# Heterotic string field theory with cyclic $L_\infty$ structure

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# Introduction

Superstring field theory is one of the approaches to define superstring theory nonperturbatively.

We need a complete superstring field theory including both the Neveu-Schwarz and Ramond sectors.

**Open superstring**: There are three complemantary approaches.

· WZW-like approach: (Berkovits (1995))
Kunitomo and Okawa (2016)

· Homotopy algebraic approach: (Erler, Konopka and Sachs (2014))
Erler, Okawa and Takezaki (2016)

· Sen's approach: (Sen (2016))
Konopka and Sachs (2016)

## Closed superstring

$\diamond$ Sen's approach:   Sen (2016)

No complete theory based on the remaining two approaches so we should construct them.

We first consider the heterotic string field theory.

## Homotopy algebraic formulation (Erler-Konopka-Sachs 2014)

String fields: $\Phi = \Phi_{NS} + \Phi_R \ \in \ \mathcal{H} = \mathcal{H}_{small}^{NS(2,-1)} + \mathcal{H}_{small}^{R(2,-1/2)}$.

Constraints: $b_0^- \Phi = L_0^- \Phi = 0\,,\ XY\Phi_R \ = \ \Phi_R\,.$

$X$ and $Y$ are defined by

$$X \ = \ -\delta(\beta_0)G_0 + \delta'(\beta_0)b_0\,, \qquad Y \ = \ -2c_0^+\delta'(\gamma_0)\,,$$

which satisfy

$$[Q, \Xi] = X\,, \qquad XYX = X\,.$$

$\Phi_R$ is restricted in the form

$$\Phi_R \ = \ \phi_R - \frac{1}{2}(\gamma_0 + 2c_0^+ G)\psi_R\,, \qquad (G = G_0 + 2\gamma_0 b_0)\,,$$

which we denote $\Phi \in \mathcal{H}^{res}$.

Note that $\phi_R$ and $\psi_R$ are field and anti-field, respectively, in BV formalism (gauge-fixed basis).

Symplectic form:

$$\begin{aligned}\Omega(\Phi_1\,,\Phi_2) \;&= \omega_s(\Phi_{1NS}\,,\Phi_{2NS}) + \omega_s(\Phi_{1R}\,, Y\Phi_{2R})\\ &= \langle\!\langle \phi_{1R}|\psi_{2R}\rangle\!\rangle + \langle\!\langle \psi_{1R}|\phi_{2R}\rangle\!\rangle\,.\\ \Big(\omega_s(\Phi_1\,,\Phi_2) \;&= (-1)^{|\Phi_1|}\langle\Phi_1|c_0^-|\Phi_2\rangle\Big)\end{aligned}$$

String products:

$$\begin{aligned}&L_n(\Phi_1\,,\cdots\,,\Phi_n) \in \mathcal{H}^{res}\,, \quad (n\geq 2)\,,\\ &L_1\Phi \;=\; Q\Phi\,.\end{aligned}$$

If $L_1 = Q,\ L_2,\ L_3,\ \cdots$ is a cyclic $L_\infty$ algebra satisfying

$$\sum_\sigma \sum_{m=1}^{n} \frac{(-1)^{\epsilon(\sigma)}}{m!(n-m)!} L_{n-m+1}(L_m(\Phi_{\sigma(1)}\,,\cdots\,,\Phi_{\sigma(m)})\,,\Phi_{\sigma(m+1)}\,,\cdots\,,\Phi_{\sigma(n)}) \;=\; 0\,,$$

$$\Omega(\Phi_1\,, L_n(\Phi_2\,,\cdots\,,\Phi_{n+1})) \;=\; -(-1)^{|\Phi_1|}\Omega(L_n(\Phi_1\,,\cdots\,,\Phi_n)\,,\Phi_{n+1})\,,$$

with proper ghost and picture numbers,

Action:

$$S \;=\; \sum_{n=0}^{\infty} \frac{1}{(n+2)!}\Omega(\Phi\,,\ L_{n+1}(\underbrace{\Phi\,,\ \cdots\,,\ \Phi}_{n+1}))$$

is invariant under

Gauge tf.:

$$\delta\Phi = \sum_{n=0}^{\infty} \frac{1}{n!} L_{n+1}(\underbrace{\Phi\,,\ \cdots\,,\ \Phi}_{n},\ \Lambda)\,.$$

## How to construct $\{L_n\}$ with proper ghost/picture numbers?

We know "bosonic" products $\{L_n^{(0)}\}$, which is an $L_\infty$ algebra with proper ghost number but no picture number.

Construct $\{L_n\}$ by suitably putting $X$ and $\xi_0$ in $\{L_n^{(0)}\}$ so as to be proper picture number.

