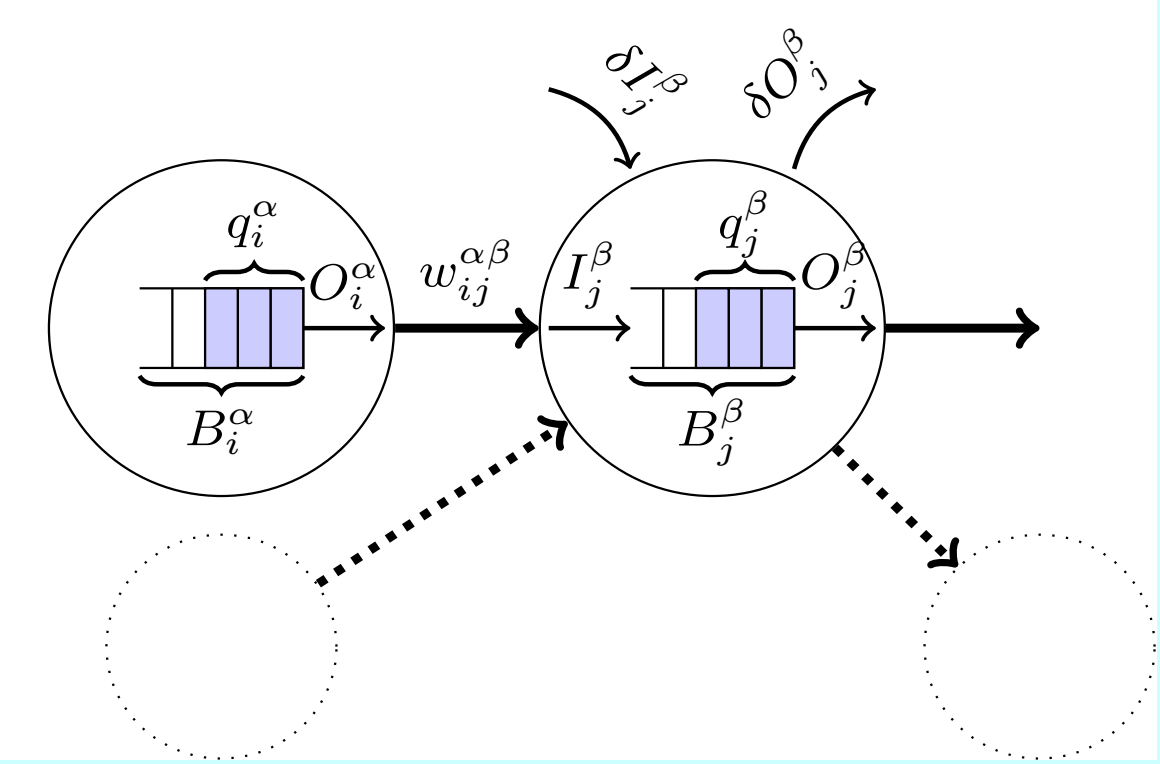
Slower is faster effects and multiplex Braess' paradox

Manfredi, DiTucci, Latora, PRL 120, 068301 (2018)

Multimodal mobility networks



Mobility networks with finite storage capacity



$q_i^\alpha(t)$ Queue length at node

B_i^α Node capacity (buffer size)

$q_i^\alpha(t) \le B_i^\alpha$

Cost function

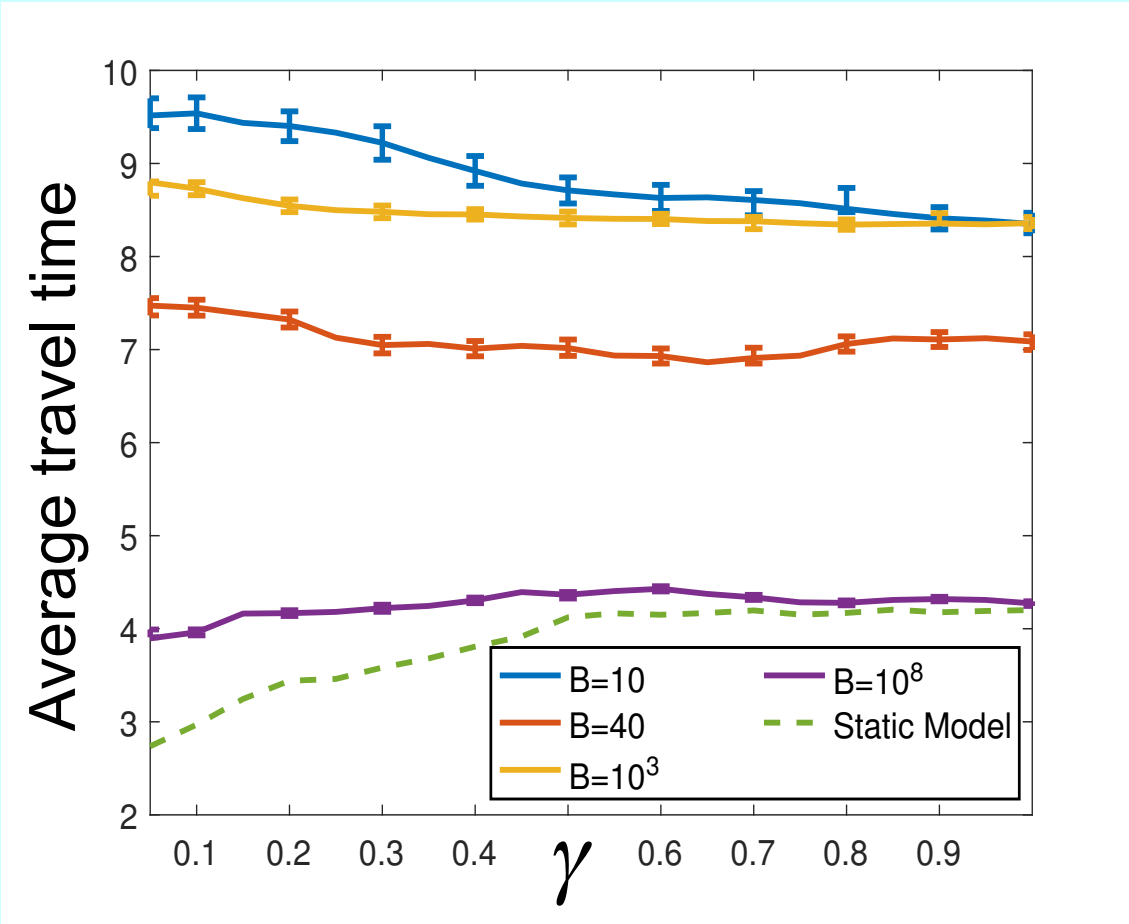
$$w_{ij}^{\alpha\beta}(t) = c \cdot \gamma_{ij}^{\alpha\beta} + \frac{q_j^\beta(t)}{B_j^\beta - q_j^\beta(t)}$$

