When space and time are not on equal footing

Anisotropic scale invariance

Daniel K. Brattan[1]

[1]Istituto Nazionale di Fisica Nucleare - Sezione di Genova
Via Dodecaneso, 33 - 16146 - Genova - Italy

danny.brattan@gmail.com

Scale invariance in physics (part I)

- Property of systems where there are no scales. Can occur at low and high energies

$$(i\gamma^\mu\partial_\mu + m)\,\psi = 0 \quad \xrightarrow{E\gg m} \quad i\gamma^\mu\partial_\mu\psi = 0\ ,$$
$$\partial^2\psi + m^2\psi = 0 \quad \xrightarrow{E\ll m} \quad i\partial_t\psi + \frac{1}{2m}\vec{\partial}^2\psi = 0\ .$$

- Well known scale transformations:

$$\text{Galilean}: \qquad t \mapsto \Lambda^2 t\ , \qquad x \mapsto \Lambda x\ ,$$
$$\text{relativistic}: \qquad t \mapsto \Lambda t\ , \qquad x \mapsto \Lambda x\ .$$

- Important for renormalisation group where fixed points are scale invariant.

Scale invariance in physics (part II)

- More general scale transformation:

$$\text{anisotropic/Lifshitz}: \qquad t \mapsto \Lambda^z t \,, \qquad x \mapsto \Lambda x \,,$$

  where “z” is called the dynamical critical exponent.
- Found in multiple physical systems:
  - Multicritical points of certain materials.[1]
  - Strongly correlated electron systems.[2]
- More speculatively these symmetries may be relevant for particle physics[3] and quantum gravity.[4]

[1] Hornreich, Luban, and Shtrikman 1975; Grinstein 1981.

[2] Fradkin et al. 2004; Vishwanath, Balents, and Senthil 2004; Ardonne, Fendley, and Fradkin 2004.

[3] Alexandre 2011.

[4] Reuter 1998; Kachru, Liu, and Mulligan 2008; Horava 2009b,a; Gies et al. 2016.

Scale anomalies in low energy effective theories

- Two simple examples

$$\hat{H}_D = \gamma^0 \gamma^j \hat{p}_j - \frac{\lambda}{r} \, , \qquad \hat{H}_S = \frac{\hat{p}^2}{2m} - \frac{\lambda}{r^2} \, . \tag{1}$$

models for e.g. nuclear physics,[5] Aharonov-Bohm[6] and graphene[7] with a charged impurity.

- Boundary conditions break scale invariance → bound states.
- $\lambda < \lambda_c$: continuous scale invariance (CSI) at low energies.
- $\lambda > \lambda_c$: residual discrete scale invariance (DSI)

$$r \mapsto e^{-\frac{2\pi}{\nu}} r \qquad E = E_0 e^{-\frac{2\pi n}{\nu}} \, , \qquad n \in \mathbb{Z} \, . \tag{2}$$

[5] Efimov 1970; Efimov 1971; Braaten and Hammer 2006.

[6] Jackiw and Pi 1990; Bergman and Lozano 1994.

[7] Ovdat et al. 2017.

Some questions...

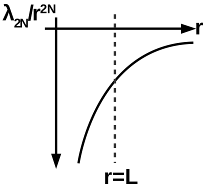
1. What do theories with Lifshitz scale invariance look like?
2. When is the scale symmetry anomalous?
3. What are the consequences of such an anomaly?

Scale covariant Hamiltonians and boundary conditions

- **Question:** what is the spectrum of a scale invariant Hamiltonian?
- Consider Hamiltonians

$$\hat{H}_N = \hat{p}^{2N} + \sum_{i=1}^{2N} \frac{\lambda_i}{r^i} d_r^{2N-i} \, ,$$
$$\hat{H}_N \to \Lambda^{-2N} \hat{H}_N \, .$$

- Requires regularisation due to singularities.
- Hard cut-off ($r = L$) implies vanishing probability current at wall.