For NS sector: Erler-Konopka-Sachs (2014)

We modify EKS construction and extend it to the one for both NS and Ramond sectors.

Proper picture number: $p(\Phi_{NS}) = -1\,,\ p(\Phi_R) = -\frac{1}{2}$

Picture number of $L_{n+2}$ is determined so that $L^{(p)}_{n+2}$ is closed in $\mathcal{H}^{NS(-1)}_{small} \oplus \mathcal{H}^{R(-1/2)}_{small}$ :

$$p((L^{(p)}_{n+2}(\cdots))_{NS}) = -1\,, \qquad p((L^{(p)}_{n+2}(\cdots))_{R}) = -\frac{1}{2}\,.$$

Consider $L^{(p)}_{n+2}|_{2r}$ with $2r$ =(Ramond #)=(# of R inputs)−(# of R outputs).

(1) Output is NS state when $2r$ of $n+2$ inputs are R states:

$$\left(-\frac{1}{2}\right) \times 2r + (-1) \times (n+2-2r) + p = -1$$

(2) Output is R state when $2r+1$ of $n+2$ inputs are R states:

$$\left(-\frac{1}{2}\right) \times (2r+1) + (-1) \times (n+1-2r) + p = -\frac{1}{2}$$

We find $p = n - r + 1$ or $L^{(p)}_{p+r+1}|_{2r}$ .

Cyclicity: Ramond # is not suitable to consider cyclicity. Instead

cyclic Ramond # = (# of R inputs) + (# of R outputs) (denoted as $|^{2r}$) is useful:

$$\Omega\Big(\Phi_{NS1}\,, L_n^{(p)}|^{2r}_{2r}(\Phi_{NS2}\,,\cdots\,,\Phi_{R(n+1)})\Big) \sim \Omega\Big(\Phi_{R(n+1)}\,, L_n^{(p)}|^{2r}_{2r-2}(\Phi_{NS1}\,,\Phi_{NS2}\,,\cdots)\Big)$$

So first consider $B^{(p)}_{p+r+1}|^{2r}$ $(p+r+1\geq 2)$

(1') Output is NS when $2r$ inputs are R, $B^{(p)}_{p+r+1}|^{2r}_{2r}$ :

$$\left(-\frac{1}{2}\right)\times 2r+(-1)\times((p+r+1)-2r)+p=-1\,,\quad (p\#\ \text{deficit}=0)$$

(2') Output is R when $2r-1$ inputs are R, $B^{(p)}_{p+r+1}|^{2r}_{2r-2}$ :

$$\left(-\frac{1}{2}\right)\times(2r-1)+(-1)\times((p+r+1)-2r+1)+p=-\frac{1}{2}-1\,,\quad (p\#\ \text{deficit}=1)$$

$B_{n+2}$ cannot be $L_{n+2}$ but such a combination often appears as difference of (nonlinear extension of) $Q$ and $\eta$. First order EoM (Berkovits)(HK), general form of WZW-like theory (Erler), Democratic theory (Kroyter), $etc.$

## Coalgebraic representation

Symmetrized tensor product: $(\Phi_i \in \mathcal{H})$

$$\Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_n = \sum_\sigma (-1)^{\epsilon(\sigma)} \Phi_{\sigma(1)} \otimes \Phi_{\sigma(2)} \otimes \cdots \otimes \Phi_{\sigma(n)} \ \in \mathcal{H}^{\wedge n}$$

Symmetrized tensor algebra: $\mathcal{SH} = \mathcal{H}^{\wedge 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\wedge 2} \oplus \cdots$

Linear map: $L_n \,:\, \mathcal{H}^{\wedge n} \to \mathcal{H}$ w/ $L_n(\Phi_1 \wedge \cdots \wedge \Phi_n) = L_n(\Phi_1, \cdots, \Phi_n)$

Coderivation: $\boldsymbol{L}_n \,:\, \mathcal{SH} \longrightarrow \mathcal{SH}$

$$\begin{aligned}
\boldsymbol{L}_n \Phi_1 \wedge \cdots \wedge \Phi_m &= 0\,, && \text{for } m < n\,,\\
\boldsymbol{L}_n \Phi_1 \wedge \cdots \wedge \Phi_m &= L_n(\Phi_1 \wedge \cdots \wedge \Phi_m)\,, && \text{for } m = n\,,\\
\boldsymbol{L}_n \Phi_1 \wedge \cdots \wedge \Phi_m &= (L_n \wedge \mathbb{I}_{m-n})\Phi_1 \wedge \cdots \wedge \Phi_m\,, && \text{for } m < n\,.
\end{aligned}$$

