Crossing symmetry of superstring amplitudes

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Outline: 1. Introduction

Crossing symmetry (1)

Sept. 1968: Veneziano amplitude (model for hadronic interactions)

- key assumption: crossing symmetry
- birth of string theory

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Today

Proof of crossing symmetry for superstring amplitudes at all loops.

In collaboration with: Corinne de Lacroix, Ashoke Sen [1810.07197]

Restriction (technical): 4-point amplitudes of stable states
(massless, BPS, stable non-BPS)

Crossing symmetry (2)

Crossing symmetry:

- relations between amplitudes with exchange of particles/anti-particles in initial/final states
- often assumed or observed

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- relations between amplitudes with exchange of particles/anti-particles in initial/final states
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Why a general proof?

- be sure that observed examples are not accident of simple amplitudes
- learn about fundamental properties of QFT
  (related to causality and locality)

Method

Idea of proof in QFT [Bros-Epstein-Glaser, '64-65]:

1. prove analyticity of S-matrix in "primitive domain" Δ
2. analytic extension $\mathcal{H}(\Delta)$
3. show that 2) $\Rightarrow$ crossing symmetry

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Idea of proof in QFT [Bros-Epstein-Glaser, '64-65]:

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Remarks:

- 1) is non-perturbative (full S-matrix)
- 2) and 3) are general statements from theory of several complex variables
- string theory: prove 1) perturbatively

Implication of analyticity

Starting point for other results:

- analyticity in Jost–Lehman–Dyson domain
- analyticity of elastic forward amplitude ($t = 0$, $s \in \mathbb{C}$)
- dispersion relations (?)
- positivity bounds on amplitudes (?)
- CPT theorem (?)

Outline: 2. Crossing symmetry: QFT

Amplitude and Green functions

4-point scattering process

- $p_a = (E_a, \boldsymbol{p}_a) \in \mathbb{C}$, $a = 1, \ldots, 4$: external momenta
- momentum conservation: $p_1 + \cdots + p_4 = 0$
- on-shell condition: $p_a^2 = -m_a^2$

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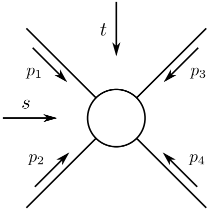
Green functions:

off-shell $\quad G(p_1, \ldots, p_4) =$



truncated $\quad \tilde{G}(p_1, \ldots, p_4) = G(p_1, \ldots, p_4) \prod_{a=1}^{4} (p_a^2 + m_a^2)$

on-shell $\quad A(p_1, \ldots, p_4) = \lim_{p_a^2 \to -m_a^2} \tilde{G}(p_1, \ldots, p_4)$

QFT: G calculated as sum of Feynman diagrams

Physical amplitudes

Mandelstam variables

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 + p_3)^2, \qquad u = -(p_1 + p_4)^2$$

mass-shell: $s + t + u = \sum_a m_a^2$

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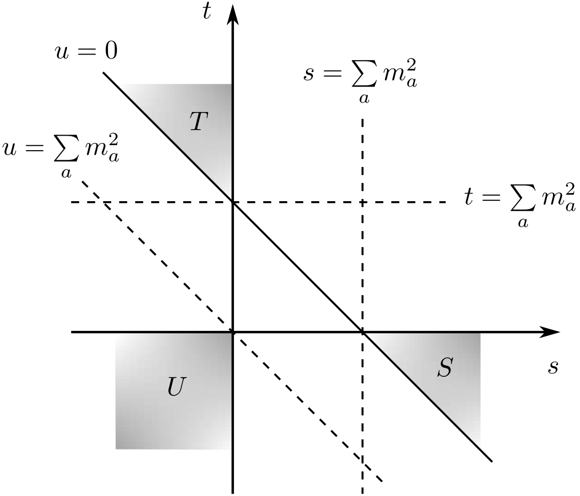
Physical regions

- S (s-channel): $\quad s \geq \sum_a m_a^2, \quad t, u \leq 0$
- T (t-channel): $\quad t \geq \sum_a m_a^2, \quad s, u \leq 0$
- U (u-channel): $\quad u \geq \sum_a m_a^2, \quad s, t \leq 0$

Physical amplitudes

$$A_{S,T,U}(p_1, \ldots, p_4) = \lim_{p_a \in S,T,U} A(p_1, \ldots, p_4)$$

Mandelstam plane



$p_a \in \mathbb{R}$ on-shell

Statement of crossing symmetry

Crossing symmetry

The processes $\begin{array}{l} S : 1+2 \to 3+4 \\ T : 1+\bar{3} \to \bar{2}+4 \\ U : 1+\bar{4} \to 3+\bar{2} \end{array}$ (and CPT conjugates) are equivalent under analytic continuation on the complex mass-shell

$$A_S(s,t) = A_T(t,s), \qquad A_S(s,u) = A_U(u,s)$$

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$$A_S(s,t) = A_T(t,s), \qquad A_S(s,u) = A_U(u,s)$$

- looks natural from LSZ: $A_{S,T,U}$ all come from a single function A
- but: not guaranteed that A is analytic in a domain with paths between between S, T, U