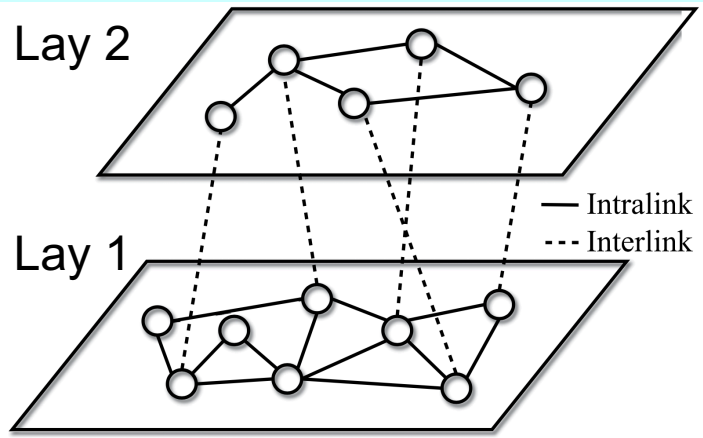
Time to traverse the link

Congestion cost

The slower is faster effect

2 layers, $N^1=150$, $N^2=30$



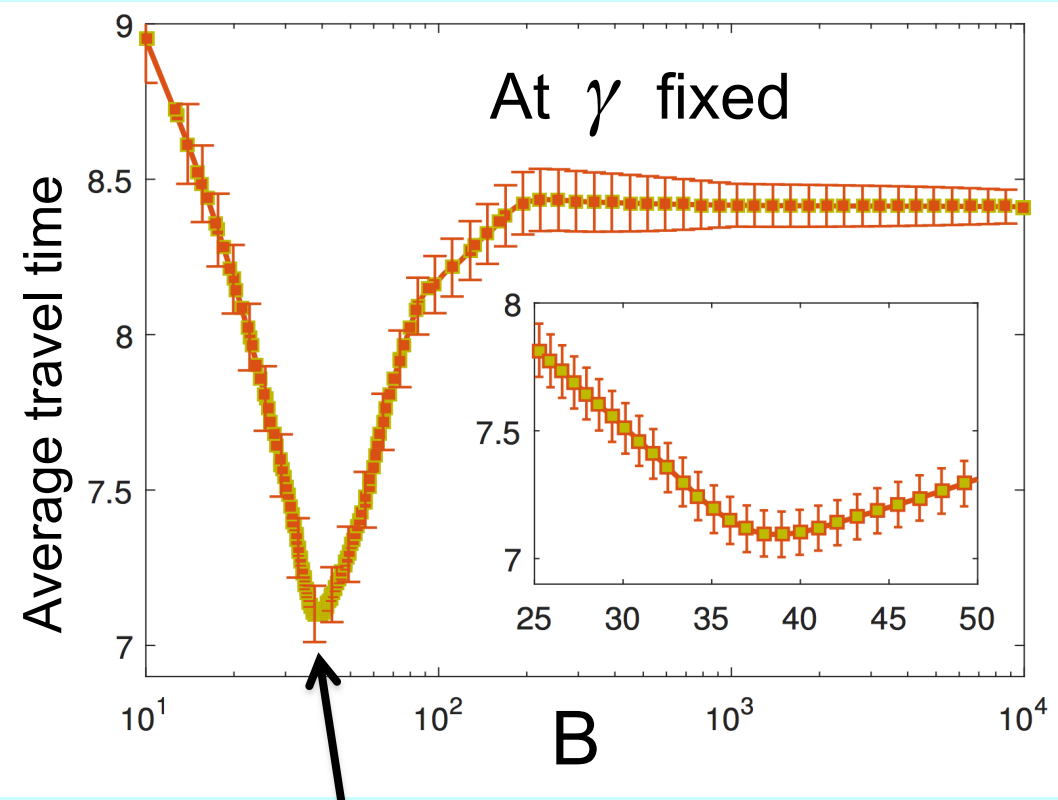


$$\gamma_{ij}^{11} = \gamma_{ij}^{12} = 1 \qquad \forall i, j$$

$$\gamma_{ij}^{22} = \gamma \qquad 0 < \gamma \leq 1$$

Manfredi, DiTucci, Latora, PRL 120, 068301 (2018)

Braess' paradox

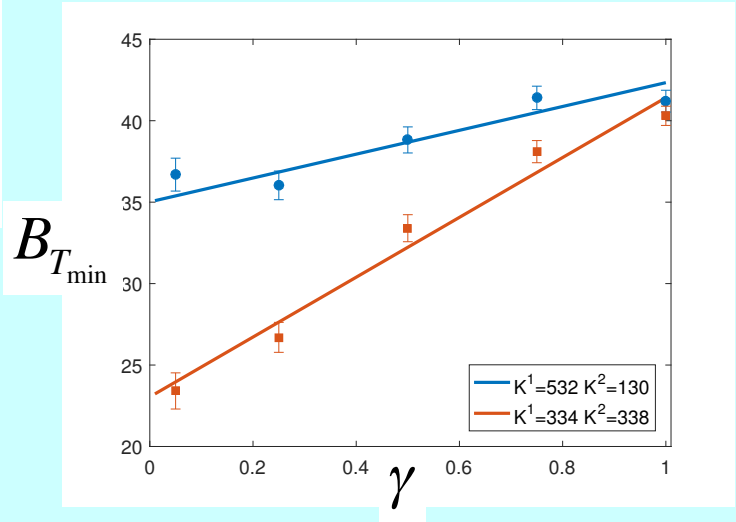


Optimal value of the buffer
(which depends on γ)

