Manifest T-Duality from a World-Sheet Perspective

Franco Pezzella

INFN - Naples Section - Italy

with V. E. Marotta (Heriot-Watt Edimbourgh) and P. Vitale (Naples Univ. *Federico II*)

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String Field Theory and String Perturbation Theory
GGI - Florence - May, 6 2019

Plan of the talk

1. Introduction and Motivation

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Dualities

- Two non-linear sigma models with different Lagrangian densities may correspond to the same classical theory: under certain conditions there exists an involution, known as *duality*, generating transformations which change the geometry of the target manifold but leave physics unchanged.
- Duality symmetries play an important role in string theory as tools for disentangling its full symmetry structure. They led to important insights in understanding the geometry of space-time from the string point of view.
- Helpful information could be inferred from the sigma model constructed by making the duality simmetry manifest at the Lagrangian level.

Reminding T-Duality in String Theory

- T-duality is an old subject in string theory. It implies that in many cases two different geometries for the extra-dimensions are physically equivalent and that string physics at a very small scale cannot be distinguished from the one at a large scale. It is also a clear indication that ordinary geometric concepts can break down at the string scale.
- T-duality has been a valuable guide for finding new phenomena such as D-branes, mirror symmetry or exotic solutions.
- In the case of circular compactification of the space coordinate x^a , T-duality is encoded, for bosonic closed strings, in the simultaneous transformations $R \leftrightarrow \alpha'/R$ and $p_a \leftrightarrow w^a/\alpha'$ under which the string coordinate $X^a = X^a_L + X^a_R \leftrightarrow \tilde{X}_a \equiv X^a_L - X^a_R$, with w^a playing the role of momentum mode for $\tilde{X}_a$. These transformations leave the mass spectrum invariant.

O(D,D)-Duality in String Theory

- In toroidal compactifications on T^d (with constant backgrounds $G_{\mu\nu}$ and $B_{\mu\nu}$) T-duality is described by $O(d,d;\mathbb{Z})$ transformations.
- Already at the classical level the indefinite orthogonal group $O(D,D)$ naturally appears from the constraints in the Hamiltonian description of the usual bosonic string model.
- With $*$ the Hodge operator with respect to $h = \mathrm{diag}(-1,1)$, the action is:

$$S[X;G,B] = \frac{T}{2}\int \left[G_{ab}(X)dX^a \wedge *dX^b + B_{ab}(X)dX^a \wedge dX^b\right]$$

- Varying S with respect to X^a yields the equation of motion:

$$d * dX^a + \Gamma^a{}_{bc}dX^b \wedge *dX^c = \frac{1}{2}G^{am}H_{mbc}dX^b \wedge dX^c$$

with $H = dB$ and $\Gamma^a{}_{bc} = \frac{1}{2}G^{am}(\partial_b G_{mc} + \partial_c G_{mb} - \partial_m G_{bc})$ the coefficients of the Levi-Civita connection.

O(D,D)-Duality in String Theory

- The dynamics of the theory is determined by the equations of motion for the coordinates X^a accompanied with the constraints (in the conformal gauge):

$$G_{ab}(\dot{X}^a\dot{X}^b + X'^a X'^b) = 0 \qquad G_{ab}\dot{X}^a X'^b = 0$$

deriving from $T_{\alpha\beta} = 0$.

- The Hamiltonian density H is determined through a Legendre transformation with respect to the canonical momentum $P_a = \frac{\partial L}{\partial \dot{X}^a} = \frac{1}{2\pi\alpha'}\left(G_{ab}\dot{X}^b + B_{ab}X'^b\right)$ and $\dot{X}^a$.

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- The first constraint implies $P_a\dot{X}^a = W_a X'^a$. H can therefore also result from a Legendre transformation with respect to the canonical winding $W_a = \frac{\partial L}{\partial X'^a} = -\frac{1}{2\pi\alpha'}\left(G_{ab}X'^b + B_{ab}\dot{X}^b\right)$ and X'^a.