QFT proof (1)

Outline of proof [Bros-Epstein-Glaser '64-65] [Bros '86]:

1. assumptions: $m_a^2 > 0$, asymptotic states = stable particles
2. define the "primitive domains"

$$\Delta_k = \bigcap_{A_\alpha} \Big[\{ \operatorname{Im} P_{(\alpha)} \neq 0, (\operatorname{Im} P_{(\alpha)})^2 \leq 0 \} \\ \cup \{ \operatorname{Im} P_{(\alpha)} = 0, -P_{(\alpha)}^2 < M_\alpha^2 \} \\ \cap \{ \operatorname{Im} p_\alpha^i = 0, i = k, \ldots, D-1 \} \Big]$$

$$A_\alpha \subset \{1, \ldots, n\}, \quad P_{(\alpha)} = \sum_{a \in A_\alpha} p_a, \quad M_\alpha\text{: production threshold}$$

In words: p_a with k possible complex components s.t. all P_α have: 1) non-zero imaginary timelike part, or 2) real momentum squared below multi-particle threshold in channel A_α

QFT proof (2)

3. prove analyticity inside Δ_D of S-matrix from micro-causality (fields commute at spacelike separations) [Araki, Burgoyne, Ruelle, Steimann, '60-61]
   problem: $\Delta_D \cap \text{mass-shell} = \emptyset$
4. compute the "envelope of holomorphy" $\mathcal{H}(\Delta_2)$ (= analytic extension)
   $\rightarrow \mathcal{H}(\Delta_2) \cap \text{mass-shell} \neq \emptyset$
5. show $\exists$ a path in $\mathcal{H}(\Delta_2) \cap \text{mass-shell}$ between all pairs of $\mathrm{i}\epsilon$-neighbourhoods of physical regions

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Notes:

- only $\mathcal{H}(\Delta_2)$ is necessary
- 4) and 5) $\Leftarrow$ theory of several complex variables only
- work with the complete S-matrix

Primitive domain and mass-shell

Proof that $\Delta_D \cap \text{mass-shell} = \emptyset$:

1. complex mass-shell:

$$\operatorname{Re} p_a \cdot \operatorname{Im} p_a = 0, \qquad (\operatorname{Re} p_a)^2 - (\operatorname{Im} p_a)^2 + m_a^2 = 0$$

2. if $\operatorname{Im} p_a$ timelike, $(\operatorname{Im} p_a)^2 \leq 0$, then need $\operatorname{Re} p_a$ timelike, $(\operatorname{Re} p_a)^2 < 0$, for 2nd condition, but violates 1st condition

3. if $\operatorname{Im} p_a = 0$, then $P_{(\alpha)} \geq M_\alpha^2$

Envelope of holomorphy

More on the envelope of holomorphy:

- consider $f(z_1, \ldots, z_n)$ analytic in Δ
- analyticity in several variables $\Rightarrow$ constrain shape of Δ
- if shape not arbitrary: analyticity in $\Delta \Rightarrow$ analyticity in $\mathcal{H}(\Delta)$
- given Δ, $\mathcal{H}(\Delta)$ is independent of f
- typically: use edge-of-the-wedge theorem (Bogoliubov)

Outline: 3. Crossing symmetry: string theory

SFT in a nutshell

SFT = standard QFT s.t.:

- infinite number of fields (of all spins)
- infinite number of interactions
- non-local interactions $\propto \mathrm{e}^{-\# k^2}$
- reproduce worldsheet amplitudes (if well-defined)

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$\rightarrow$ study singularities of Green functions from Feynman diagrams in momentum space

Momentum representation (1)

- string field Fourier expansion

$$|\Psi\rangle = \sum_A \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \phi_A(k)\, |A, k\rangle$$

k, D-dimensional momentum
A, discrete labels (Lorentz indices, group repr., KK modes...)

- 1PI action

$$S = \int \mathrm{d}^D k\, \phi_A(k) K_{AB}(k) \phi_B(-k) + \sum_n \int \mathrm{d}^D k_1 \cdots \mathrm{d}^D k_n\, V^{(n)}_{A_1,\ldots,A_n}(k_1, \ldots, k_n) \phi_{A_1}(k_1) \cdots \phi_{A_n}(k_n)$$

Momentum representation (2)

Propagator

$$K_{AB}(k)^{-1} = \frac{-\mathrm{i}\, M_{AB}}{k^2 + m_A^2}\, Q_A(k)$$

- M_{AB} mixing matrix for states of equal mass
- Q_A polynomial

Momentum representation (3)

Vertices

$$-\mathrm{i}V^{(n)}_{A_1,\ldots,A_n}(k_1,\ldots,k_n) = -\mathrm{i}\int \mathrm{d}t\,\mathrm{e}^{-g^{\{A_a\}}_{ij}(t)\,k_i\cdot k_j - c\sum_{a=1}^{n} m_a^2} \times P_{A_1,\ldots,A_n}(k_1,\ldots,k_n;t)$$