Analytical prediction:

$$B_{T_{\min}} \approx \frac{\bar{\gamma} Q \bar{h}}{N^1 + N^2 - \frac{Q\bar{h}}{O}}$$

$$\bar{\gamma} = \frac{\gamma^{11} K^1 + \gamma^{22} K^2 + \gamma^{12} K^{12}}{K^1 + K^2 + K^{12}}$$



Manfredi, DiTucci, Latora, PRL 120, 068301 (2018)

DIFFERENT DYNAMICS

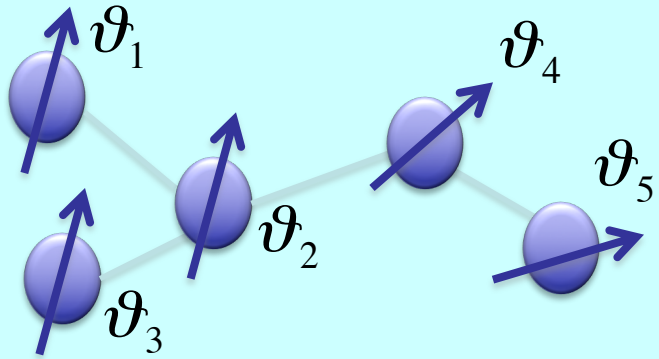
Two different types of dynamics

Explosive synchronisation induced by multiplexity

Nicosia, Skardal, Arenas, Latora, PRL 118, 138302 (2017)

DYNAMICS 1: Synchronization of coupled oscillators

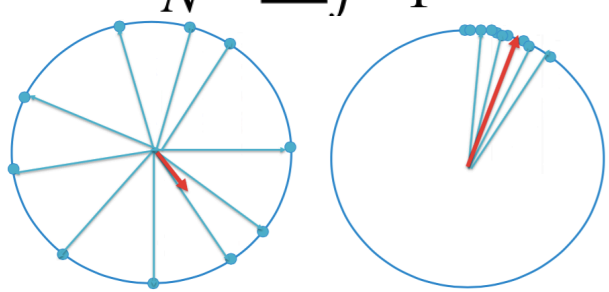
N Kuramoto oscillators

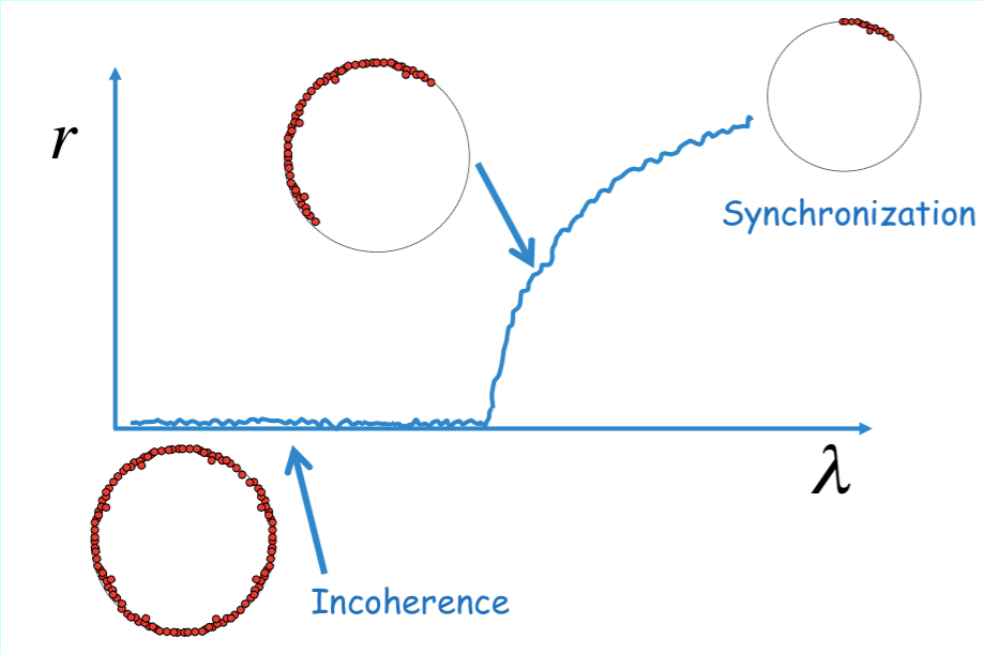


$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i)$$

Order parameter

$$r = \frac{1}{N} \left| \sum_{j=1}^{N} e^{i\theta_j} \right|$$

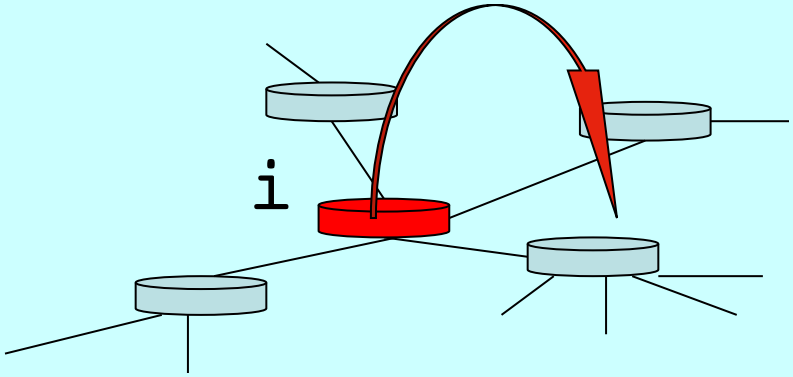


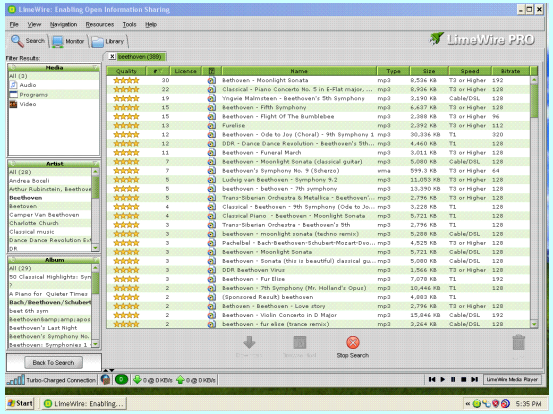


Phase transition (continuous)