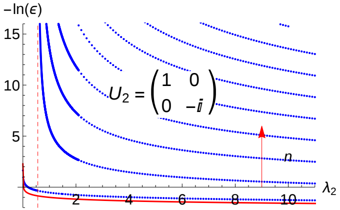


Self-adjoint boundary conditions

- Probability current is a bilinear form and can be diagonalised for boundary conditions.
- Conditions for vanishing of current

$$\vec{\Psi}^+(L) = U_N \vec{\Psi}^-(L) \; .$$

- Compute energy levels and find a (low energy) CSI to DSI transition.



$$\left(d_r^4 - \frac{\lambda_2}{r^4}\right)\Psi(r) = -\epsilon^4 \Psi(r)$$

Deriving the renormalisation group (RG) equation

- **Question:** As L changes can we keep a given state in the spectrum?

$$\vec{\Psi}^{+}(L) = U_N(L)\vec{\Psi}^{-}(L) \, .$$

- **Ans:** Yes. Boundary condition must change as:

$$\begin{aligned} 0 &= [L d_L U_N(L) - C_{+-} + U_N(L) C_{--} - C_{++} U_N(L) \\ &\quad + U_N(L) C_{-+} U_N(L)] \, \vec{\psi}^{-}(L) \, . \end{aligned} \tag{3}$$

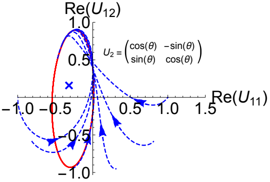
- For $EL^{2N} \ll 1$, the equation is translationally invariant[8]

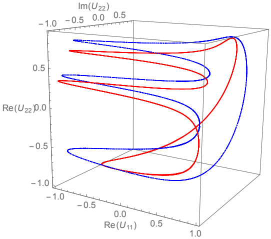
$$\begin{aligned} -iLU_N^{-1} d_L U_N &= iC_{--} - iU_N^{-1} C_{+-} + iC_{-+} U_N \\ &\quad - iU_N^{-1} C_{++} U_N \, . \end{aligned} \tag{4}$$

[8] Mueller and Ho 2004; Camblong et al. 2000; Kaplan et al. 2009.

The end of RG flow

- There are many ways for an RG flow to end[9].
- DSI for $\hat{H}_S = d_r^2 - \lambda_2/r^2$ shown to imply limit cycle behaviour[10].
- First example of an isolated limit cycle in quantum theory.
- New types of flow suggested: limit tori.
- Energy spectra for limit tori mildly displaced from a limit cycle.





[9] Wilson and Kogut 1974; Cambel 1993.

[10] Braaten and Phillips 2004; Kolomeisky and Straley 1992.

Summary

Conditions on $\{\Delta_i\}$		Characteristic RG picture
all roots on symmetry line ($\text{Re}\,[z] = N - 1/2$)	$\text{Im}\,[\Delta_i]\,/\,\text{Im}\,[\Delta_j] \in \mathbb{Q}$	Many limit cycles with DSI
	$\text{Im}\,[\Delta_i]\,/\,\text{Im}\,[\Delta_j] \notin \mathbb{Q}$	Many limit tori
some roots off the symmetry line ($\text{Re}\,[z] = N - 1/2$)	for $\text{Re}\,[\Delta_i] = N - 1/2$ if $\text{Im}\,[\Delta_i]\,/\,\text{Im}\,[\Delta_j] \in \mathbb{Q}$	Isolated limit cycles with DSI
	for $\text{Re}\,[\Delta_i] = N - 1/2$ if $\text{Im}\,[\Delta_i]\,/\,\text{Im}\,[\Delta_j] \notin \mathbb{Q}$	Isolated limit torus
no roots on symmetry line ($\text{Re}\,[z] = N - 1/2$)		2^N fixed points

- **Question:** what happens when $L \to 0$? **Ans:** Zero modes, $\psi \sim r^{\Delta}$.
- Presence of DSI can be determined from zero mode power laws.
- DSI to CSI transitions are a feature of local scale invariance.
- They result in geometric towers of energy and RG flow limit cycles.

Supersymmetry to the rescue?

