Strings on Celestial Sphere

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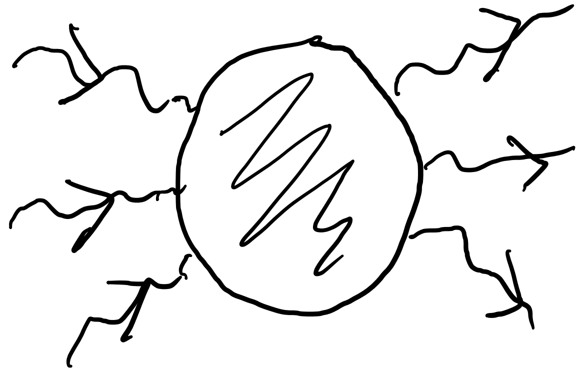
w/Stephan Stieberger
1806.05688
1812.0180
w/Wei Fan, Angelos Fotopoulos
1903.01676
see also my review
"A Course in Amplitudes"
1703.05670

Amplitudes : $\langle \text{in} | \text{out} \rangle$



gauge bosons
gravitons
$m=0$

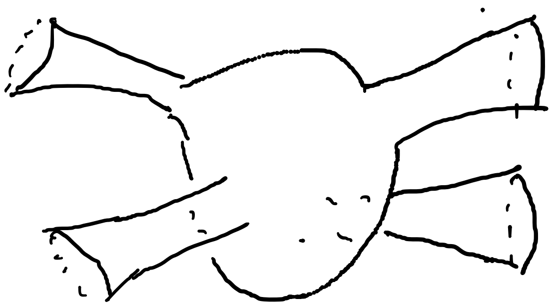
YM, Einstein's GR

$$\langle p_1, h_1, a_1 \ldots | \ldots p_n, h_n, a_n \rangle \equiv \mathcal{A}(p, h, a)$$

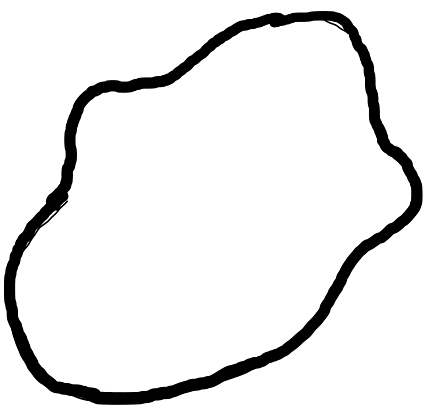
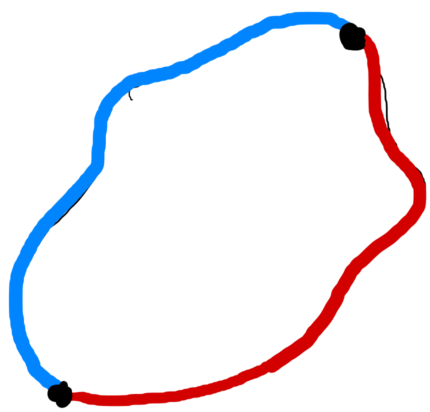
$$= (2\pi)^4 \, i \, \delta^{(4)}\left(\sum p_{out} - \sum p_{in}\right) \mathcal{M}$$

Feynman, recursions,
scattering eqs, ...

Why Strings?

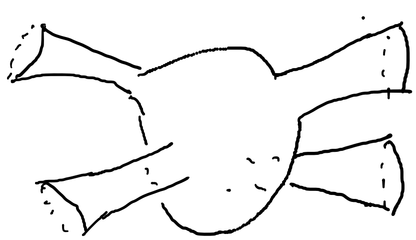


- Einstein $= YM^2$: KLT, color-kinematic collinear equivalence...

 = 

graviton "=" 2 gauge bosons ?

- Supersoft in UV

$s \rightarrow$  $\sim e^{-\alpha' s}$

$\alpha' \sim M^{-2}$ Regge slope

Why Celestial Sphere?

