Aspects in Coded Aperture Imaging
Hadamard Matrices Basic Theory

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Outline

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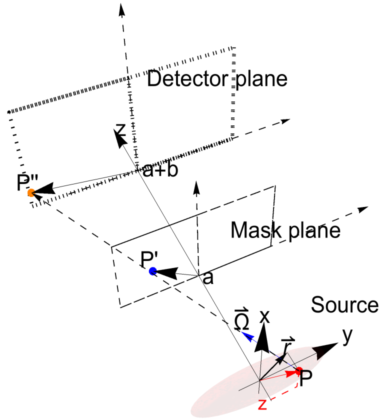
Geometrical setting



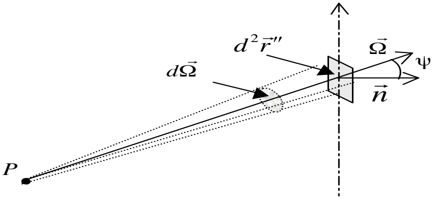
Physical Settings

- The light **source function** $\mathcal{O}\left(x, y, z, \hat{\Omega}\right) = \mathcal{O}\left(\boldsymbol{r}, z, \hat{\Omega}\right)$: averaged number of photons per unit volume and solid angle emitted at $(\boldsymbol{r}, z)$ in the $\hat{\Omega}$ direction.
- In principle $\mathcal{O}$ may be a function on time. We will consider only static situations.
- The **aperture function** $A\left(\boldsymbol{r}'\right)$: mask transmission function at $z = a$.
- The ideal mask is supposed to be of null thickness and in the vacuum.
- $A\left(x', y'\right) \rightarrow \{0, 1\}$ (incoherent mask)
- The **image density function** $\mathcal{P}\left(\boldsymbol{r}''\right)$: average number of photons getting the detector on the plane $z = a + b$, in the infinitesimal area $dx''\, dy''$ centred in the point (x'', y'').
- The **directional attenuation factor** $\mathcal{M}\left(\boldsymbol{r}, z, \hat{\Omega}\right) = \exp\left[- \int_P^{P''} \mu_\lambda\left(s, \hat{\Omega}\right)\, ds\right]$

Imaging

$$\mathcal{P}\left(\boldsymbol{r}''\right) = \int_{Source} \mathcal{O}\left(\boldsymbol{r}, z, \hat{\Omega}\right) A\left(\boldsymbol{r}'\right) \mathcal{F}\left(\boldsymbol{r}, z, \hat{\Omega}\right) \mathcal{M}\left(\boldsymbol{r}, z, \hat{\Omega}\right) d\boldsymbol{r}\, dz\,,$$

$$\boldsymbol{r}' = \frac{b\,\boldsymbol{r} + (a-z)\,\boldsymbol{r}''}{a+b-z}, \quad \mathcal{F}\left(\boldsymbol{r}, z, \hat{\Omega}\right) = \frac{\cos^3\psi\left(\boldsymbol{r}, z, \hat{\Omega}\right)}{(a+b-z)^2}$$



Simplifications

- the *source is isotropic* : $\mathcal{O}\left(\boldsymbol{r}, z, \hat{\Omega}\right) = \frac{\mathcal{O}(\boldsymbol{r},z)}{4\pi}$,
- the *source is planar* and *it is located at* $z = 0$: $\mathcal{O}\left(\boldsymbol{r}, z\right) = O\left(\boldsymbol{r}\right)\,\delta\left(z\right)$,
- the medium is perfectly transparent: $\mathcal{M}\left(\boldsymbol{r}, z, \hat{\Omega}\right) \equiv 1$
- $\tilde{O}\left(\boldsymbol{\xi}\right) = \Re\left(O\right) = O\left(-\frac{b}{a}\boldsymbol{r}\right)$