- **Question:** superconformal inverse square potential is scale invariant. Is this true for anisotropic scale invariance?
- Relevant to e.g. novel forms of supersymmetry breaking[11] and black hole physics.[12]
- Want to write $\hat{H}_N = \hat{h}_N^{(\mathrm{b})} = \hat{q}_N^\dagger \hat{q}_N$.
- Ability to do this encoded in zero modes of $\hat{h}_N^{(\mathrm{b})}$:

$$\hat{h}_N^{(\mathrm{b})}\psi(r) = 0 \qquad \Rightarrow \qquad \psi(r) \sim r^\Delta \,. \tag{5}$$

- Define a new superpartner Hamiltonian: $\hat{h}_N^{(\mathrm{f})} = \hat{q}_N \hat{q}_N^\dagger$.

[11] Falomir and Pisani 2005.

[12] Okazaki 2015.

$\mathcal{N} = 2$ formalism

- Define formally self-adjoint supercharges:

$$\hat{Q}_N^+ = \begin{pmatrix} 0 & \hat{q}_N^\dagger \\ \hat{q}_N & 0 \end{pmatrix} , \quad \hat{Q}_N^- = \begin{pmatrix} 0 & -i\hat{q}_N^\dagger \\ i\hat{q}_N & 0 \end{pmatrix} .$$

- These give the super-Hamiltonian:

$$\hat{H}_N = \left(\hat{Q}_N^+\right)^2 = \left(\hat{Q}_N^-\right)^2 = \begin{pmatrix} \hat{h}_N^{(\mathrm{b})} & 0 \\ 0 & \hat{h}_N^{(\mathrm{f})} \end{pmatrix} ,$$

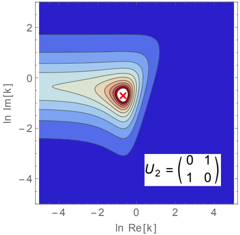
$$\left[\hat{H}_N, \hat{Q}_N^\pm\right] = 0 .$$

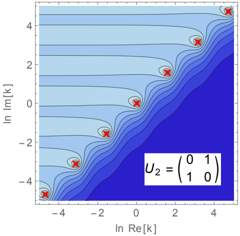
- Now make sure $\hat{Q}_N^\pm$ is self-adjoint:

$$\int_{r=L}^{\infty} dr \left[\vec{\Phi}^\dagger(r)\hat{Q}_N^+\vec{\Psi}(r) - \left(\hat{Q}_N^+\vec{\Phi}(r)\right)^\dagger \vec{\Psi}(r)\right] . \qquad (6)$$

Results of supersymmetrisation

- Self-adjoint extensions restricted to $U_m^2 = \mathbb{1}_m$.
- Bound state spectrum is empty as $\langle \Psi | \hat{H} | \Psi \rangle \geq 0$.
- However, quasi-bound states appear in spectrum unless U_m is diagonal.
- Uppermost image is a Hamiltonian with real power laws for the zero modes. Lowermost has complex modes and a geometric tower.





Onward to fields...

- **Question:** having considered non-local linear interactions, what about local non-linear interactions?
- Primarily interested in introducing interactions to the free field theory

$$\mathcal{L} = \frac{i}{2}\left(\phi^* \partial_t \phi - \partial_t \phi^* \phi\right) - \frac{1}{2m} |\vec{\partial}^2 \phi|^2 \, . \qquad (7)$$

- Quartic dispersion relations ($\epsilon \sim \vec{p}^4$)
  - ultracold gases with shaken optical lattices,[13]
  - fermions in biased bilayers of graphene,[14]
  - heavy fermion metals.[15]

[13] Miao, Liu, and Zheng 2015; Radić, Natu, and Galitski 2015; Po and Zhou 2015; Wu, Zhou, and Wu 2017.

[14] McCann and Koshino 2013.

[15] Ramires et al. 2012.