DYNAMICS 2 : Random walks

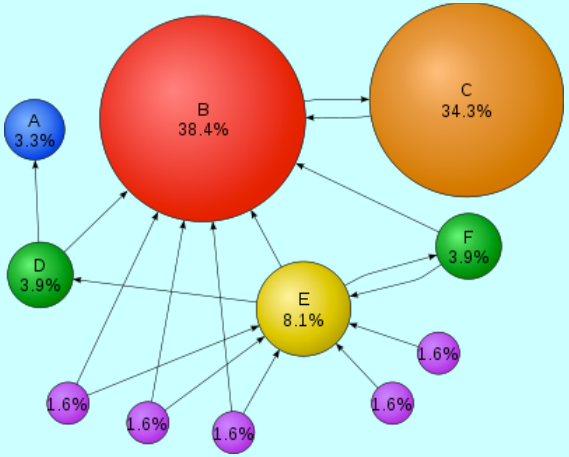
Exploration….with only local information





Searching items on p2p nets

$$\pi_{ji} = prob(i \rightarrow j) = \frac{a_{ij}}{\sum_l a_{il}} = \frac{a_{ij}}{k_i}$$



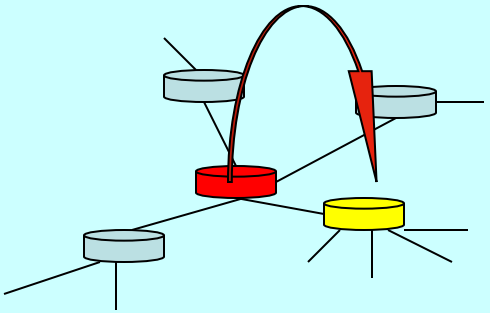
Ranking Web pages

Degree-biased random walks

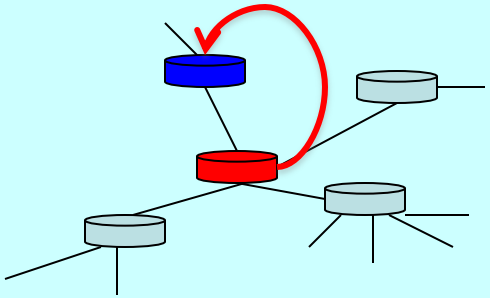
$$\pi_{ji} = \frac{a_{ij} k_j^{\alpha}}{\sum_l a_{il} k_l^{\alpha}}$$

J. Gomez-Gardenes, V. Latora,
Phys. Rev. E 78, 065102R(2008)

$\alpha > 0$ Walkers prefer high-degree nodes

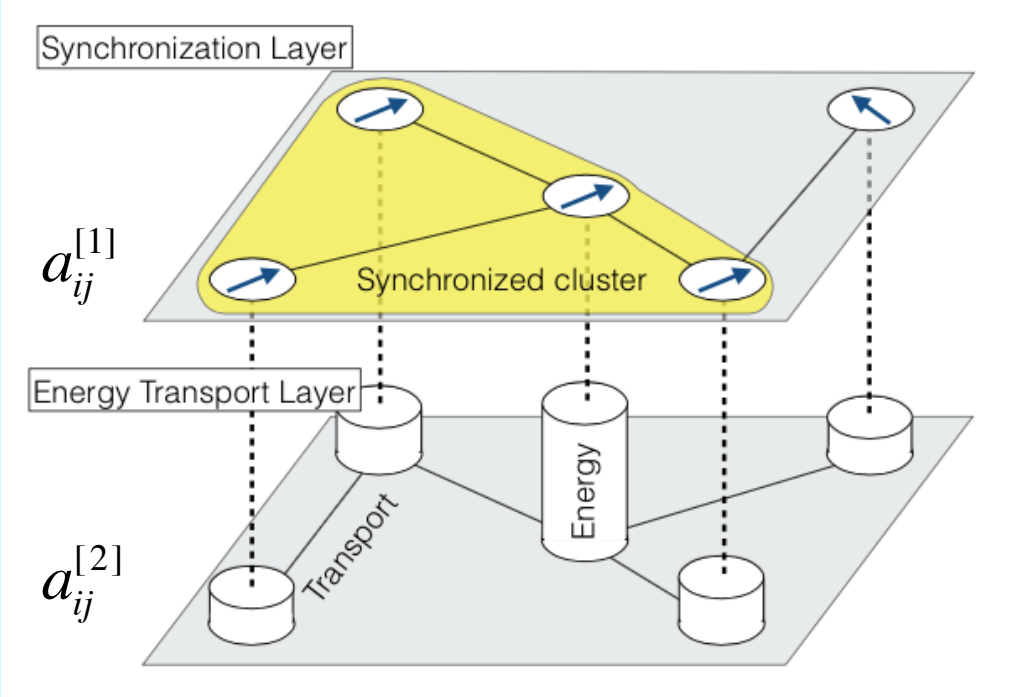


$\alpha < 0$ Walkers prefer low-degree nodes



$\alpha = 0$ we get back to unbiased (plain) random walk

OUR MODEL: Random walkers + Kuramoto



LAYER 1 Kuramoto dynamics

$$\dot{\vartheta}_i = \omega_i + \lambda \sum_j a_{ij}^{[1]} \sin(\vartheta_j - \vartheta_i)$$

$$\omega_i \propto \# walk_i^{[2]}$$

LAYER 2 Biased random walks

$$\Pi_{i \to j} \propto a_{ij}^{[2]} (s_j^{[1]})^{\alpha}$$

Bidirectional dependence

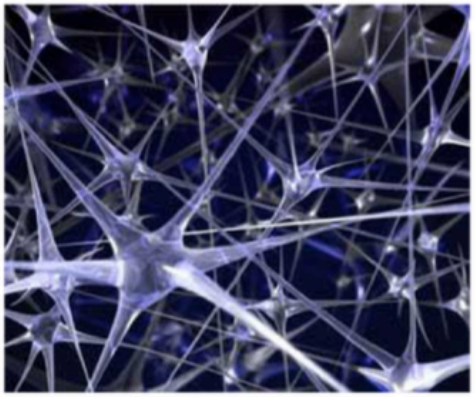
Frequencies at L1 depends on # of walkers at L2