Then we can consider coderivation $\boldsymbol{L} = \sum_{n=0}^{\infty} \boldsymbol{L}_{n+1}$. The $L_\infty$ relation is represented as nilpotency for a (degree odd) coderivation $\boldsymbol{L}$:

$$[\,\boldsymbol{L}\,, \boldsymbol{L}\,] \;=\; 0\,.$$

Consider nonlinear extensions of $\boldsymbol{Q}$ and $\boldsymbol{\eta}$ :

$$\pi_1 \boldsymbol{D} \;=\; \pi_1 \boldsymbol{Q} + \pi_1^0 \boldsymbol{B}\,, \qquad \pi_1 \boldsymbol{C} \;=\; \pi_1 \boldsymbol{\eta} - \pi_1^1 \boldsymbol{B}\,,$$

satisfying $[\boldsymbol{D},\boldsymbol{D}] = [\boldsymbol{C},\boldsymbol{C}] = [\boldsymbol{D},\boldsymbol{C}] = 0$ , where $\pi_1^0$ and $\pi_1^1$ are the projectors onto NS and R components, respectively. The dierence of these two $L_\infty$ algebra

$$\boldsymbol{D} - \boldsymbol{C} \;=\; \boldsymbol{Q} - \boldsymbol{\eta} + \boldsymbol{B}\,, \qquad \boldsymbol{B} = \sum_{p,r=0}^{\infty} \boldsymbol{B}^{(p)}_{p+r+1}\big|^{2r}\,,$$

is also an $L_\infty$ algebra

$$[\,\boldsymbol{D} - \boldsymbol{C}\,, \boldsymbol{D} - \boldsymbol{C}\,] \;=\; 0\,,$$

which can be cyclic (w.r.t. $\omega_l$ defined by BPZ inner product of $\mathcal{H}_{large}$ ):

Note: All of these $\boldsymbol{D}$ , $\boldsymbol{C}$ and $\boldsymbol{D} - \boldsymbol{C}$ are not closed in $\mathcal{H}_{small}$ .

We first construct cyclic $\boldsymbol{D}-\boldsymbol{C}$, then transform $(\boldsymbol{D},\boldsymbol{C})$ to $(\boldsymbol{L},\boldsymbol{\eta})$:

$$\hat{\boldsymbol{F}}^{-1}\boldsymbol{D}\hat{\boldsymbol{F}} \;=\; \pi_1\boldsymbol{L}\,, \qquad \hat{\boldsymbol{F}}^{-1}\boldsymbol{C}\hat{\boldsymbol{F}} \;=\; \pi_1\boldsymbol{\eta}\,,$$

with $\hat{\boldsymbol{F}}^{-1} = \pi_1\mathbb{I}_{\mathcal{SH}} - \Xi\pi_1^1\boldsymbol{B}$, where

$$\pi_1\boldsymbol{L} \;=\; \boldsymbol{Q} + \pi_1^0\boldsymbol{b} + X\pi_1^1\boldsymbol{b}\,,$$

with $\pi_1\boldsymbol{b} = \pi_1\boldsymbol{B}\hat{\boldsymbol{F}}$. Now $(\boldsymbol{L},\boldsymbol{\eta})$ are two (anti)commutative $L_\infty$ algebras

$$[\,\boldsymbol{L}\,,\boldsymbol{L}\,] \;=\; [\,\boldsymbol{\eta}\,,\boldsymbol{\eta}\,] \;=\; [\,\boldsymbol{\eta}\,,\boldsymbol{L}\,] \;=\; 0\,,$$

and thus $\boldsymbol{L}$ is closed in $\mathcal{H}_{small}$. We can show that if $\boldsymbol{B}$ is cyclic w.r.t. $\omega_l$ then $\boldsymbol{b}$ is cyclic w.r.t. $\omega_s$ which implies $\boldsymbol{L}$ is cyclic w.r.t. $\Omega$:

$$\Omega(\Phi_1\,, L_n(\Phi_2\,,\cdots\,,\Phi_{n+1})) \;=\; \omega_s(\Phi_1\,, b_n(\Phi_2\,,\cdots\,,\Phi_{n+1}))\,.$$

# How to construct $\boldsymbol{B}$

R number is additive in commutator but cyclic R number is not:

$$
\begin{aligned}
[\,\boldsymbol{l}|_{2r}\,, \boldsymbol{l}'|_{2s}\,] &= [\,\boldsymbol{l}|_{2r}\,, \boldsymbol{l}'|_{2s}\,]|_{2r+2s}\,, \\
[\,\boldsymbol{l}|^{2r}\,, \boldsymbol{l}'|^{2s}\,] &= [\,\boldsymbol{l}|^{2r}\,, \boldsymbol{l}'|^{2s}\,]|^{2r+2s} + [\,\boldsymbol{l}|^{2r}\,, \boldsymbol{l}'|^{2s}\,]|^{2r+2s-2}\,.
\end{aligned}
$$