$$P\left(\boldsymbol{r}''\right) = \int_{Source'} \tilde{O}\left(\boldsymbol{\xi}\right)\, A\left(\frac{a}{a+b}\left(\boldsymbol{r}'' - \boldsymbol{\xi}\right)\right) \cos^3\psi\left(\boldsymbol{\xi}, \boldsymbol{r}''\right) d\boldsymbol{\xi}\,.$$

$$\cos^3\psi\left(\boldsymbol{\xi}, \boldsymbol{r}''\right) = \cos^3\left(\arctan\left(\frac{|\boldsymbol{r}'' + \frac{a}{b}\boldsymbol{\xi}|}{a+b}\right)\right) = \left[1 + \frac{|\boldsymbol{r}'' + \frac{a}{b}\boldsymbol{\xi}|^2}{(a+b)^2}\right]^{-\frac{3}{2}}$$

Far Field

$$|\boldsymbol{r}'' - \boldsymbol{r}| \ll a + b,$$

$$P\left(\boldsymbol{r}''\right) = \tilde{O} \otimes \tilde{A} = \int_{Source'} \tilde{O}\left(\boldsymbol{\xi}\right) \tilde{A}\left(\boldsymbol{r}'' - \boldsymbol{\xi}\right) d\boldsymbol{\xi}\,,$$

Decoding Kernel G : $\tilde{A} \otimes G = \delta\left(\boldsymbol{\eta} - \boldsymbol{\xi}\right)$

$$\tilde{O}\left(\boldsymbol{\eta}\right) = P \otimes G$$

If the mask function A has a discretised form, as for the coding URA masks (generalizations), a discrete approximation of the decoding kernel G is ready provided

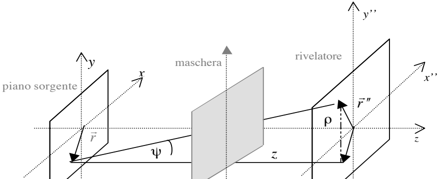
Near Field

an expansion parameter

$$\zeta = \frac{a}{b}\frac{|\boldsymbol{\xi}|}{|\boldsymbol{r}''|} = \frac{|\boldsymbol{r}|}{|\boldsymbol{r}''|},$$

$$\left[1 + \frac{|\boldsymbol{r}'' + \frac{a}{b}\boldsymbol{\xi}|^2}{(a+b)^2}\right]^{-\frac{3}{2}} = \left(1 + \frac{|\boldsymbol{r}''|^2}{(a+b)^2}\right)^{-3/2}\left[1 + \sum \frac{f_n\left(\frac{|\boldsymbol{r}''|}{(a+b)}, \hat{r}''\cdot\hat{\xi}\right)}{\left(1+\left(\frac{|\boldsymbol{r}''|}{(a+b)}\right)^2\right)^n}\zeta^n\right] \approx$$

$$\left(\mathbf{1} + \frac{|\boldsymbol{r}''|^{\mathbf{2}}}{(a+b)^{\mathbf{2}}}\right)^{-\mathbf{3}/\mathbf{2}}\left[\mathbf{1} - \frac{\mathbf{3}\left(\frac{|\boldsymbol{r}''|}{(a+b)}\right)^{\mathbf{2}}}{\mathbf{1}+\left(\frac{|\boldsymbol{r}''|}{(a+b)}\right)^{\mathbf{2}}}\left(\hat{r}''\cdot\hat{\xi}\right)\zeta - \frac{\mathbf{3}\left(\frac{|\boldsymbol{r}''|}{(a+b)}\right)^{\mathbf{2}}\left(\mathbf{1}+\left(\mathbf{1}-\mathbf{5}\left(\hat{r}''\cdot\hat{\xi}\right)^{\mathbf{2}}\right)\left(\frac{|\boldsymbol{r}''|}{(a+b)}\right)^{\mathbf{2}}\right)}{\left(\mathbf{1}+\left(\frac{|\boldsymbol{r}''|}{(a+b)}\right)^{\mathbf{2}}\right)^{\mathbf{2}}}\zeta^{\mathbf{2}}\right]$$