Walk at L2 depends on local synchronisation at L1

Nicosia, Skardal, Arenas, Latora, PRL 118, 138302 (2017)

Coupled networks in the brain

LAYER 1 Kuramoto dynamics **LAYER 2 Biased random walks**



+

Neurons (Activity) **Blood vessels (Energy)**

Random walkers + Kuramoto dynamics

$$\begin{cases} \dot{x}_i = F_{\omega_i}(\mathbf{x}, A^{[1]}) \\ \dot{y}_i = G_{\chi_i}(\mathbf{y}, A^{[2]}) \end{cases}$$

$$\begin{cases} \dot{\omega}_i = f(\omega_i, y_i) \\ \dot{\chi}_i = g(\chi_i, x_i) \end{cases}$$

$$\dot{x}_i = \omega_i + \lambda \sum_{j=1}^{N} a_{ij}^{[1]} \sin(x_j - x_i)$$

$$\dot{y}_i = \sum_{j=1}^{N} \left(\frac{a_{ji}^{[2]} \chi_i^{\alpha}}{\sum_l a_{jl}^{[2]} \chi_l^{\alpha}} - \delta_{ij} \right) y_j$$

$$\dot{\omega}_i = \frac{1}{\tau_\omega} \left(N y_i(t) - \omega_i \right)$$

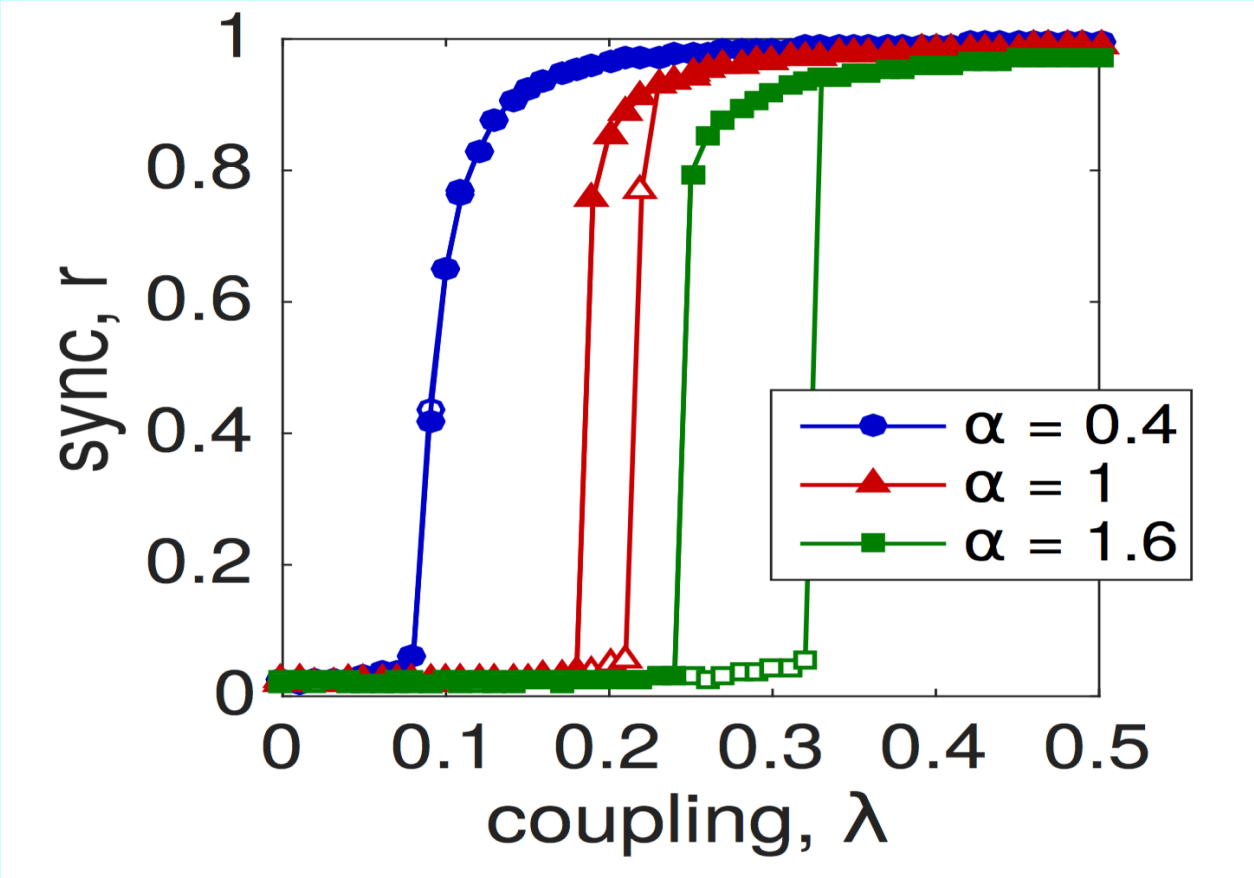
$$\dot{\chi}_i = \frac{1}{\tau_\chi} \left(s_i^{\mathrm{dyn}} - \chi_i \right)$$

Dynamical strength of i:
Level of local synchronisation of oscillator *i*

Explosive synchronization in multiplex nets

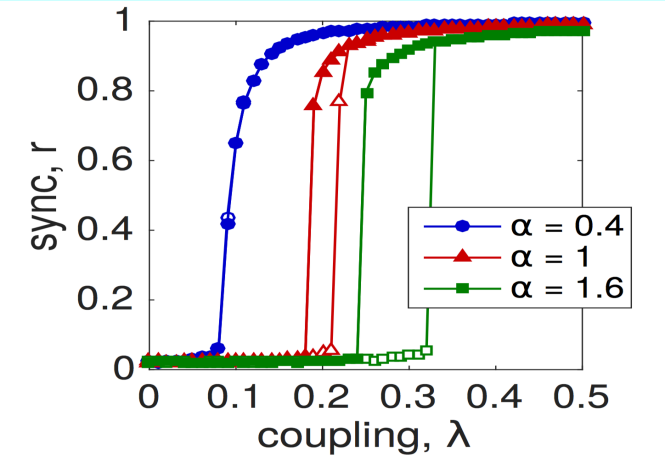
LAYER 1 **Kuramoto dynamics on SF graph**