We define $[\cdot\,,\,\cdot]^1$ and $[\cdot\,,\,\cdot]^2$ by

$$
\begin{aligned}
[\,\boldsymbol{l}|^{2r}\,, \boldsymbol{l}'|^{2s}\,]^1 &\equiv [\,\boldsymbol{l}|^{2r}\,, \boldsymbol{l}'|^{2s}\,]|^{2r+2s}\,, \\
[\,\boldsymbol{l}|^{2r}\,, \boldsymbol{l}'|^{2s}\,]^2 &\equiv [\,\boldsymbol{l}|^{2r}\,, \boldsymbol{l}'|^{2s}\,]|^{2r+2s-2}\,.
\end{aligned}
$$

If $\boldsymbol{l} = \sum_r \boldsymbol{l}|^{2r}$ and $\boldsymbol{l}' = \sum_s \boldsymbol{l}'|^{2s}$

$$
\begin{aligned}
[\,\boldsymbol{l}\,, \boldsymbol{l}'\,] &= [\,\boldsymbol{l}\,, \boldsymbol{l}'\,]^1 + [\,\boldsymbol{l}\,, \boldsymbol{l}'\,]^2 \\
&= \sum_{r,s} [\,\boldsymbol{l}|^{2r}\,,\, \boldsymbol{l}'|^{2s}\,]|^{2r+2s} + \sum_{r,s} [\,\boldsymbol{l}|^{2r}\,,\, \boldsymbol{l}'|^{2s}\,]|^{2r+2s-2}\,.
\end{aligned}
$$

Preparation: Bosonic products $L^{(0)}_{n+1}|^{2r}$ :

$$(1) \quad \left(-\frac{1}{2}\right) \times 2r + (-1) \times (n+1-2r) \;=\; -(n-r) - 1\,,$$

$$(2) \quad \left(-\frac{1}{2}\right) \times (2r+1) + (-1) \times (n+1-2r-1) \;=\; -(n-r+1) - \frac{1}{2}\,.$$

So they have "p # deficit" $m = n - r(+1)$. We define a generating function counting the p # deficit:

$$\boldsymbol{L}^{(0)}(s) \;=\; \boldsymbol{Q} + \sum_{m,r=0}^{\infty} s^m \boldsymbol{L}^{(0)}_{m+r+1}|^{2r} \;\equiv \boldsymbol{Q} + \boldsymbol{L}^{(0)}_B(s)\,,$$

which reduces to the bosonic $L_\infty$ algebra as $\boldsymbol{L}^{(0)} = \boldsymbol{L}^{(0)}(s=1)$ : $[\boldsymbol{L}^{(0)}, \boldsymbol{L}^{(0)}] = 0$ . $\boldsymbol{L}^{(0)}_B(s)$ satisfies

$$\begin{aligned} [\,\boldsymbol{Q}\,, \boldsymbol{L}^{(0)}_B(s)\,] + \frac{1}{2}[\,\boldsymbol{L}^{(0)}_B(s)\,, \boldsymbol{L}^{(0)}_B(s)\,]^1 + \frac{s}{2}[\,\boldsymbol{L}^{(0)}_B(s)\,, \boldsymbol{L}^{(0)}_B(s)\,]^2 \;&=\; 0\,, \\ [\,\boldsymbol{\eta}\,, \boldsymbol{L}^{(0)}_B(s)\,] \;&=\; 0\,. \end{aligned} \qquad (1)$$

On the other hand, if we introduce a generating function

$$\boldsymbol{B}(t) \;=\; \sum_{p,r=0}^{\infty} t^p \boldsymbol{B}_{p+r+1}^{(p)}\big|^{2r}\,, \qquad (\boldsymbol{B}_0^0|^0 \equiv 0)$$

$[\boldsymbol{D}-\boldsymbol{C}, \boldsymbol{D}-\boldsymbol{C}] = 0$ is equivalent to

$$\begin{aligned} [\,\boldsymbol{Q}\,, \boldsymbol{B}(t)\,] + \frac{1}{2}[\,\boldsymbol{B}(t)\,, \boldsymbol{B}(t)\,]^1 \;&= 0\,, \\ [\,\boldsymbol{\eta}\,, \boldsymbol{B}(t)\,] - \frac{t}{2}[\,\boldsymbol{B}(t)\,, \boldsymbol{B}(t)\,]^2 \;&= 0\,. \end{aligned} \tag{2}$$