Near Field

$$P\left(\mathbf{r}''\right) = \left(1+\frac{|\mathbf{r}''|^2}{(a+b)^2}\right)^{-3/2}\left[P^0\left(\mathbf{r}''\right)+P^1\left(\mathbf{r}''\right)+P^2\left(\mathbf{r}''\right)\right] = \left(1+\frac{|\mathbf{r}''|^2}{(a+b)^2}\right)^{-3/2} P_C\left(\mathbf{r}''\right)$$

$$\tilde{O}_C\left(\boldsymbol{\eta}\right) = P_C \otimes G = \tilde{O}^0\left(\boldsymbol{\eta}\right) + \tilde{O}^1\left(\boldsymbol{\eta}\right) + \tilde{O}^2\left(\boldsymbol{\eta}\right)$$

$$\tilde{O}^0\left(\boldsymbol{\eta}\right) = \tilde{O}\left(\boldsymbol{\eta}\right)$$

$$\tilde{O}^1\left(\boldsymbol{\eta}\right) = K_1\ G^{\psi}\boldsymbol{R}_O \cdot \boldsymbol{R}_{G^{\psi}}$$

$$\boldsymbol{R}_O = \left(\boldsymbol{\xi}\tilde{O}\left(\boldsymbol{\xi}\right)\right) \otimes A,\ G^{\psi} = \cos^2\psi\left(0,\mathbf{r}''\right) G\left(\mathbf{r}''+\boldsymbol{\eta}\right), \quad \boldsymbol{R}_{G^{\psi}} = \int \mathbf{r}'' G^{\psi} d\mathbf{r}''.$$

$$\tilde{O}^2\left(\boldsymbol{\eta}\right) = K_2\,\mathcal{I}_O \int |\mathbf{r}''|^4 \cos^4\psi\left(0,\mathbf{r}''\right) G\left(\mathbf{r}''+\boldsymbol{\eta}\right) d\mathbf{r}''$$

$$\mathcal{I}_O = \left(\left(\hat{r}'' \cdot \hat{\xi}\right)^2 \tilde{O}\left(\boldsymbol{\xi}\right)\right) \otimes A$$

URA

Point Sread Function : $PSF = A \otimes G \approx \delta(\boldsymbol{r})$ in an appropriate sense.

$$A = (a(i,j)), \quad a(i,j) = 0, 1$$

a family of nearly square binary arrays with flat PACF sidelobes

$$\phi(l,k) = \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} a(i,j)\, a(i + l \bmod N_x, j + k \bmod N_y) .$$

satisfying the property

$$\phi(l,k) = \begin{cases} K & (l,k) = (0,0) \\ \lambda & \text{otherwise} \end{cases} ,$$

Existence criterion:

$$\left[\sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} a(i,j)\right]^2 = \sum_{l=0}^{N_x-1} \sum_{m=0}^{N_y-1} \phi(l,m)$$

$$K + \lambda(N-1) = K^2 \quad \Rightarrow \quad \lambda = \frac{K(K-1)}{N-1}, \quad N = N_x N_y$$

Abelian difference sets

Difference set
Let be $\emptyset \neq D \subset G$ with $|D| = k$ and $(G, +)$ a finite group.
D is called a $(|G|, k, \lambda)$ - difference set in $(G, +)$, if every nonzero element of G can be written as a difference of elements $d_1 + (-d_2)$, with $d_1, d_2 \in D$, in exactly λ different ways.

If $(G, +)$ is abelian, then D is called an *abelian difference set*.
If $(G, +)$ is cyclic, then D is called a *cyclic difference set*.

Main Theorem

- $(G, +)$ finite abelian group of type $\left(p_1^{r_1}, p_2^{r_2}, \ldots, p_M^{r_M}\right)$ p_i prime
- D be a $(|G|, k, \lambda)$-difference set of $(G, +)$.