Kinematics: momentum spinors

$$\lambda = \sqrt{\omega}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

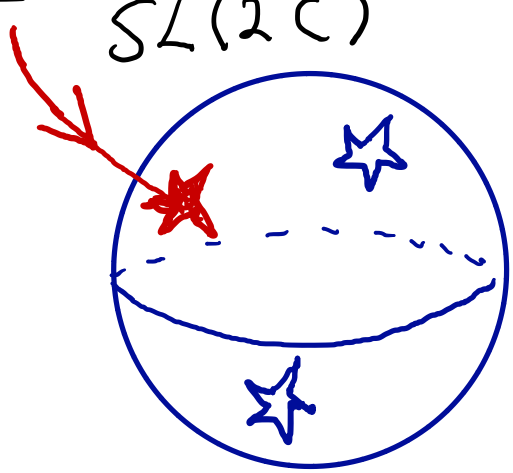
energy

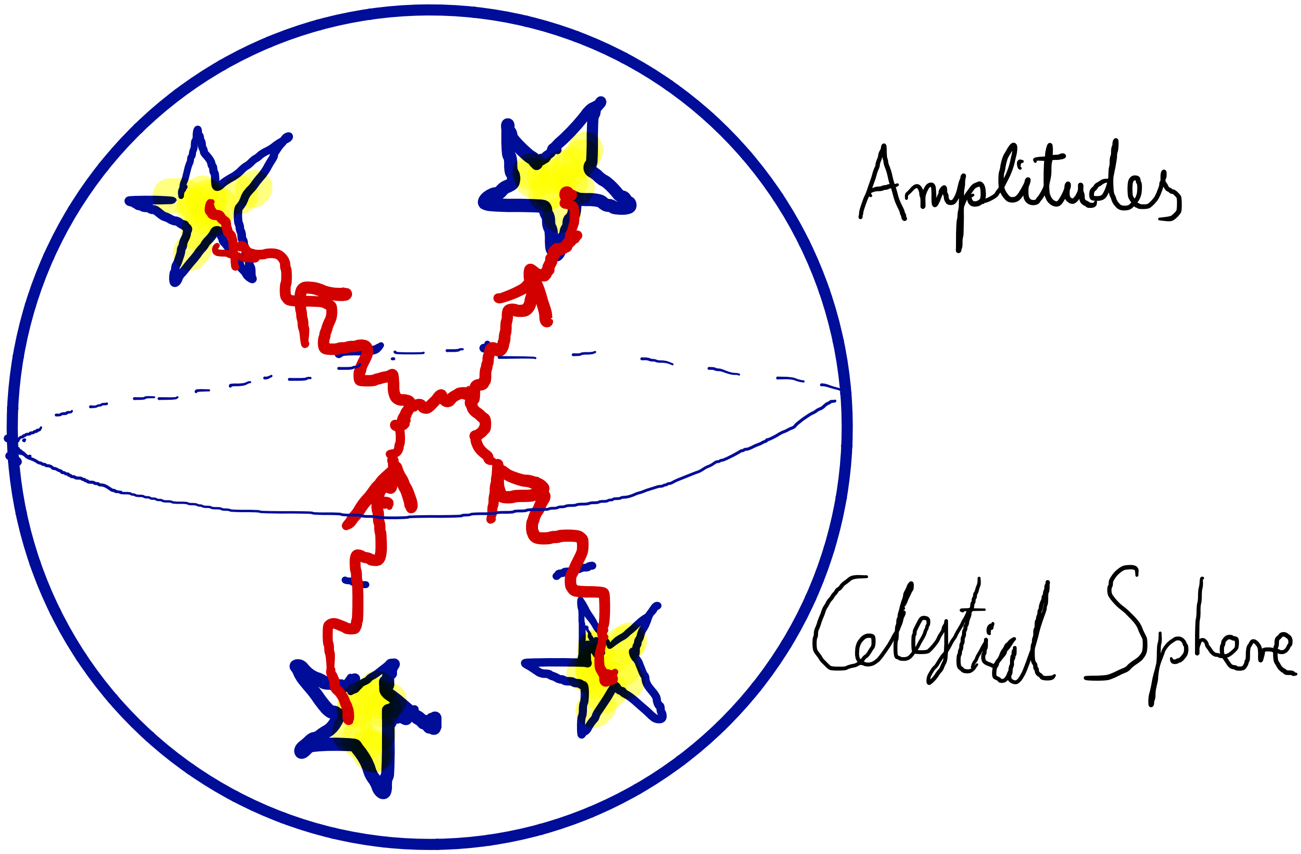
$\langle \lambda_i | \lambda_j \rangle \equiv \langle ij \rangle$