O(D,D) Invariance of the Hamiltonian Density

- The Hamiltonian density $H_d = \frac{1}{4\pi\alpha'} G_{ab}(\dot{X}^a\dot{X}^b + X'^a X'^b)$ can be written equivalently as:

$$\begin{aligned} H_d &= \frac{1}{4\pi\alpha'} \begin{pmatrix} \partial_\sigma X \\ 2\pi\alpha' P \end{pmatrix}^t \mathcal{H}(G,B) \begin{pmatrix} \partial_\sigma X \\ 2\pi\alpha' P \end{pmatrix} \\ &= \frac{1}{4\pi\alpha'} \begin{pmatrix} \partial_\tau X \\ -2\pi\alpha' W \end{pmatrix}^t \mathcal{H}(G,B) \begin{pmatrix} \partial_\tau X \\ -2\pi\alpha' W \end{pmatrix} \end{aligned}$$

  where the *generalized metric* is introduced:

$$\mathcal{H}(G,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

- H_d results to be proportional to the squared length of the generalized vectors A_P and A_W in $TM \bigoplus T^*M$, as measured by the generalized metric $\mathcal{H}$:

$$\begin{aligned} A_P(X) &= \partial_\sigma X^a \frac{\partial}{\partial x^a} + 2\pi\alpha' P_a dx^a \\ A_W(X) &= \partial_\tau X^a \frac{\partial}{\partial x^a} - 2\pi\alpha' W_a dx^a \end{aligned}$$

Constraints and Generalized Vectors

- In terms of A_P the constraints become:

$$A_P^t \mathcal{H} A_P = 0 \qquad A_P^t \eta A_P = 0.$$

The first constraint sets H_d to zero and the second constraint completely determines the dynamics and it is rewritten in terms of the matrix $\eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$, i.e. the invariant metric of the group $O(D, D)$ defined by the $D \times D$ matrices $\mathcal{T}$ satisfying the condition $\mathcal{T}^t \eta \mathcal{T} = \eta$. In particular the generalized metric is an element of $O(D, D)$, i.e. $\mathcal{H}^t \eta \mathcal{H} = \eta$.

- All the admissible generalized vectors satisfying $A_P^t \eta A_P = 0$ are related by an $O(D, D)$ transformation via $A'_P = \mathcal{T} A_P$.
- For A'_P to solve the first constraint a compensating transformation $\mathcal{T}^{-1}$ has to be applied to $\mathcal{H}$, i.e. $\mathcal{H}' = (\mathcal{T}^{-1})^t \mathcal{H} (\mathcal{T}^{-1})$.

$O(D,D)$ AND CONSTANT BACKGROUNDS

- In the presence of constant backgrounds (G, B), the equations of motion for the string coordinates are a set of conservation laws on the world-sheet:

$$\partial_\alpha J^\alpha_a = 0 \quad \text{with} \quad J^\alpha_a = h^{\alpha\beta} G_{ab} \partial_\beta X^b + \epsilon^{\alpha\beta} B_{ab} \partial_\beta X^b$$

- Locally, one can express such currents as:

$$h^{\alpha\beta} G_{ab} \partial_\beta X^b + \epsilon^{\alpha\beta} B_{ab} \partial_\beta X^b \equiv \epsilon^{\alpha\beta} \partial_\beta \tilde{X}_a \quad \rightarrow \quad \text{dual coordinates}$$

- In terms of the coordinates $\tilde{X}_a$'s:

$$\tilde{S}[X; G, B] = \frac{T}{2} \int \left[\tilde{G}_{ab} d\tilde{X}^a \wedge * d\tilde{X}^b + \tilde{B}_{ab}(X) d\tilde{X}^a \wedge d\tilde{X}^b \right]$$

with $\tilde{G} = (G - BG^{-1}B)^{-1}$ and $\tilde{B} = -G^{-1} B \tilde{G}$.

$O(D, D; \mathbb{R}) \to O(d, d; \mathbb{Z})$

- If the closed string coordinates are defined on a compact target manifold T^d, the dual coordinates satifisfy the same periodicity conditions and then T-duality maps two theories of the same type into one another $\to$ exact symmetry.
- For closed strings, toroidal compactification means:

$$X^a(\sigma,\tau) \equiv X^a(\sigma+\pi,\tau) + 2\pi L^a \quad L^a = \sum_{i=1}^{d} w_i R_i e_i^a$$

with w_i being the winding numbers and e_i^a vector basis on T^d.