Further if we extend $\boldsymbol{B}$'s to those with arbitrary p # deficit and define a generating function

$$\boldsymbol{B}(s,t) \;=\; \sum_{m,p,r=0}^{\infty} s^m t^p \boldsymbol{B}^{(p)}_{m+p+r+1}|^{2r} \;\equiv\; \sum_{p=0}^{\infty} t^p \boldsymbol{B}^{(p)}(s)\,,$$

the ralation (2) can be extendet to

$$\begin{aligned}
\boldsymbol{I}(s,t) \;&\equiv [\,\boldsymbol{Q}\,, \boldsymbol{B}(s,t)\,] + \frac{1}{2}[\,\boldsymbol{B}(s,t)\,, \boldsymbol{B}(s,t)\,]^1 \;+ \frac{s}{2}[\,\boldsymbol{B}(s,t)\,, \boldsymbol{B}(s,t)\,]^2 \;=\; 0\,,\\
\boldsymbol{J}(s,t) \;&\equiv [\,\boldsymbol{\eta}\,, \boldsymbol{B}(s,t)\,] - \frac{t}{2}[\,\boldsymbol{B}(s,t)\,, \boldsymbol{B}(s,t)\,]^2 \;=\; 0\,.
\end{aligned}$$

Here $\boldsymbol{B}(t) = \boldsymbol{B}(0,t)\,, \boldsymbol{B} = \boldsymbol{B}(0,1)$ and $\boldsymbol{B}^{(0)}(s) = \boldsymbol{B}(s,0)\,.$

Differential equation: If we introduce a degree even coderivation (gauge products)

$$\boldsymbol{\lambda}(s,t) \;=\; \sum_{m,p,r}^{\infty} s^m t^p \boldsymbol{\lambda}^{(p+1)}_{m+p+r+2}\big|^{2r} \;\equiv\; \sum_{p=0}^{\infty} t^p \boldsymbol{\lambda}^{(p+1)}(s)\,,$$

and postulate differential equations

$$\begin{aligned}
\partial_t \boldsymbol{B}(s,t) \;&=\; [\boldsymbol{Q}, \boldsymbol{\lambda}(s,t)] + [\boldsymbol{B}(s,t), \boldsymbol{\lambda}(s,t)]^1 + s[\boldsymbol{B}(s,t), \boldsymbol{\lambda}(s,t)]^2\,,\\
[\boldsymbol{\eta}, \boldsymbol{\lambda}(s,t)] \;&=\; \partial_s \boldsymbol{B}(s,t) + t[\boldsymbol{B}(s,t), \boldsymbol{\lambda}(s,t)]^2\,.
\end{aligned} \tag{3}$$

Then we can show

$$\begin{aligned}
\partial_t \boldsymbol{I}(s,t) \;&=\; [\,\boldsymbol{I}(s,t)\,, \boldsymbol{\lambda}(s,t)\,]^1 + s\,[\,\boldsymbol{I}(s,t)\,, \boldsymbol{\lambda}(s,t)\,]^2\,,\\
\partial_t \boldsymbol{J}(s,t) \;&=\; -\,\partial_s \boldsymbol{I}(s,t) - t\,[\,\boldsymbol{I}(s,t)\,, \boldsymbol{\lambda}(s,t)\,]^2\\
&\qquad + [\,\boldsymbol{J}(s,t)\,, \boldsymbol{\lambda}(s,t)\,]^1 + s\,[\,\boldsymbol{J}(s,t)\,, \boldsymbol{\lambda}(s,t)\,]^2\,.
\end{aligned}$$

Therefore if $\boldsymbol{I}(s,0) = \boldsymbol{J}(s,0) = 0$ then $\boldsymbol{I}(s,t) = \boldsymbol{J}(s,t) = 0\,.$ However equations $\boldsymbol{I}(s,0) = \boldsymbol{J}(s,0) = 0$ are the same as (1), those for $\boldsymbol{L}_B^{(0)}(s)$, and thus satisfied by taking $\boldsymbol{B}^{(0)}(s) = \boldsymbol{L}_B^{(0)}(s)$.

Differential eqs. (3) can be written as

$$
\begin{aligned}
(n+1)\boldsymbol{B}^{(n+1)}(s) \; &= [\, \boldsymbol{Q} \,, \boldsymbol{\lambda}^{(n+1)}(s) \,] + \sum_{n'=0}^{n} [\, \boldsymbol{B}^{(n-n')}(s) \,, \boldsymbol{\lambda}^{(n'+1)}(s) \,]^1 \\
&\qquad + \sum_{n'=0}^{n} s \, [\, \boldsymbol{B}^{(n-n')}(s) \,, \boldsymbol{\lambda}^{(n'+1)}(s) \,]^2 \,, \\
[\, \boldsymbol{\eta} \,, \boldsymbol{\lambda}^{(n+1)}(s) \,] \; &= \partial_s \boldsymbol{B}^{(n)}(s) + \sum_{n'=0}^{n-1} [\, \boldsymbol{B}^{(n-n'-1)} \,, \boldsymbol{\lambda}^{(n'+1)}(s) \,]^2 \,,
\end{aligned}
$$

and can be solved iteratively.