$$p^\mu = \omega q^\mu\,, \quad \text{with } q^\mu = \frac{1}{2}\left(1+|z|^2, z+\bar{z}, -i(z-\bar{z}), 1-|z|^2\right),$$

$$z = \frac{\lambda_1}{\lambda_2}$$: direction (projective coord.)

$$z \xrightarrow[SL(2\,C)]{\text{Lorentz}} \frac{az+b}{cz+d}$$





Celestial CFT

Strominger

We want to understand scattering amplitudes as conformal correlators of "some" 2D CFT ($\rightarrow$ holography?)

$$\mathcal{A}(p; \ldots) = \mathcal{A}(z, \omega; \ldots)$$

"not good"

$$\omega \rightarrow (cz + d)\,(\bar{c}\bar{z} + \bar{d})\,\omega$$

Conformal primary fields

$$\phi_{h,\bar{h}}(z,\bar{z}) \qquad h+\bar{h}=\Delta \quad \text{dimension}$$
$$h-\bar{h}=s \quad \text{spin}$$

$$\phi_{h,\bar{h}}\left(\frac{az+b}{cz+d},\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right)=(cz+d)^{2h}(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}\phi_{h,\bar{h}}(z,\bar{z})$$

Conformal wave-packets

$$\psi(x;p)=e^{\pm ipx}$$

Mellin transf.

p

$$\longrightarrow \varphi_{\Delta}(x;z,\bar{z})=\int_0^\infty d\omega\,\omega^{\Delta-1}e^{\pm i\omega q x}$$

$$\sim \frac{1}{(xq)^{\Delta}} \qquad h=\bar{h}=\frac{\Delta}{2}$$

$$\Delta = 1+i\lambda \quad \text{principal conf. series}$$

Pasterski & Shao

More precisely,

Mellin transf.

$$A^{\Delta\pm}_{\mu J} = g(\Delta) V^{\Delta\pm}_{\mu J} + \partial_\mu a^{\Delta\pm}_J$$

"large" gauge transf.

$$g(\Delta) = (\pm i)^\Delta \frac{\Delta - 1}{\Gamma(\Delta+1)} = (\pm i)^{1+i\lambda} \frac{i\lambda}{\Gamma(2+i\lambda)}$$

$\lambda \to 0$ is an interesting limit

"Soft Conformal Limit"

Donnay, Pühm, Strominger
FFT

- $\Delta \to 1$, $g \to 0$
- $A^{\Delta=1}_\mu$ is pure (large) gauge
- $h = 1$ or $\bar{h} = 1$

$$\mathcal{O}^a_{\lambda, J=\pm 1}(z,\bar{z}) \xrightarrow[\lambda\to 0]{} \begin{array}{ll} j^a(z) & J=+1 \\ \bar{j}^a(\bar{z}) & J=-1 \end{array}$$

gluon operators

holomorphic currents

$$\mathcal{A}(z,\omega,\ldots) \longrightarrow \tilde{\mathcal{A}}(z,h,\bar{h}\ldots)$$