LAYER 2 **Biased random walks on ER random graph**

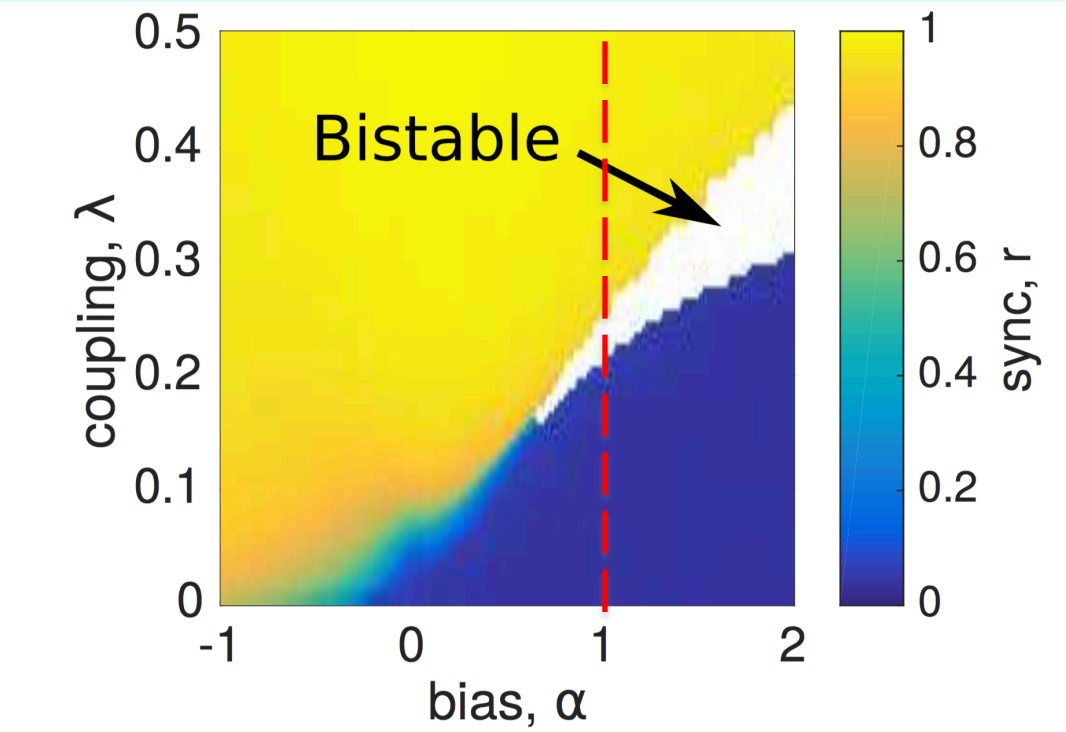


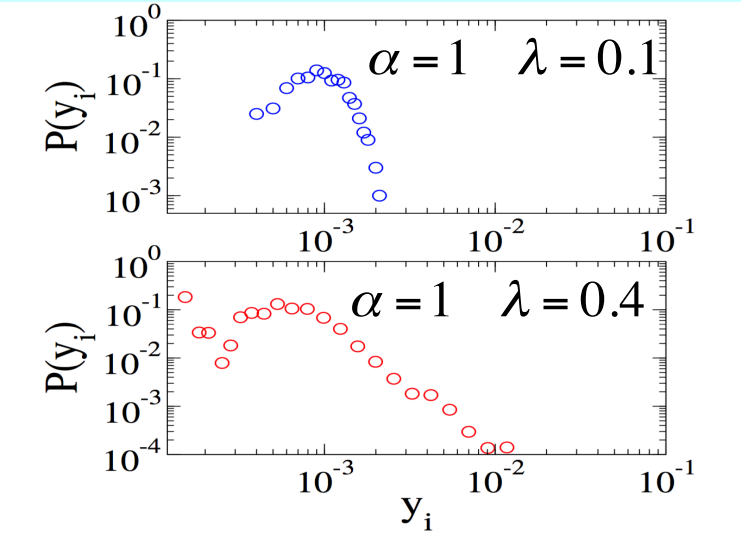
Discontinuous phase transition for large values of the bias exponent (fast switching in the brain?)