## Consistency:

If we read (3) as

$$\begin{aligned} [\boldsymbol{Q}, \boldsymbol{\lambda}(s,t)] &= \boldsymbol{N}(s,t)\,, \qquad [\boldsymbol{\eta}, \boldsymbol{\lambda}(s,t)] \;=\; \boldsymbol{K}(s,t)\,, \\ \boldsymbol{N}(s,t) &= \partial_t \boldsymbol{B}(s,t) - [\boldsymbol{B}(s,t), \boldsymbol{\lambda}(s,t)]^1 - s[\boldsymbol{B}(s,t), \boldsymbol{\lambda}(s,t)]^2\,, \\ \boldsymbol{K}(s,t) &= \partial_s \boldsymbol{B}(s,t) + t[\boldsymbol{B}(s,t), \boldsymbol{\lambda}(s,t)]^2\,, \end{aligned} \tag{4}$$

the "integrability" conditions

$$\begin{aligned} [\,\boldsymbol{Q}\,, \boldsymbol{N}(s,t)\,] \;=\; [\,\boldsymbol{\eta}\,, \boldsymbol{K}(s,t)\,] \;&=\; 0\,, \\ [\,\boldsymbol{\eta}\,, \boldsymbol{N}(s,t)\,] + [\,\boldsymbol{Q}\,, \boldsymbol{K}(s,t)\,] \;&=\; 0\,, \end{aligned}$$

follow from $\boldsymbol{I}(s,t) = \boldsymbol{J}(s,t) = 0$.

For $n=0$ we have

$$\boldsymbol{B}^{(1)}(s) \;=\; [\,\boldsymbol{Q}\,,\boldsymbol{\lambda}^{(1)}(s)\,] + [\,\boldsymbol{L}_B^{(0)}(s)\,,\boldsymbol{\lambda}^{(1)}(s)\,]^1 + s\,[\,\boldsymbol{L}_B^{(0)}(s)\,,\boldsymbol{\lambda}^{(1)}(s)\,]^2\,, \tag{5a}$$

$$[\,\boldsymbol{\eta}\,,\boldsymbol{\lambda}^{(1)}(s)\,] \;=\; \partial_s \boldsymbol{L}_B^{(0)}(s)\,. \tag{5b}$$

(5b) can be solved as

$$\begin{aligned}\pi_1\boldsymbol{\lambda}^{(1)}(s) \;&=\; \sum_{m,r=0}^{\infty}(m+1)s^m\frac{1}{m+r+3}\Big(\xi L^{(0)}_{Bm+r+2}|^{2r} - L^{(0)}_{Bn+r+2}|^{2r}(\xi\wedge\mathbb{I}_{m+r+1})\Big)\\ &\equiv \pi_1\xi\circ\partial_s\boldsymbol{L}_B^{(0)}(s)\,.\end{aligned}$$

Then $\boldsymbol{B}^{(1)}(s)$ is obtained by substituting this into (5a).

Similarly $\boldsymbol{B}^{(n)}(s)$ for $^\forall n$ can be determined recursively:

$$\begin{aligned}
\boldsymbol{\lambda}^{(n+1)}(s) &= \xi \circ \left( \partial_s \boldsymbol{B}^{(n)}(s) + \sum_{n'=0}^{n-1} [\, \boldsymbol{B}^{(n-n'-1)} \,, \boldsymbol{\lambda}^{(n'+1)}(s) \,]^2 \right) \,, \\
(n+1)\boldsymbol{B}^{(n+1)}(s) &= [\, \boldsymbol{Q} \,, \boldsymbol{\lambda}^{(n+1)}(s) \,] + \sum_{n'=0}^{n} [\, \boldsymbol{B}^{(n-n')}(s) \,, \boldsymbol{\lambda}^{(n'+1)}(s) \,]^1 \\
&\quad + \sum_{n'=0}^{n} s \, [\, \boldsymbol{B}^{(n-n')}(s) \,, \boldsymbol{\lambda}^{(n'+1)}(s) \,]^2 \,.
\end{aligned}$$

# WZW-like formulation

(Berkovits 1995, Berkovits-Okawa-Zwiebach 2004)

Gauge invariant action in WZW-like formulation can also be obtained by "field redefinition".