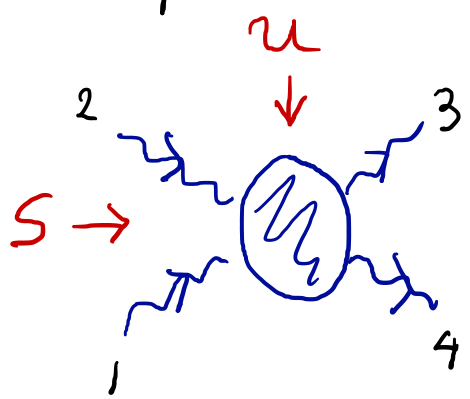
$$\tilde{\mathcal{A}} = \int d\omega_1\, \omega_1^{i\lambda_1} \ldots d\omega_n\, \omega^{i\lambda_n}\, \delta^{(4)}(\omega_1 q_1 + \ldots \omega_n q_n)\, \mathcal{M}(\omega_i, z_i, \bar{z}_i)$$

Celestial amplitude

Example: YM $2 \to 2$

Pasterski, Shao, Strominger

Schreiber, Volovich, Zlotnikov



2 kinematic variables

$$s = (p_1+p_2)^2 \quad u = (p_3-p_2)^2$$

$$\downarrow$$

c.m. energy $E=\sqrt{s}$, scattering angle θ

$$\downarrow \qquad\qquad \downarrow$$

$$\omega \qquad\qquad \sin^2\frac{\theta}{2} = -\frac{u}{s}$$

Mellin $\downarrow$ $\qquad\qquad \downarrow$ $z_{ij} = z_i - z_j$

$$\underbrace{\lambda_1+\lambda_2+\lambda_3+\lambda_4}_{\text{conf. dimension}} \qquad \sin^2\frac{\theta}{2} = \frac{z_{13}z_{24}}{z_{12}z_{34}} = \underbrace{a = \bar{a}}_{\text{conf. inv. cross-ratio}}$$

$$\mathcal{M}(\underset{1}{-},\underset{2}{-},\underset{3}{+},\underset{4}{+}) = \frac{\ldots\langle 12\rangle^3\ldots}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \frac{\omega_1\omega_2}{\omega_3\omega_4}\frac{z_{12}^3}{z_{23}z_{34}z_{41}}\ldots$$

$$= \frac{1}{a}\frac{z_{12}\bar{z}_{34}}{\bar{z}_{12}z_{34}}\,\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right)$$

color factor & couplings

← conf. prefactor

$$\tilde{A}(-,-,+,+) = \left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)(1-a)^{2/3}a^{-7/3}\times$$

$$\times g(\lambda_1)g(\lambda_2)g(\lambda_3)g(\lambda_4)\times\int_0^\infty \omega^{i(\lambda_1+\lambda_2+\lambda_3+\lambda_4)-1}d\omega$$

↑ normalization

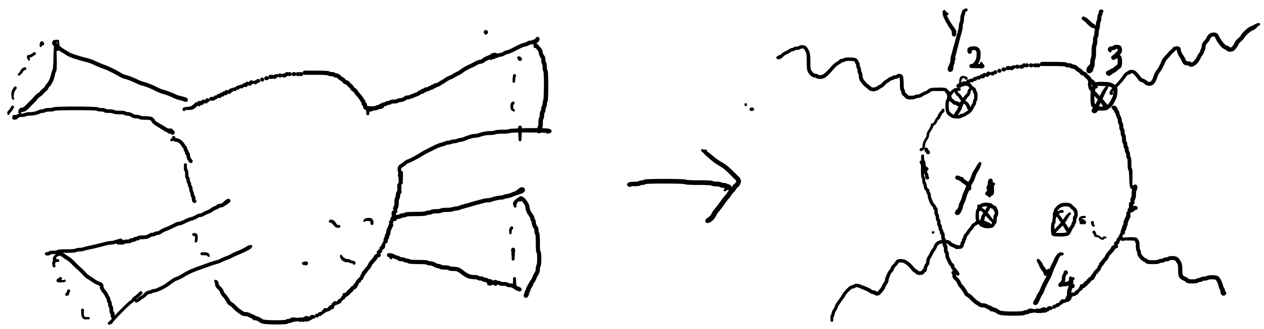
depends (again) on 2 variables: $a,\ \sum_{i=1}^4\lambda_i$