There exists an M-dim. incoherent binary array $\mathbf{a}$, with $p_1^{r_1} \times p_1^{r_1} \times \cdots \times p_M^{r_M}$ components, such that the $PACF\,(\mathbf{a})$ is

$$\phi\left(l_1, l_2, \ldots, l_M\right) = \begin{cases} k & l_1 = l_2 = \cdots = l_M = 0, \\ \lambda & \text{otherwise.} \end{cases}$$

Abelian difference sets

Proof

$\forall$ cyclic component $C\left(p_i^{r_i}\right)$ of the group $(G,+)$ $\exists$ generating element μ_i : $\forall g \in G \exists!$ representation

$$g = (x_1\ \mu_1, x_2\ \mu_2, \ldots, x_M\ \mu_M)\,, \qquad x_i \in \left\{0, 1, \ldots, p_i^{r_i} - 1\right\}, \quad i \in \{0, 1, \ldots, M\}$$

Define the M-dimensional array

$$a\,(x_1, x_2, \ldots, x_M) = \begin{cases} 1 & g \in D, \\ 0 & \text{otherwise.} \end{cases}$$

- There are exactly k elements of the matrix a equal to 1.
- $\phi\,(0, 0, \ldots, 0) = k$
- sidelobes

$$\phi\,(l_1, l_2, \ldots, l_M) = \sum_{x_1=0}^{p_1^{r_1}-1} \cdots \sum_{x_M=0}^{p_M^{r_M}-1} a\,(x_1, x_2, \ldots, x_M)\ a\,(x_1 + l_1, x_2 + l_2, \ldots, x_M + l_M)$$

$$\sum_{(x_1\ \mu_1, x_2\ \mu_2, \ldots, x_M\ \mu_M) \in D} a\,(x_1 + l_1, x_2 + l_2, \ldots, x_M + l_M)\,.$$

Abelian difference sets

$((x_1 + l_1)\ \mu_1, (x_2 + l_2)\ \mu_2, \ldots, (x_2 + l_2)\ \mu_M) \in D.$
However,

$$((x_1 + l_1)\ \mu_1, (x_2 + l_2)\ \mu_2, \ldots, (x_2 + l_2)\ \mu_M) - (x_1\ \mu_1, x_2\ \mu_2, \ldots, x_M\ \mu_M) = (l_1\ \mu_1, l_2\ \mu_2, \ldots, l_M\ \mu_M) \in G / \{0\}$$

which is a fixed element of G.

But, by the definition of a difference set, such elements appears exactly λ times. $\square$

Quadratic Residue URA's

$GF(p^m)$ a Galois field with p a prime number and μ a primitive element .
$\forall \alpha \in GF(p^m)$:

- $\alpha = \mu^j$, for a $j \in \{0, 1, \ldots, p^m - 1\}$.
- $\alpha = \sum_{i=0}^{m-1} x_i \, \mu^i$, with $x_i \in \{0, 1, \ldots, p-1\}$.
-
$$a\left(x_1, x_2, \ldots, x_m\right) = \begin{cases} 0 & x_1 = x_2 = \cdots = x_m = 0, \\ 1 & \sum_{i=0}^{m-1} x_i \, \mu^i = \mu^j, \quad j \text{ even}, \\ -1 & \sum_{i=0}^{m-1} x_i \, \mu^i = \mu^j, \quad j \text{ odd}. \end{cases} \tag{1}$$

- $GF(p_1), GF(p_2), \quad p_2 - p_1 = 2$

$$a\left(x, y\right) = \begin{cases} 0 & x = 0, \\ 1 & x \neq 0 \quad y = 0, \\ 0 & x \neq 0 \quad y \neq 0 \quad a_1\left(x\right) a_2\left(y\right) = +1, \\ 1 & x \neq 0 \quad y \neq 0 \quad a_1\left(x\right) a_2\left(y\right) = -1, \end{cases} \tag{2}$$

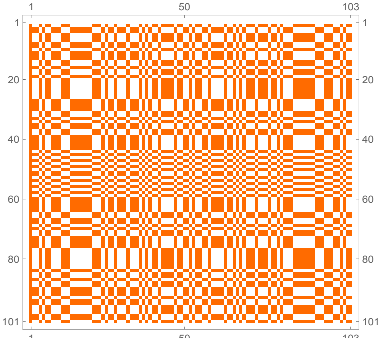
$$a_i\left(x\right) = \begin{cases} 0 & x = 0, \\ 1 & x = \mathrm{mod}_{p_i} \mu_i^2 \\ -1 & x = \mathrm{mod}_{p_i} \mu_i^1 \end{cases}$$, for some generating element μ_i of $GF(p_i)$.