Schrödinger scalar with $|\phi|^4$ - a historical review

- Compare our theory of interest to well known action

$$\mathcal{L} = \frac{i}{2}\left(\phi^* \partial_t \phi - \partial_t \phi^* \phi\right) - \frac{1}{2m}|\vec{\partial}\phi|^2 - \frac{\lambda}{4}|\phi|^4 \,. \qquad (8)$$

- Known features:
  1. classically has Schrödinger conformal symmetry and
  2. this is broken at quantum level to give a bound state of two particles.
- Differences to our field theory:
  1. ours has scale but no conformal invariance,
  2. there are many more permissible interactions,
  3. one of which is exactly marginal and
  4. there is a bound state of three particles.

Free theory

- **Question:** what is the most general, scale invariant action with Lifshitz symmetry?
- The classical Hamiltonian density $\mathcal{H}$ is

$$\frac{1}{2m}\left|\vec{\partial}^2\phi + \zeta_4|\phi|^2\phi\right|^2 + \lambda_4|\phi|^2|\vec{\partial}\phi|^2 + \frac{1}{3!^2}\left(\lambda_6 - \frac{3!^2|\zeta_4|^2}{2m}\right)|\phi|^6\,. \quad (9)$$

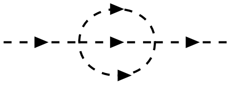
- Free propagator

$$G_F(r,t) \quad = \quad -i\theta(t)\int_0^\infty \frac{d\|\vec{k}\|}{(2\pi)}\|\vec{k}\|J_0(\|\vec{k}\|r)e^{i\left(\frac{\vec{k}^4}{2m}t\right)}\,. \quad (10)$$

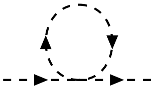
- Spatial correlator

$$\left\langle 0|[\phi(t,\vec{x}),\phi^\dagger(t,\vec{y})]|0\right\rangle \quad \sim \quad \frac{\delta(\|\vec{x}-\vec{y}\|)}{\|\vec{x}-\vec{y}\|}\,. \quad (11)$$

Exact two-point function



Charge conservation violating



Vanishes under normal ordering

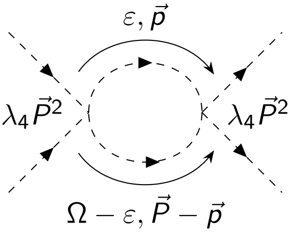
- The bare propagator takes the form

$$\Gamma_B^2(\vec{p}, \varepsilon) = \varepsilon - \frac{1}{2m}\vec{p}^4 - \Sigma(\vec{p}, \varepsilon) - \Pi(\vec{p}, \varepsilon)\ . \qquad (12)$$

- Thus the exact 1PI-propagator is

$$\Gamma_R^2(\vec{p}, \varepsilon) = \varepsilon - \frac{1}{2m}\vec{p}^4\ . \qquad (13)$$

2-body interactions - part I



.

- Need to evaluate

$$\int^{\tilde{\Lambda}} \frac{dx d\theta}{(2\pi)^2} \frac{x}{1 - (X_2^2 - 2X_2 x \cos(\theta) + x^2)^2 - x^4 + i\epsilon} \tag{14}$$

where $x = \|\vec{p}\|/(2m\Omega)^{1/4}$, $\tilde{\Lambda} = \Lambda/(2m\Omega)^{1/4}$.

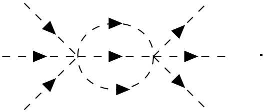
2-body interactions - part II

- Resum

$$-i\Gamma_4^{\mathrm{1PI}} \quad = \quad \frac{-i\lambda_4 P^2}{1+2mi\lambda_4 X_2^2 I_1(X_2)} = P^2 f\left(\frac{\Omega}{P^4}\right) \; . \quad (15)$$

- Lifshitz scaling allows for a non-trivial dependence on the kinematic parameter - unlike relativistic and Schrödinger cases.
- The other two body interaction is not so pleasant $\rightarrow$ logarithmic and quadratic divergences.
- Logarithmic term introduces a scale.

