Open, closed and heterotic strings at one loop: Elliptic MZVs versus modular graph forms

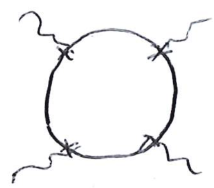
based on:
- 1412.5535 : J. Brödel, C. Mafra, N. Matthes, OS
- 1803.00527 : J. Brödel, OS, F. Zerbini
- 1811.02548 : J. Gerken, A. Kleinschmidt, OS

Long-term goals

- physics : $\xrightarrow{\alpha'\text{-expand}}$ closed-string amplitudes ("hard")
  from (single-valued) open-string input ("easier")

$$\text{e.g.}\quad \overbrace{\frac{1}{\pi}\int_{\mathbb{C}} \frac{d^2z}{z\bar z(1-\bar z)}\, |z|^{2s_{12}}|1-z|^{2s_{23}}}^{\frac{1}{s_{12}}+2\zeta_3 s_{23}(s_{12}+s_{23})+O(\alpha'^4)} = \mathrm{sv}\overbrace{\int_0^1 \frac{dz}{z}\, z^{s_{12}}(1-z)^{s_{23}}}^{\frac{1}{s_{12}}-\zeta_2 s_{23}+\zeta_3 s_{23}(s_{12}+s_{23})+O(\alpha'^3)}$$

closed-string tree: sphere



open-string tree: disk

tree level: single-valued MZVs [Brown, Schnetz, etc..]

$$\mathrm{sv}\,\zeta_{2k}=0\,,\quad \mathrm{sv}\,\zeta_{2k+1}=2\zeta_{2k+1}\,,\quad \mathrm{sv}\,\zeta_{3,5}=-10\zeta_3\zeta_5\,,\ \text{etc.}$$

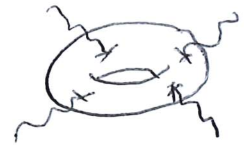
- mathematics : non-hol. modular forms
  from (iterated integrals over) holomorphic Eisenstein series

Main characters:

Genus-one integrals I_{1234}, $B^{(w,0)}_{1234}$

@ fixed surface τ and $z_1 = 0$

inequiv.



(II) $$\mathcal{M}^{1\text{-loop}}_{\mathrm{IIA/B}} \sim \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\, B^{(0,0)}_{1234}(s_{ij},\tau)$$

symmetrize in z_2, z_3, z_4 then "esv"

$$B^{(0,0)}_{1234}(s_{ij},\tau) = \int_{(\mathrm{torus}(\tau))^3} \left(\prod_{j=2}^{4} \frac{d^2 z_j}{\mathrm{Im}\,\tau}\right) \exp\left(\sum_{i<j}^{4} 2 s_{ij}\, G(z_{ij},\tau)\right)$$

bos Green fct.

$$\log\left|\frac{\theta_1(z_i-z_j,\tau)}{\theta_1'(0,\tau)}\right|^2 - 2\pi\,\frac{(\mathrm{Im}(z_i-z_j))^2}{\mathrm{Im}\,\tau}$$

(I) $$A^{1\text{-loop}}_{\mathrm{open}}(1,2,3,4) \sim \int_0^{i\infty} d\tau\, I_{1234}(s_{ij},\tau)$$

$$I_{1234}(s_{ij},\tau) = \int_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4\, \exp\left(\sum_{i<j}^{4} s_{ij}\, G(z_{ij},\tau)\right)$$

no symmetrization just "esv"

(III) $$A^{1\text{-loop}}_{\mathrm{het}}(1,2,3,4) \sim \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \left\{ \frac{G_{10}(\tau)}{\eta^{24}(\tau)}\, B^{(2,0)}_{1234}(s_{ij},\tau) + \ldots \right\}$$

$B^{(2,0)}_{1234} \rightarrow$ as $B^{(0,0)}_{1234}$, but with elliptic fct. $V_2(z_1,z_2,z_3,z_4|\tau)$ ~~of~~ in the integrand

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I) α'-expansion of I_{1234} ($\int dz_j$ order by order in $s_{ij} G(z_{ij},\tau)$)

$\Downarrow$ 1412.5535

elliptic multiple zeta values (eMZVs)