- t moduli parameters
- $P_{\{A_a\}}$ polynomial
- $c > 0 \rightarrow$ damping in sum over states
- g_{ij} positive definite
- no singularity for $k_i \in \mathbb{C}$ (finite)
- $\lim_{k^0\to\pm\mathrm{i}\infty} V^{(n)} = 0$
- $\lim_{k^0\to\pm\infty} V^{(n)} = \infty$

Green function

Truncated Green function = sum of Feynman diagrams of the form

$$\mathcal{F}(p_1, \ldots, p_n) \sim \int \mathrm{d}T \prod_s \mathrm{d}^D \ell_s \, \mathrm{e}^{-G_{rs}(T)\,\ell_r\cdot\ell_s - 2H_{ra}(T)\,\ell_r\cdot p_a - F_{ab}(T)\,p_a\cdot p_b}$$
$$\times \prod_i \frac{1}{k_i^2 + m_i^2} \, \mathcal{P}(p_a, \ell_r; T)$$

T, moduli parameters, $\mathcal{P}$, polynomial in (p_a, ℓ_r)

- momenta:
  - external $\{p_a\}$
  - internal $\{k_i\}$
  - loop $\{\ell_s\}$

  k_i = linear combination of $\{p_a, \ell_s\}$
- G_{rs} positive definite
  - integrations over spatial loop momenta $\boldsymbol{\ell}_r$ converge
  - integrations over spatial loop momenta ℓ_r^0 diverge

Momentum integration

Prescription = generalized Wick rotation [1604.01783, Pius-Sen]:

1. define Green function for Euclidean internal/external momenta
2. analytic continuation of external energies + integration contour s.t.
   - keep poles on the same side
   - keep ends at $\pm i\infty$

$\to$ analyticity for $\boldsymbol{p}_a \in \mathbb{R}$, p_a^0 in first quadrant $\operatorname{Im} p_a^0 > 0, \operatorname{Re} p_a^0 \geq 0$

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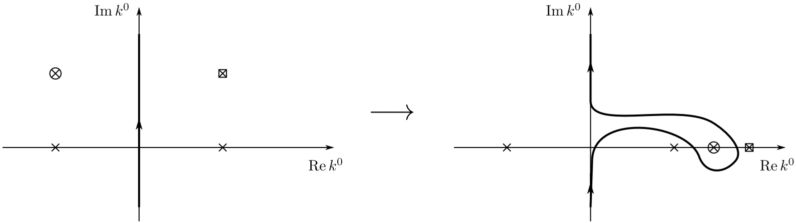

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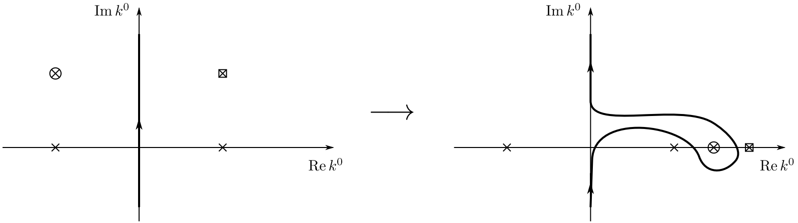

$\Rightarrow$ Cutkosky rules, unitarity, spacetime and moduli space $i\epsilon$-prescriptions [Pius-Sen] [Sen]
Timelike Liouville theory [Bautista-Dabholkar-HE, to appear]

Analyticity for string theory (1)

Result

Prove analyticity inside Δ_2 of n-point superstring Green functions at all loop orders. This implies crossing symmetry for $n = 4$.

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Comments:

- Feynman graphs $\rightarrow$ perturbative computations
- valid for states with any spin
- technical assumptions: mass gap, stable external states
- regularization of massless states
- envelope of holomorphy only for $n = 4$

$\rightarrow$ put the proof at the same level as in QFT

Analyticity for string theory (2)

Method to study singularity:

1. start with some $p_a = p_a^{(1)}$, $\ell_r^0 \in \mathrm{i}\mathbb{R}$, $\boldsymbol{\ell}_r \in \mathbb{R}$ s.t. no singularity
2. find a path $p_a = p_a^{(1)} \to$ desired $p_a = p_a^{(2)}$
3. deform the integral contour as the poles move
4. assume $\exists$ singularity
   pinching = collision of two poles from opposite sides
5. display an inconsistency

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Proceed by steps:

1. analyticity in Δ_1: go from $p_a = 0$ to desired $\mathrm{Re}\, p_a$ and $\mathrm{Im}\, p_a^0$ (keep $\mathrm{Im}\, \boldsymbol{p}_a = 0$)
2. analyticity in Δ_2: go from $p_a \in \Delta_1$ to desired $\mathrm{Im}\, p_a^1$ (keep $\mathrm{Im}\, p_a^i = 0\ \forall i \geq 2$)

Outline: 4. Conclusion

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Results:

- analyticity of superstring *n*-point amplitudes in Δ_2
- proof of crossing symmetry for 4-point superstring amplitudes at the same level as in QFT
- show that, in some sense, string theory behaves like local QFT
- new proof of analyticity valid for more general QFTs
- starting point for studying other properties

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Outlook:

- CPT theorem
- prove analyticity in Δ_D
- prove analyticity in $\mathcal{H}(\Delta_2)$ just from Feynman diagrams