$$h_+ = \bar{h}_- = 1+\frac{i\lambda}{2} \qquad \bar{h}_+ = h_- = \frac{i\lambda}{2}$$

$$\int_0^\infty \omega^{i\Sigma\lambda_i-1} = 2\pi\,\delta\Big(\sum_{i=1}^4\lambda_i\Big)$$

consequence of 4D conf. inv. of YM at tree level

String theory (heterotic or type I open)



$$\mathcal{M}_{H,I} \sim \int dy_1\, dy_2\, dy_3\, dy_4 \; V(y_1)\,V(y_2)\,V(y_3)\,V(y_4)$$

vertex operators
integrated on world-sheet

$$\to \quad \mathcal{M}_{H,I} = \mathcal{M} \cdot F_{H,I}(\alpha' s, \alpha' u)$$

string formfactor

$$\alpha' \to 0 \;(s, u \ll M^2) \Rightarrow F \to 1$$

string theory $\to$ QFT

e.g.

$$F_I = \frac{\Gamma(1-\alpha' s)\,\Gamma(1-\alpha' u)}{\Gamma(1-\alpha' s-\alpha' u)}$$

massive string poles at $s = nM^2$

supersoft UV:

$$F_I \xrightarrow[\substack{s\to\infty \\ u/s=-a}]{} e^{-\alpha' s}$$

supersoft in UV

$$\tilde{A}_I(-,-,+,+) = \left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)(1-a)^{2/3}a^{-7/3}$$

$$\underbrace{\times\int_0^\infty d\omega\,\omega^{i\Sigma\lambda-1}\,F_I(\alpha'\omega, a)}_{(\alpha')^{\frac{-i\Sigma\lambda}{2}}\,I(\Sigma\lambda,\ a=\sin^2\frac{\theta}{2})}$$

$s\sim\omega$

trivial α'-dependence, reduced to a factor!

So how to reach YM limit? Where are strings?

$$I\left(\sum\lambda,\ a=\sin^2\frac{\theta}{2}\right) = \underbrace{2\pi\,\delta\left(\sum\lambda\right)}_{YM}$$

$$\left.+\ldots\sum_{k=1}^{\infty} c_k\left(i\sum\lambda\right)\cdot\left(\sin^2\frac{\theta}{2}\right)^k\right\}\text{small angle expansion}$$

coefficients are related to residues of massive string poles

YM recovered at $\theta\to 0$ (forward scattering)

Another limit: $\sum \lambda \to \infty$

F_I originates from integrating vertex position over world-sheet (boundary)