3-body interactions and bound states



- Hard to evaluate even numerically. Zero $\vec{P}$:

$$\int d^2 x_1 I_1(x_1) \sim \ln \Lambda \tag{16}$$

- Resum six-point function

$$\Gamma_6^{\mathrm{1PI}}\Big|_{\vec{P}=\vec{0}} = \frac{\tilde{\lambda}}{1 + \frac{m\tilde{\lambda}}{3!\pi}\left(\frac{1}{32}\ln\left(-\frac{\Lambda^4}{2m\Omega}\right) - \mathrm{Re}[\phi]\right)} \tag{17}$$

Bound state: $\Omega = -\frac{\Lambda^4}{2m}\exp\left(\left(\frac{192}{m\tilde{\lambda}}\right)\pi - 32\,\mathrm{Re}[\phi]\right).$

Summary

- Unlike the Schrödinger theory there is an exactly marginal deformation for the $z = 4$ Lifshitz theory

$$V_{\text{int.}} \sim \lambda_4 |\phi|^2 \|\vec{\partial}\phi\|^2 \tag{18}$$

- We expect this to extend to general "z" for a scalar. However, fermion interaction depends on relative momentum across vertex.
- We conjecture that there are bound states of $(N+1)$-particles for the polynomial interaction in a Lifshitz theory with $z = 2N$.

Lifshitz anyons and the Aharonov-Bohm effect

- **Problem:** realisable potentials given by external gauge field

$$S = \int d^d x dt \, \frac{i}{2} \left(\Psi^* D_t \Psi - (D_t \Psi)^* \Psi \right) + \| \vec{D} \Psi \|^{2N} \quad (19)$$

with $D_\mu = \partial_\mu - ieA_\mu^{\text{ext}}$.

- **A solution:** couple to Chern-Simon's.[16]
- **Extensions:**
  - Non-abelian gauge fields.
  - Competition with anyonic superfluidity.
  - Identification of anomaly coefficients?

[16] Bergman 1994.

While we are on the subject - supersymmetric field theory...

- $(2+1)$-dimensional supersymmetric Schrödinger theory has two types of supersymmetry

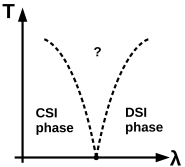
$$\text{kinematic}: \quad \left\{\hat{q}, \hat{q}^\dagger\right\} = \hat{N}_{\text{boson}} + \hat{N}_{\text{fermion}} \ ,$$
$$\text{dynamical}: \quad \left\{\hat{Q}, \hat{Q}^\dagger\right\} = \hat{H} \ .$$

- Combine to give a type of $\mathcal{N} = 2$ SUSY with spatial momentum being a central charge $\rightarrow$ supersymmetric BPS vortices[17] $\rightarrow$ quantum Hall effect.
- **Question:** is there a similar phenomenon for Lifshitz scaling symmetry?
- **Problem:** central charge $\sim P^N$ which is a non-linear operator.

[17] Tong and Turner 2015.

Thermal partition functions, thermal corrections and the Casmir effect

- **Problem:** what are the finite temperature consequences of the QCP transition?
  - Mutual fixed point annihilation at $T = 0$ gives BKT-like transition.
  - Application: may be responsible from transition[18] in QED_3. New phase[19] ?
  - Virial expansion and Tan contact term[20] .
- Need $\lambda \sim \lambda(T)$.



[18] Coleman and Schofield 2005.

[19] Herbut 2016.

[20] Ordonez 2016.

Fractional quantum mechanics

- **Problem:** more types of scale invariance in quantum theory than represented by integer powers e.g.

$$\left[(-\Delta)^{\frac{\alpha}{2}} + V(r)\right]\psi(\vec{r}) = E\psi(\vec{r}) \,. \tag{20}$$

  Is there DSI for these systems?
  - Inherently non-local test on behaviour of CSI to DSI transitions.
  - Can occur in systems where $m(\vec{k}) \sim k^{\alpha}$.
- **Solution:**
  - Attempt to generalise general solutions to equation of motion (hard).
  - Multi-fractional spacetime[21] approach.

[21] Calcagni 2012.

Thanks for listening!