Explosive synchronization in multiplex nets



Discontinuous PT at L1

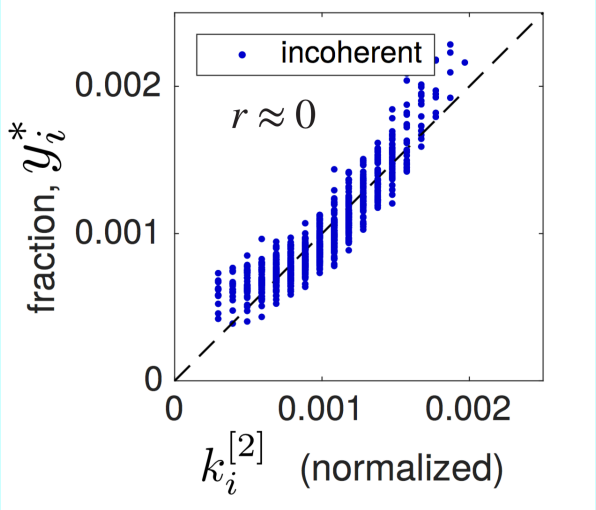


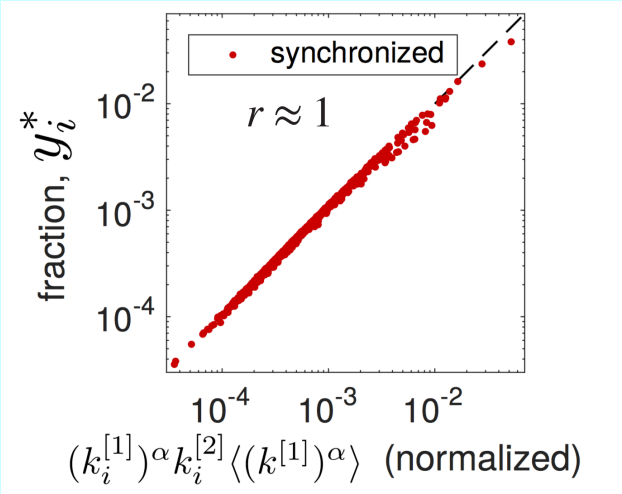


Homogeneous-heterogeneous distribution of random walkers at L2

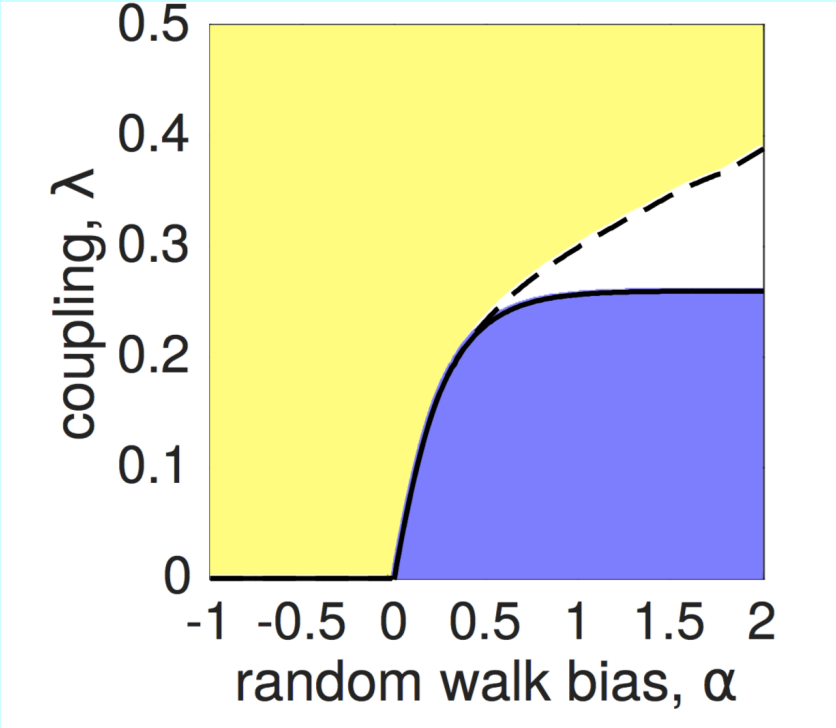
Analytical derivation

$$\# walks_i^{[2]} \propto k_i^{[2]}$$





$$\# walks_i^{[2]} \propto k_i^{[2]} (k_i^{[1]})^\alpha \left\langle (k^{[1]})^\alpha \right\rangle$$



Heterogeneous mean-field

Nicosia, Skardal, Arenas, Latora, PRL 17

Summing up

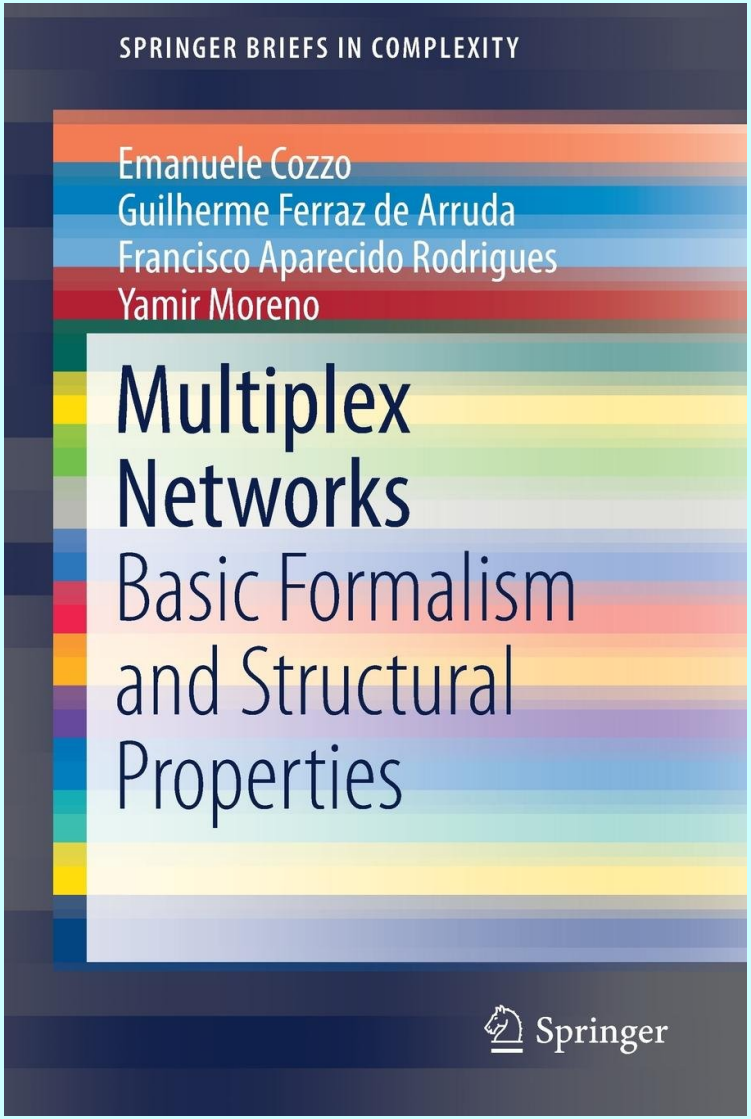
Intrinsically multiplex structural properties