$$F_I \sim \int dx \; x^{-\alpha' s} (1-x)^{-\alpha' u}$$

$$\int_0^\infty d\omega \, \omega^{i\sum\lambda - 1} F_I(\alpha'\omega, a)$$

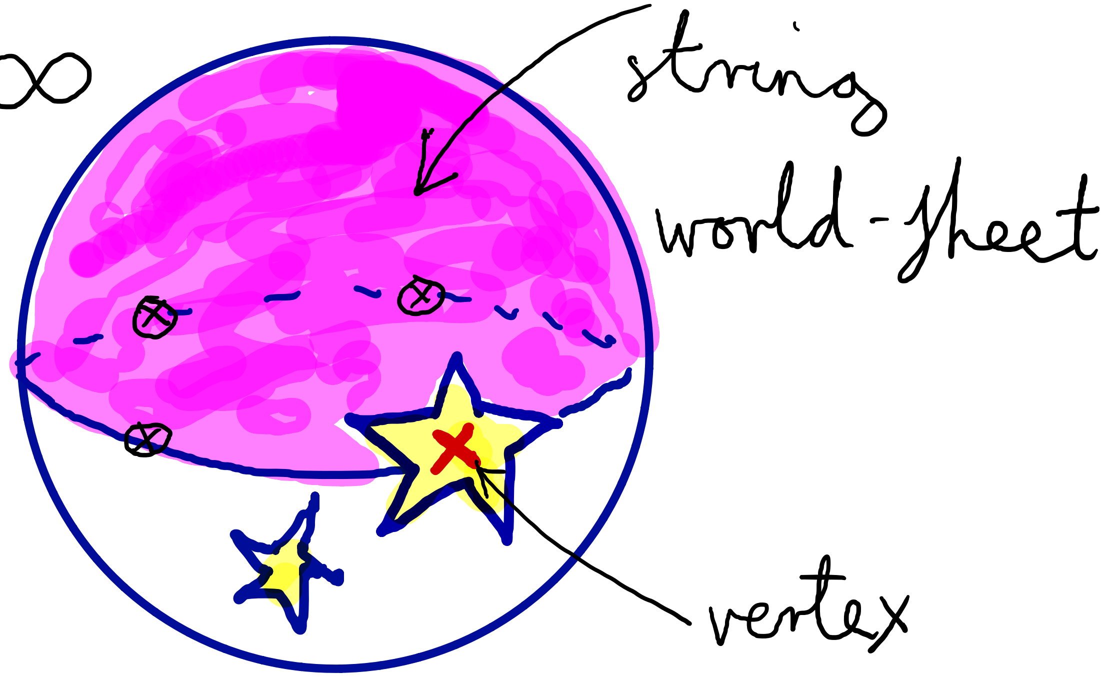
$$\overset{d\omega \leftrightarrow dx}{\sim} \int \frac{dx}{x} \left[\ln x - a \ln(1-x) \right]^{i\sum\lambda}$$

stationary phase $\leftrightarrow$ Gross-Mende saddle p.

$$x = a$$

vertex position mapped on CS

$\sum \lambda \to \infty$

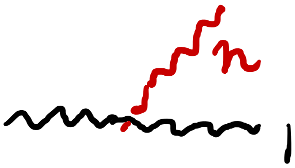


String world-sheet on celestial sphere

$$\text{String CFT} \overset{?}{\longleftrightarrow} \text{Celestial CFT}$$

Soft vs. Conformally Soft & Ward identities

$$\mathcal{M}_{J_1\ldots(J_n=+1)} \xrightarrow[\omega_n\to 0]{} \frac{1}{\omega_n}\left(\frac{1}{z_{n-1n}}+\frac{1}{z_{n1}}\right)\mathcal{M}_{J_1,\ldots J_{n-1}}$$



$$\tilde{\mathcal{A}}_{J_1\ldots J_n} = \ldots\, g(\lambda_n)\int \underbrace{d\omega_n\,\omega_n^{i\lambda_n}\,\mathcal{M}}_{\frac{1}{\lambda_n}}$$

$$\xrightarrow[\lambda_n\to 0]{} \lambda_n \times \frac{1}{\lambda_n} \sim \text{finite}$$

↑ from $\omega_n \approx 0$

Conformally Soft ≡ Soft FFT

Pate, Raclariu, Strominger, Nandan, Schreiber, Volovich, Zlotnikov

$$\tilde{\mathcal{A}}_{J_1\ldots(J_n=+1)} \xrightarrow[\lambda_n\to 0]{} \left(\frac{1}{z_{n-1n}}+\frac{1}{z_{n1}}\right)\tilde{\mathcal{A}}_{J_1\ldots J_{n-1}}$$

On the other hand, $\lambda = 0$ gluon is associated to a holomorphic current :

$$\mathcal{O}^{a}_{\lambda=0, J=+1}(z,\bar{z}) = j^{a}(z)$$

$$\langle j^a(z)\mathcal{O}^{b_1}_{\lambda_1 J_1}(z_1,\bar{z}_1)\mathcal{O}^{b_2}_{\lambda_2 J_2}(z_2,\bar{z}_2)\dots\mathcal{O}^{b_n}_{\lambda_n J_n}(z_n,\bar{z}_n)\rangle =$$

$$= \sum_{i=1}^{n}\sum_{c} \frac{f^{ab_i c}}{z-z_i}\langle\mathcal{O}^{b_1}_{\lambda_1 J_1}(z_1,\bar{z}_1)\dots\mathcal{O}^{c}_{\lambda_i J_i}(z_i,\bar{z}_i)\dots\mathcal{O}^{b_n}_{\lambda_n J_n}(z_n,\bar{z}_n)\rangle$$