- The equations of motion for the coordinates $\chi^A = (X^a, \tilde{X}_a)$ $(A = 1, \ldots 2d), (a = 1, \ldots d)$ can be combined into a single $O(d, d; \mathbb{Z})$-invariant equation:

$$\mathcal{H}\partial_\alpha \chi = \eta \epsilon_{\alpha\beta}\partial^\beta \chi$$

- For $G_{ab} = \eta_{ab}$ and $B = 0$, the equations of motion become the duality conditions: $\partial_\alpha X^a = \epsilon_\alpha{}^\beta \partial_\beta \tilde{X}^a$.

T-dual invariant bosonic string formulation

- In the case of toroidal compactification $O(D,D) \to O(d,d;\mathbb{Z})$. This group becomes a *symmetry* not only of the mass spectrum and the vacuum partition function but also of the scattering amplitudes.
- The presence of the $O(d,d;\mathbb{Z})$ symmetry suggests to extend the usual formulation of String Theory, based on the Polyakov action, by introducing this symmetry at the level of the world-sheet sigma-model. It would be interesting, therefore, looking for a manifestly $O(d,d;\mathbb{Z})$-dual invariant formulation of string theory.
- The introduction of *both* the coordinates X^a *and* the dual ones $\tilde{X}_a$ is required. Such formulation is based on a doubling of the string coordinates in the target space.

Doubling Coordinates: Motivation

- The main goal of this new action would be to explore more closely aspects of stringy geometry. In fact, if interested in writing down the complete effective field theory of such generalized sigma-model, one should consider, correspondingly to the introduction of X^a and $\tilde{X}_a$, a dependence on such coordinates of the fields associated with string states. In this way, the string effective field theory becomes a Double Field Theory.

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$$\Phi(k_a, w^a) \leftrightarrow \Phi(X^a, \tilde{X}_a)$$

- Double Field Theory is a proposal to incorporate T-duality as a symmetry of an effective field theory for a string living in a space-time which is the product of a Minkowski space M with a d-dimensional torus, $M \times T^d$.

Double Field Theory

- In its simplest formulation, the field content involves the d-dimensional metric G_{ab}, the Kalb-Ramond field B_{ab} and a scalar dilaton ϕ, with G_{ab} and B_{ab} *unified* in a single object: a generalized symmetric $O(d,d)$ metric $\mathcal{H}_{MN}$, with $M, N = 1, \cdots, 2d$.
- A consistent Double Field Theory, invariant under *double diffeormorphisms*, is obtained if the so-called *section condition* is imposed: $2\partial^a\tilde{\partial}_a f = \partial^M\partial_M f = 0$ on fields and their products and on the gauge parameters characterizing the standard linearized diffeomorphisms and the antisymmetric tensor gauge transformations.

- In the decompactification limit, when the dual coordinates are projected out through the section condition, the DFT action has to reproduce the action of the universal massless bosonic sector of supergravity:

$$S = \int dX \sqrt{G} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right]$$

Supergravity with manifest T-duality

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- The main purpose of DFT is to provide an answer to the question: How the supergravity action becomes when G, B and ϕ are dependent on X^a and $\tilde{X}_a$? Which symmetries and what properties would it have? This could shed light on aspects of string (super)gravity unexplored thus far.

Doubled Worlsheet

- The Floreanini-Jackiw Lagrangians, [R. Floreanini and R. Jackiw,1987]associated with chiral and anti-chiral scalar fields ϕ_+ and ϕ_-:

$$\mathcal{L}_\pm(\phi_\pm) = \pm\frac{1}{2}\dot{\phi}_\pm\phi'_\pm - \frac{1}{2}\phi'^2_\pm$$

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inspired the following general string "sigma model": [Tseytlin, Nucl. Phys. and Phys. Lett., 1991]

$$S = -\frac{T}{2}\int d^2\sigma e\left[C_{ij}\nabla_0\chi^i\nabla_1\chi^j + M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right]$$