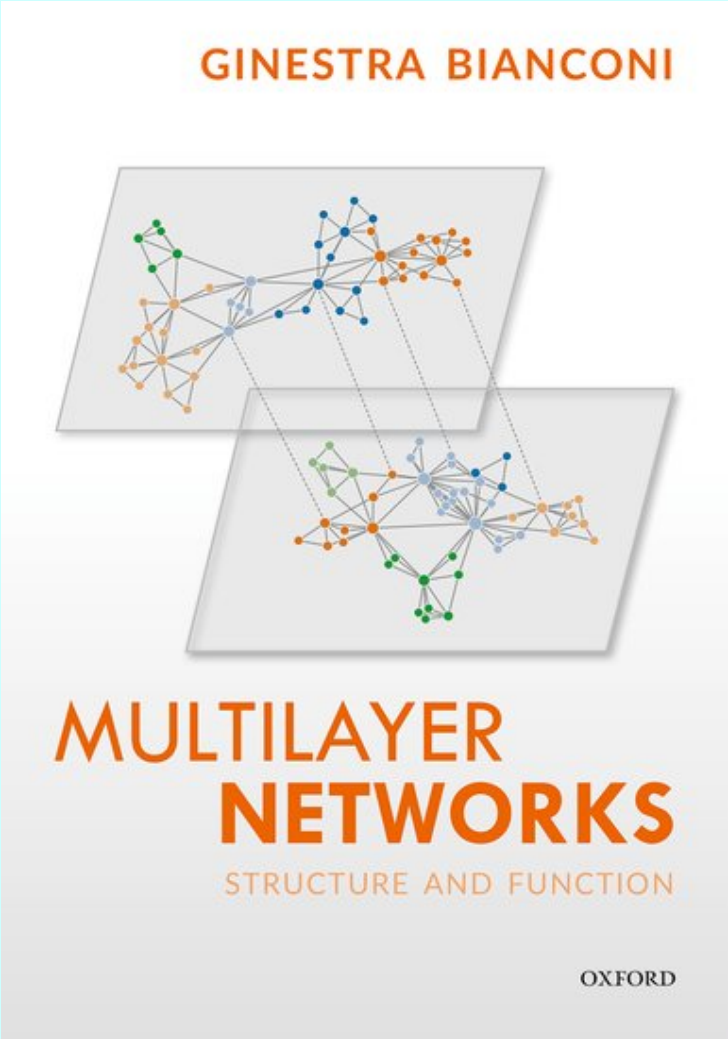
Reducibility of a multiplex

Multiplexity-induced collective behaviors

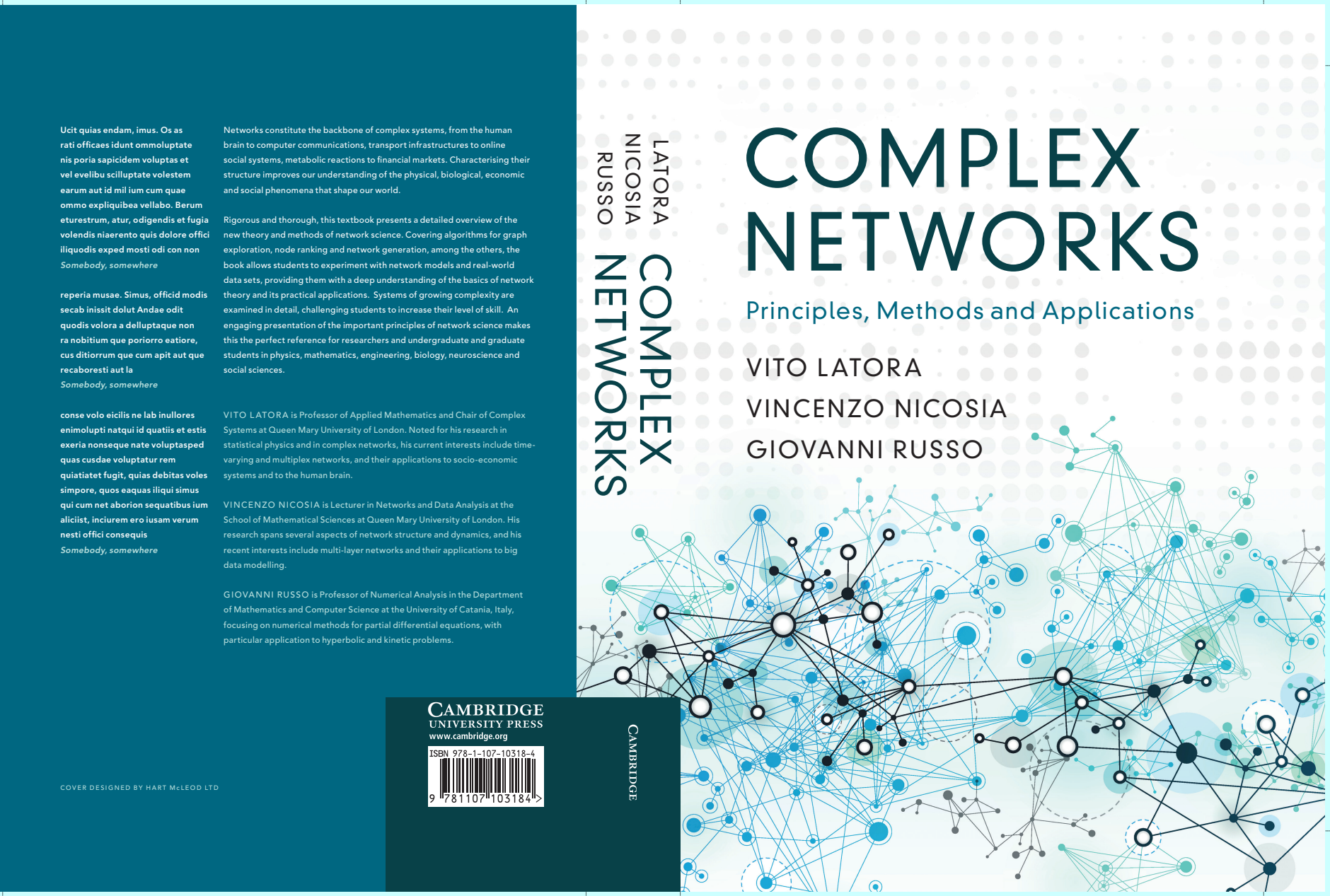
Battiston et al, When multiplex really matters: the emergent dynamics of multiplex networks, soon on ArXiv

Books on multilayer networks





But do not forget:



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Networks constitute the backbone of complex systems, from the human brain to computer communications, transport infrastructures to online social systems, metabolic reactions to financial markets. Characterising their structure improves our understanding of the physical, biological, economic and social phenomena that shape our world.

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