Conformally Soft Limit $\equiv$ Ward identity

Conformally soft gluon $\sim$ Goldstone mode

In CFT, Ward identities follow from OPE

j^a is the (global) gauge current

OPE

$$\mathcal{O}^a_{\lambda_1+}(z,\bar z)\,\mathcal{O}^b_{\lambda_2+}(w,\bar w) \underset{z\to w}{\sim} \;?$$

collinear limit

Can be extracted from well-known collinear limits of YM amplitudes:

FFT

$$\mathcal{O}^a_{\lambda_1+}(z,\bar z)\,\mathcal{O}^b_{\lambda_2+}(w,\bar w) = \frac{C_{(++)+}(\lambda_1,\lambda_2)}{z-w}\sum_c f^{abc}\mathcal{O}^c_{(\lambda_1+\lambda_2)+}(w,\bar w) + \ldots,$$

$$C_{(++)+}(\lambda_1,\lambda_2) = 1 + \frac{\lambda_1\lambda_2}{(1+i\lambda_1)(1+i\lambda_2)} \qquad \text{etc.}$$

$$\left\{\begin{array}{ccc} \text{OPE} & \rightarrow & \text{Ward identity} \\ \text{collinear} & \rightarrow & \text{soft} \end{array}\right\}$$

Symmetries of 2D celestial CFT as inherited from 4D

ST

$$\tilde{\mathcal{A}}(z_i, \bar{z}_i, h_i, \bar{h}_i)$$

4D Lorentz $\to$ 2D conformal generators

$$\mathfrak{su}(1,1)\left\{\begin{aligned}
L_1 &\equiv M_{23} + iM_{10} = (1-z^2)\partial_z - 2zh \\
L_2 &\equiv M_{20} + iM_{13} = (1+z^2)\partial_z + 2zh \\
L_3 &\equiv M_{21} + iM_{03} = 2(z\partial_z + h)\ ,
\end{aligned}\right.$$

4D translations:

involve shifts

$\Delta \to \Delta + 1$

$\Rightarrow$ finite translation to $\Delta = \infty$???

$$\left\{\begin{aligned}
P_0 &= (1+|z|^2)e^{(\partial_h + \partial_{\bar{h}})/2} \\
P_1 &= (z+\bar{z})e^{(\partial_h + \partial_{\bar{h}})/2} \\
P_2 &= -i(z-\bar{z})e^{(\partial_h + \partial_{\bar{h}})/2} \\
P_3 &= (1-|z|^2)e^{(\partial_h + \partial_{\bar{h}})/2}
\end{aligned}\right.$$

Conclusions

Celestial amplitudes are 2D CFT correlators of primary field operators of the principal series $\Delta_i = 1 + i\lambda_i$

They depend on celestial coordinates

In 4D conf. invariant theory

$$\sum_i \lambda_i = 0$$

Celestial superstring amplitudes have trivial α'-dependence but a very interesting limit of $\sum_i \lambda_i \to \infty$ in which the world-sheet wraps on celestial sphere.

This raises an interesting question what is the relation between world-sheet & celestial CFTs

Celestial amplitudes with gravitons are UV divergent in Einstein's GR but well defined in superstring theory due to supersoft UV behavior

Celestial CFT

- principal series $\Delta = 1 + i\lambda$
- $\Delta = 1$ conformal soft limit $\leftrightarrow$ soft $\omega = 0$
- soft theorems $\longleftrightarrow$ Ward identities of symmetry current
- we know some OPEs, in particular those underlying Ward identities

Many things to be done...