Quadratic Residue URA's

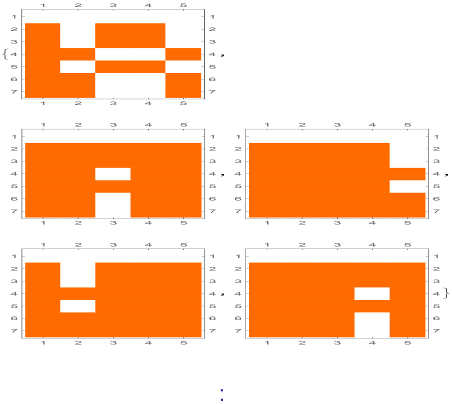
$$g(x,y) = \begin{cases} 1 & a(x,y) = 1 \\ -1 & a(x,y) = 0 \end{cases}.$$

By direct computation

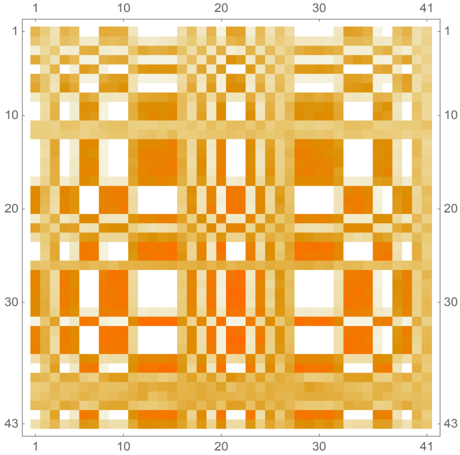
$$\sum_{i=0}^{p_1-1} \sum_{j=0}^{p_2-1} a(i,j)\, g(i+l, j+k) = \begin{cases} \frac{p_1 p_2 - 1}{2} & \text{mod}_{p_1} l = 0 \text{ and } \text{mod}_{p_2} k = 0, \\ 0 & \text{otherwise.} \end{cases}.$$



Quadratic Residue URA's

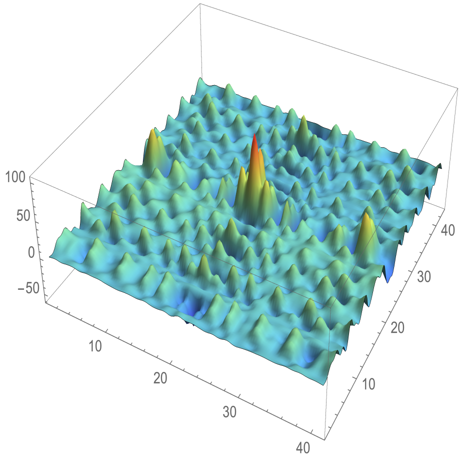


Reconstruction of a Point-like Source



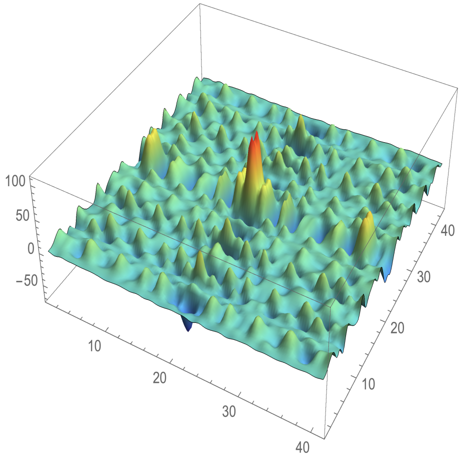
: a= 7.5 cm, b = 3.0 cm, m = 1.4, $\psi_{max} = 0.67$