$\cap$ 1301.3042, 1507.02254

$\mathbb{Q}[\mathrm{MZV}, \frac{1}{2\pi i}]$ combinations of iterated Eisenstein int's

$$\mathcal{E}_0(k_1, k_2, \ldots, k_r; \tau) = (2\pi i)^{1-k_r} \int_\tau^{i\infty} d\tau' \, G^0_{k_r}(\tau') \, \mathcal{E}_0(k_1, \ldots, k_{r-1}; \tau')$$

convergent if $k_1 \geq 4$

$$G_k(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^k} = 2\zeta_k + G^0_k(\tau)$$

$O(q)$: $\frac{2(2\pi i)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn}$

with $\mathcal{E}_0(\emptyset;\tau) = 1$ and $k_j \in \mathbb{N}_0$ and $G^0_0 = -1$

e.g. $\mathcal{E}_0(k, 0, \ldots, 0; \tau) = \frac{-2}{(k-1)!} \sum_{m,n=1}^{\infty} \frac{m^{k-1}}{(mn)^p} q^{mn}$

$$\begin{aligned} I_{1234}(s_{ij};\tau) &= \frac{1}{6} + \frac{3 s_{13}}{2\pi^2}\left(\zeta_3 - 6\,\mathcal{E}_0(4,0,0;\tau)\right) \\ &+ \frac{1}{\pi^2}(s_{13}^2 + 2 s_{12} s_{23})\left(60\,\mathcal{E}_0(6,0,0,0;\tau) - \frac{\zeta_4}{2}\right) \\ &- (s_{12}^2 + s_{13}^2 + s_{14}^2)\left(\mathcal{E}_0(4,0;\tau) - \frac{\zeta_2}{12}\right) + O(\alpha'^3) \end{aligned}$$

II) By Pierre Vanhove's talk: mod invariant series

$$B^{(0,0)}_{1234}(s_{ij};\tau) = 1 + E_2 \sum_{j=2}^{4} s_{1j}^2 + \frac{1}{3}(5E_3 + \zeta_3)\sum_{j=2}^{4} s_{1j}^3 + \underbrace{O(\alpha'^4)}$$

non-hol. Eisenstein series

$$E_k(\tau) = \left(\frac{\operatorname{Im}\tau}{\pi}\right)^k \sum_{(m,n)\neq(0,0)} \frac{1}{|m\tau+n|^{2k}}$$

more general "modular graph fct's" (MGF's) far beyond E_k [1512.06779]

MGF's $\longrightarrow$ $y := \pi \operatorname{Im}\tau$, svMZVs & $\operatorname{Re}\mathcal{E}_0(\ldots;\tau)$, e.g.

$$E_2 = \underbrace{\frac{y^2}{45} + \frac{\zeta_3}{y}}_{\text{dominant @ } \tau\to i\infty} \underbrace{- 12\operatorname{Re}\mathcal{E}_0(4,0;\tau) - \frac{6}{y}\operatorname{Re}\mathcal{E}_0(4,0,0;\tau)}_{O(q) \text{ or } O(\bar q), \text{ suppressed @ } \tau\to i\infty}$$

which open-string quantity to compare with?

$\hookrightarrow$ $I_{1234}(s_{ij};\tau)$ after
- symmetrizing $\int_{0<z_2<z_3<z_4<1}$ over z_2, z_3, z_4
- modular transf. $\tau \to -\frac{1}{\tau}$

$$\sum_{\rho\in S_3} I_{1\rho(234)}(s_{ij}, -\tfrac{1}{\tau}) = 1 + \sum_{j=2}^{4} s_{1j}{}^2 \Big\{ -\frac{T^2}{180} + \frac{\zeta_2}{3} + \frac{i\zeta_3}{T} + \frac{3\zeta_4}{2T^2}$$

$T = \pi\tau$

$$\underbrace{-6\,\mathcal{E}_0(4,0;\tau) - \frac{6i}{T}\,\mathcal{E}_0(4,0,0;\tau)}\Big\} + O(s_{ij}^3)$$

expressed $\mathcal{E}_0(4,0;-1/\tau)$ via $\mathcal{E}_0(\ldots;\tau)$

Conjectural mapping [1803.00527]

$$\boxed{\mathcal{B}^{(0,0)}_{1234}(s_{ij};\tau) = \mathrm{esv}\sum_{\rho\in S_3} I_{1\rho(234)}(s_{ij}, -\tfrac{1}{\tau})}$$

with proposed elliptic sv map

$$\mathrm{esv}\begin{cases} \zeta_{n_1,n_2,\ldots,n_r} \to \mathrm{sv}\,\zeta_{n_1,n_2,\ldots,n_r} \\ \tau \to (\tau-\bar\tau) = 2i\,\mathrm{Im}\,\tau \\ \mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) = 2\,\mathrm{Re}\,\mathcal{E}_0(k_1,k_2,\ldots,k_r;\tau) \end{cases}$$