First we note that if we project $\boldsymbol{B}(s,t)$ and $\boldsymbol{\lambda}(s,t)$ onto the pure NS sector,

$$\boldsymbol{B}(s,t)|^0 \equiv \sum_{m,n=0}^{\infty} s^m t^n \boldsymbol{B}^{(n)}_{m+n+1}|^0 = \sum_{m,n=0}^{\infty} s^m \boldsymbol{B}^{[m]}(t)\,,$$

$$\boldsymbol{\lambda}(s,t)|^0 \equiv \sum_{m,n=0}^{\infty} s^m t^n \boldsymbol{\lambda}^{(n+1)}_{m+n+2}|^0 = \sum_{m,n=0}^{\infty} s^m \boldsymbol{\lambda}^{[m]}(t)\,,$$

diff. eqs. (3) reduces to those introduced by EKS, and thus, by construction, $\boldsymbol{B}(s,t)|^0$ and $\boldsymbol{\lambda}(s,t)|^0$, reduce to those of EKS.

This implies that the $L_\infty$ algebra restricted to the pure NS sector, $\boldsymbol{Q}+\boldsymbol{B}^{[0]}|^0$, can be written in the form of a similarity tf.

$$\boldsymbol{Q} + \boldsymbol{B}^{[0]}|^0 = \hat{\boldsymbol{g}}^{-1}\,\boldsymbol{Q}\,\hat{\boldsymbol{g}}\,,$$

generated by the cohomomorphism

$$\hat{\boldsymbol{g}} \;=\; \vec{\mathcal{P}}\exp\left(\int_0^1 dt\,\boldsymbol{\lambda}^{[0]}(t)|^0\right)\,.$$

Then $\boldsymbol{L}$ is transformed by (the inverse of) this similarity transformation as

$$\pi_1\tilde{\boldsymbol{L}} \;\equiv\; \pi_1\hat{\boldsymbol{g}}\,\boldsymbol{L}\,\hat{\boldsymbol{g}}^{-1} = \pi_1\boldsymbol{Q} + \pi_1^0\tilde{\boldsymbol{b}} + X\pi_1^1\tilde{\boldsymbol{b}}\,,$$

where

$$\tilde{\boldsymbol{b}} \;=\; \hat{\boldsymbol{g}}\,(\boldsymbol{b} - \boldsymbol{B}^{[0]}|^0)\,\hat{\boldsymbol{g}}^{-1}\,.$$

Here $\hat{\boldsymbol{g}}$ preserves the cyclicity, and thus $\tilde{\boldsymbol{b}}$ is also cyclic w.r.t $\omega_l$.

By this transformation, the small Hilbert space condition $\eta\Phi = 0$ is mapped to

$$0 \;=\; \pi_1 \hat{\boldsymbol{g}} \boldsymbol{\eta} (e^{\wedge(\Phi_{NS}+\Phi_R)}) \;=\; \pi_1^0 \boldsymbol{L}^\eta (e^{\wedge \pi_1^0 \hat{\boldsymbol{g}}(e^{\wedge \Phi_{NS}})}) + \pi_1^1 \eta \Phi_R\,,$$

with $\boldsymbol{L}^\eta \equiv \hat{\boldsymbol{g}} \boldsymbol{\eta} \hat{\boldsymbol{g}}^{-1}$.

The NS component has the same form as the Maurer-Cartan eq. for the pure-gauge string field $G_\eta(V)$ in the WZW-like formulation:

$$\boldsymbol{L}^\eta (e^{\wedge G_\eta(V)}) = 0\,.$$

This suggests the identification

$$\pi_1^0 \hat{\boldsymbol{g}} (e^{\wedge \Phi_{NS}}) \;=\; G_\eta(V)\,, \qquad \Phi_R \;=\; \Psi\,. \tag{6}$$

between $(\Phi_{NS}, \Phi_R)$ and string fields $(V, \Psi)$ in the WZW-like formulation.

If we identify the associated fields $B_d(V(t))$ $(d = t, \delta)$ with

$$B_d(V(t)) \;=\; \pi_1^0 \hat{\boldsymbol{g}} \boldsymbol{\xi_d}(e^{\wedge \Phi_{NS}(t)})\,,$$

where $\boldsymbol{\xi_\delta}$ is the one coderivation $\boldsymbol{\xi_\delta}$ derived from $\xi\delta$ , then the identities

$$\begin{aligned}
dG_\eta(V(t)) \;&=\; \pi_1^0 \boldsymbol{L}^\eta(e^{\wedge G_\eta(t)} \wedge B_d(V(t)))\,,\\
\partial_t B_\delta(V(t)) - \delta B_t(V(t))&\\
&+ \pi_1^0 \boldsymbol{L}^\eta(e^{\wedge G_\eta(t)} \wedge\, B_t(V(t)) \wedge\, B_\delta(V(t))) \;=\; 0\,.
\end{aligned}$$

characterizing the associated field follows from the identification (6).