Reconstruction of a Point-like Source



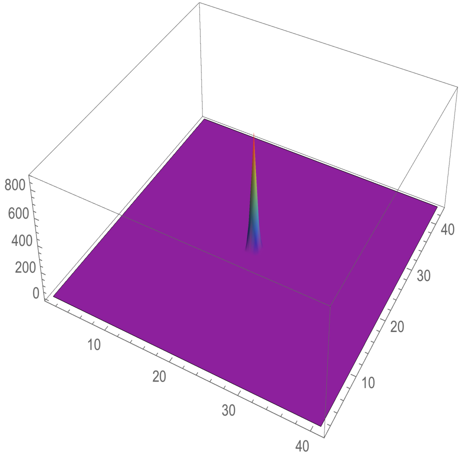
: a= 7.5 cm, b = 3.0 cm,

Reconstruction of a Point-like Source



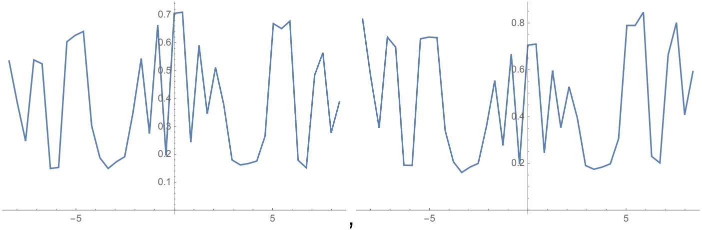
: a= 8.5 cm, b = 3.0 cm,

Reconstruction of a Point-like Source



: $a \to \infty$

Reconstruction of a Point-like Source - 0-ord corrections



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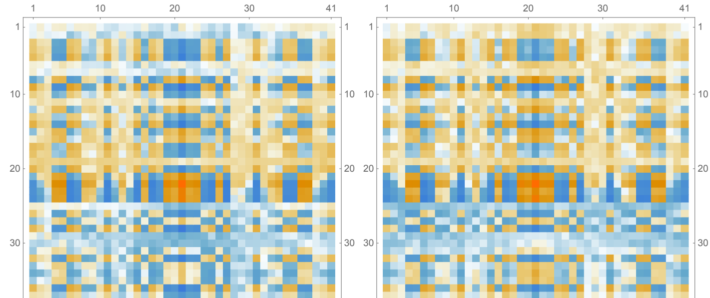
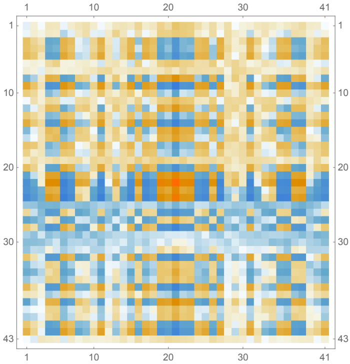


Reconstruction of a Point-like Source - 1-ord corrections



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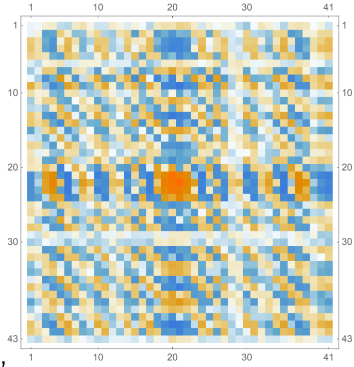
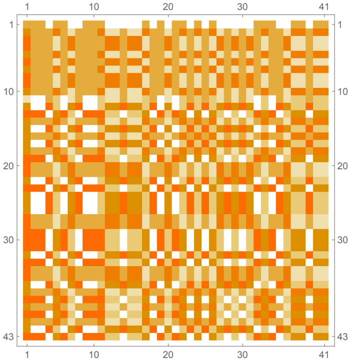


: $\cos^3$ corrections, 1-ord corrections

Composite Point-like Source



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