- ${e^a}_\alpha \rightarrow$ zweibein defined on the string world-sheet; $C_{ij} = C_{ji}$ and $M_{ij} = M_{ji}$; $\nabla_a\chi^i = e^\alpha_a\partial_\alpha\chi^i$, the functions χ^i the string coordinates in an N-dimensional Riemannian target space.
- S is invariant under:
  1. diffeomorphisms:

$$\sigma^\alpha \rightarrow \sigma'^\alpha(\sigma)$$

  2. Weyl transformations:

$${e^a}_\alpha \rightarrow \lambda(\sigma){e^a}_\alpha$$

Tseytlin action

- The action S is not manifestly invariant under local Lorentz transformations: $e^a{}_\alpha \to e'^a{}_\alpha = \Lambda^a{}_b(\sigma) e^b{}_\alpha$ where $\Lambda^a{}_b$ is an arbitrary Lorentz matrix $SO(1,1)$ but such invariance has to be required on-shell being physical observables are independent on the choice of the vielbein.
- Since the variation of S under an infinitesimal local Lorentz transformation results to be:

$$\frac{\delta S}{\delta e^a{}_\alpha} \delta e^a{}_\alpha \sim \epsilon^a{}_b t_a{}^b \ ,$$

the above requirement implies:

$$\epsilon^{ab} t_{ab} = 0 \qquad t_a{}^b \equiv -\frac{2}{T} \frac{1}{e} \frac{\delta S}{\delta e^a{}_\alpha} e^b{}_\alpha$$

Constant Backgrounds

- With C and M constant, the equations of motion for χ^i are:

$$C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j = 0$$

- This causes the constraint on the ϵ-trace to become $C = M^t C M$.
- After rotating and rescaling χ^i, C can always be put in the diagonal form:

$$C = (1, \cdots, 1, -1, \cdots, -1)$$

with N_+ eigenvalues 1 and N_- eigenvalues -1 and $N = N_+ + N_-$. So the action can be interpreted as describing interacting N_+ chiral and N_- antichiral scalars:

$$\chi^i = \begin{pmatrix} \chi_-^\mu \\ \chi_+^\mu \end{pmatrix} \qquad i = 1, \ldots, 2D \quad \mu = 1, \ldots, D$$

and the absence of a quantum Lorentz anomaly requires $N_+ = N_- = D = \frac{N}{2}$. Hence, $N = 2D$.

Non-chiral coordinates

- It is possible to make a change of coordinates in the $2D$-dimensional target space defining:

$$\begin{pmatrix} X^\mu \\ \tilde{X}_\mu \end{pmatrix}$$

with

$$X^\mu \equiv \frac{1}{\sqrt{2}}\left(\chi_+^\mu + \chi_-^\mu\right) \;\; ; \;\; \tilde{X}_\mu \equiv \delta_{\mu\nu}\frac{1}{\sqrt{2}}\left(\chi_+^\nu - \chi_-^\nu\right)$$

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- C becomes off-diagonal:

$$C_{ij} = -\eta_{ij} \;\; ; \;\; \eta_{ij} = \begin{pmatrix} 0_{\mu\nu} & \mathbb{I}_\mu^{\ \nu} \\ \mathbb{I}^\mu_{\ \nu} & 0^{\mu\nu} \end{pmatrix}$$

- The expression for $\mathcal{H}$ results to be:

$$\mathcal{H} = \begin{pmatrix} (G - B\,G^{-1}B)_{\mu\nu} & (B\,G^{-1})_\mu^{\ \nu} \\ (-G^{-1}\,B)^\mu_{\ \nu} & (G^{-1})^{\mu\nu} \end{pmatrix}$$

- The $O(D, D)$-invariant metric and the Generalized Metric appear in this "double" world-sheet action.
- We will see that this seems to be a peculiarity of the "doubling".

$O(D, D)$ INVARIANCE

- The sigma-model action can be expressed, in the non-chiral basis, as:

$$S = -\frac{T}{2} \int d^2\sigma \left[\eta_{ij} \partial_0 \chi^i \partial_1 \chi^j - \mathcal{H}_{ij} \partial_1 \chi^i \partial_1 \chi^j \right].$$

- It is invariant under the combined $O(D, D)$ transformations of χ^i *and* the matrix of the couplings parameters in $\mathcal{H}$:

$$\chi' = \mathcal{T}\chi \quad ; \quad \mathcal{H}' = \mathcal{T}^{-t}\mathcal{H}\mathcal{T}^{-1} \quad ; \quad \mathcal{T}^t \eta \mathcal{R} = \eta \quad ; \quad \mathcal{T} \in O(D, D).$$

.

- The $O(D, D)$ invariant metric η is itself an element of $O(D, D)$.