same for modular forms $\forall$ mod weights

only mod. graph fct's, & issues with shuffle multiplication

need reformulation for non-zero mod. weight & compatibility with ш

see Pierre's talk

e.g. $\mathrm{esv}\,\mathcal{E}_0(4,0;-\tfrac{1}{\tau}) = -\tfrac{1}{6}E_2(\tau)$ @ $O(s_{ij}^2)$

or $\mathrm{esv}\,\mathcal{E}_0(4,4,0,0;-\tfrac{1}{\tau}) = -\tfrac{1}{36}C_{2,1,1}(\tau) + \tfrac{1}{40}E_4(\tau)$

III) Loose end : $\mathrm{esv}\, I_{1234}$ without $\sum_{\rho\in S_3}$

↳ part of $A^{1\text{-loop}}_{het}(1,2,3,4)$

$$\mathcal{B}^{(2,0)}_{1234}(s_{ij};\tau) = \int_{(\mathrm{torus}(\tau))^3}\Big(\prod_{j=2}^{4}\frac{d^2z_j}{\mathrm{Im}\,\tau}\Big)\underbrace{V_2(z_1,z_2,z_3,z_4|\tau)}\prod_{i<j}^{4} e^{s_{ij}G(z_{ij};\tau)}$$

elliptic & modular weight (2,0), [0710.3743]

up to two simult. poles in $(z_i - z_{i+1})$

Kronecker-Eisenstein ser. $F(z,\beta,\tau) = \frac{\theta_1'(0,\tau)\,\theta_1(z+\beta,\tau)}{\theta_1(z,\tau)\,\theta_1(\beta,\tau)}$

$$V_2(z_1,z_2,z_3,z_4|\tau) = F(z_{12},\beta,\tau)F(z_{23},\beta,\tau)F(z_{34},\beta,\tau)F(z_{41},\beta,\tau)\Big|_{\beta^2}$$

α'-expansion $\Rightarrow$ $\left(\nabla = 2i(\mathrm{Im}\,\tau)^2 \frac{\partial}{\partial\tau}\right)$ $\left(\begin{array}{c}\text{modular}\\ \text{graph}\\ \text{functions}\end{array}\right)$ $\left[\begin{array}{c}1811.\\ 02548\end{array}\right]$

$$B^{(2,0)}_{1234}(s_{ij},\tau) = \frac{\pi}{(\mathrm{Im}\,\tau)^2}\Big\{3s_{13}\nabla E_2 + \frac{2}{3}(s_{13}^2+2s_{12}s_{23})\nabla E_3$$
$$+ s_{13}(s_{12}^2+s_{13}^2+s_{14}^2)\left(6\nabla C_{2,1,1} - 5\nabla E_4 + 3E_2\nabla E_2\right) + O(s_{ij}^4)\Big\}$$

Expanding @ $\tau\to i\infty$

$$B^{(2,0)}_{1234}(s_{ij},\tau)\Big|_{q^0\bar{q}^0} = \pi^2 s_{13}\left(\frac{2y}{15} - \frac{3\zeta_3}{y^2}\right)$$
$$+\pi^2(s_{13}^2+2s_{12}s_{23})\left(\frac{4y^2}{945} - \frac{\zeta_5}{y^3}\right) + O(\alpha'^3)$$
$$= (2\pi i)^2\,\mathrm{esv}\left(\frac{2}{3}I_{1234}(s_{ij},-\tfrac{1}{\tau}) - \frac{1}{3}I_{1342}(s_{ij},-\tfrac{1}{\tau}) - \frac{1}{3}I_{1423}(s_{ij},-\tfrac{1}{\tau})\right)$$

- can only match antiholomorphic $\mathcal{E}_0(\ldots;i\tau)$ $+\,O(q)$
- tested up to & including $(\alpha')^3$

IV) Summary & outlook

- iterated Eisenstein int's $\mathcal{E}_0$
  - $\rightarrow$ framework to relate open & closed strings @ g=1
- modular forms of weight (0,0) & (2,0) from "esv" (open-string integrals)
  - symmetrized cylinder orderings (weight (0,0))
  - non-symmetrized ordering (weight (2,0))

need to refine $\mathcal{E}_0(\ldots) \rightarrow 2\,\mathrm{Re}\,\mathcal{E}_0(\ldots)$ part