The action in homotopy algebraic formulation can be written in the WZW-like form by using $\Phi_{NS}(t)$ with $\Phi_{NS}(1) = \Phi_{NS}$ and $\Phi_{NS}(0) = 0$. Using the cyclicity we find

$$
\begin{aligned}
S \;&=\; \int_0^1 dt\,\omega_l\big(\xi\partial_t\Phi_{NS}(t)\,,\pi_1^0\boldsymbol{L}(e^{\wedge\Phi_{NS}(t)})\big) \\
&\quad + \frac{1}{2}\,\omega_s(\Phi_R\,,YQ\Phi_R) + \sum_{r=0}^{\infty}\frac{1}{(2r+2)!}\,\omega_s\big(\Phi_R\,,\,\pi_1^1\boldsymbol{b}(e^{\wedge\Phi_{NS}}\wedge\,\Phi_R{}^{\wedge 2r+1}\big)\,,
\end{aligned}
$$

which can be mapped to the complete WZW-like action

$$
\begin{aligned}
S \;&=\; \int_0^1 dt\,\omega_l\big(B_t(V(t))\,,\,QG_\eta(V(t))\big) \\
&\quad + \frac{1}{2}\,\omega_s(\Psi\,,\,YQ\Psi) + \sum_{r=0}^{\infty}\frac{1}{(2r+2)!}\,\omega_s\big(\Psi\,,\,\pi_1^1\tilde{\boldsymbol{b}}(e^{\wedge G_\eta(V)}\wedge\Psi^{\wedge 2r+1})\big)\,.
\end{aligned}
$$

through the identification (6), using the identity for odd coderivations $\boldsymbol{l}_{1,2}$:

$$
\omega_l(\pi_1\hat{\boldsymbol{g}}\boldsymbol{l}_1(e^{\wedge\Phi})\,,\pi_1\hat{\boldsymbol{g}}\boldsymbol{l}_2(e^{\wedge\Phi}))\;=\;\omega_l(\pi_1\boldsymbol{l}_1(e^{\wedge\Phi})\,,\pi_1\boldsymbol{l}_2(e^{\wedge\Phi}))\,.
$$

Gauge tf.: Since

$$\delta G_\eta(V) \;=\; D_\eta B_\delta(V)\,,$$

the identification (6) is not one-to-one. We have an extra gauge invariance under

$$B_\delta(V) = D_\eta\Omega\,, \qquad \delta\Psi \;=\; 0\,, \tag{7}$$

in the WZW-like formulation. In addition the gauge tf. in homotopy algebric formulation,

$$\pi_1\delta(e^{\wedge(\Phi_{NS}+\Phi_R)}) \;=\; \pi_1\boldsymbol{L}(e^{\wedge(\Phi_{NS}+\Phi_R)}\wedge(\Lambda_{NS}+\Lambda_R))\,,$$

is mapped to the gauge tf. in the WZW-like formulation, except for the terms which can be absorbed into the tf. (7), as

$$\begin{aligned} B_\delta(V) =& \pi_1^0\tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)}\wedge(\Lambda-\xi\lambda)) = Q\Lambda+\pi_1^0\tilde{\boldsymbol{b}}(e^{\wedge(G_\eta+\Psi)}\wedge(\Lambda-\xi\lambda))\,,\\ \delta\Psi =& \eta\pi_1^1\tilde{\boldsymbol{L}}(e^{\wedge(G_\eta+\Psi)}\wedge(\Lambda-\xi\lambda)) = Q\lambda+X\eta\pi_1^1\tilde{\boldsymbol{b}}(e^{(G_\eta+\Psi)}\wedge(\Lambda-\xi\lambda))\,, \end{aligned}$$

with the identification of gauge parameters $\Lambda = -\pi_1^0\hat{\boldsymbol{g}}(e^{\wedge\Phi_{NS}}\wedge\xi\Lambda_{NS})$ and $\lambda = \Lambda_R\,.$ We can independently show the gauge invariance.

## Summary

$\diamond$ We have constructed complete actions and gauge tfs. for heterotic string field theory in both homotopy algebraic formulation and WZW-like formulation.

$\diamond$ We have also confirmed that tree level four point amplitudes are correctly reproduced.

$\diamond$ The extension to the type II superstring field theory is straightforward.

$\diamond$ Construction of complete action is not the end of the story but just the beginning. SFT provides a solid foundation to study various interesting properties of heterotic string.