Covariant Double Action

- In order to understand the relation to the standard formulation, one can integrate over $\tilde{X}_\mu$ by eliminating it through the use of the equations of motion.
- If, instead, one eliminates X from its equation of motion one obtains the dual model for $\tilde{X}$.
- $S[X, \tilde{X}]$ is therefore a first-order action which interpolates between $S[X]$ and $S[\tilde{X}]$ and is manifestly duality symmetric.

Hull Action

- It can be shown to be equivalent to the following covariant action (Hull, 2005):

$$T \int \left[\mathcal{H}_{ab}(X) d\chi^a \wedge * d\chi^b \right]$$

with the self-duality relation that halves the degrees of freedom from $2D$ to D (also Duff, 1987):

$$\partial_\alpha \chi^i = \epsilon_{\alpha\beta} \eta^{ij} M_{jk} (\partial^\beta \chi^k)$$

equivalent to the condition $\epsilon_{ab} t^{ab} = 0$.

Abelian T-Duality

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- T-duality can be performed along the direction of an isometry and the dual backgrounds (G, B) and $(\tilde{G}, \tilde{B})$ are related by the Buscher rules.
- For instance, isometry generated by $k_1 = \frac{\partial}{\partial X^1}$
- A gauge field is introduced satisfying $A = dX^1$ with a Lagrange multiplier λ defining a new coordinate $\tilde{X}^1$ by $d\lambda = d\tilde{X}^1$ with

$$d\tilde{X}^1 = G_{1a} * dX^a + B_{1a} dX^a$$

This is the current associated to the isometry generated by $k = \frac{\partial}{\partial X^1}$.

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- They gauged the non-Abelian isometries of the sigma-models and constrained the field strength F to vanish. The dual action was then obtained by integrating out the gauge fields and the Lagrange multipliers had become coordinates of the dual manifold.
- The resulting non-Abelian dual was found which, however, turned out to lack the isometry that would make it possible to perform the duality transformation back to the original model. But, even without isometries, this dual is still equivalent to an apparently different sigma-model.

Poisson-Lie T-Duality and Drinfeld Double

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- Such a structure is the Drinfeld double, in which the duality transformation simply exchanges the roles of the two groups forming the double. In this case it is said that the models are indeed dual in the sense of the *Poisson-Lie T-Duality*.

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- Isotropic subspace: the value of the form on two arbitrary vectors belonging to the subspace vanishes.
- Maximally isotropic: the subspace cannot be enlarged while preserving the property of isotropy.

The Manin Triple

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- In general, $M(D)$ has always at least two points, i.e. $\mathcal{G} + \tilde{\mathcal{G}} = \mathcal{D}$ and $\tilde{\mathcal{G}} + \mathcal{G} = \mathcal{D}$.

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- The set $M(D)$ of the Manin triples corresponding to a given Drinfeld double plays the role of the modular space of σ-models mutually connected by Poisson-Lie T-duality transformation.
- In the Abelian case the Drinfeld double is $D = U(1)^{2d}$ and its modular space is $O(d, d)$.
- In general, $M(D)$ has always at least two points, i.e. $\mathcal{G} + \tilde{\mathcal{G}} = \mathcal{D}$ and $\tilde{\mathcal{G}} + \mathcal{G} = \mathcal{D}$.
- The $O(d, d)$-duality of String Theory can be understood from this geometric point of view!

Classification of T-dualities

- A classification of T-dualities is given by the underlying Drinfeld doubles:

  1. Abelian doubles, corresponding to the standard Abelian T-duality;
  2. semi-Abelian doubles, with a decomposition $\mathcal{G} + \tilde{\mathcal{G}} = \mathcal{D}$ such that $\tilde{\mathcal{G}}$ is Abelian, corresponding to the *non-Abelian T-duality* between an isometric σ-model with target $\mathcal{G}$ and a non-isometric σ-model with the target $\tilde{\mathcal{G}}$ viewed as the Abelian group.

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- Geometric structures can be understood in terms of GG/DG.

THE 3-D IRR ON THE CONFIGURATION SPACE $SU(2)$

- Action:

$$S_0 = -\frac{1}{4}\int_{\mathbb{R}} \mathrm{Tr}(g^{-1}dg \wedge^* g^{-1}dg) = -\frac{1}{4}\int_{\mathbb{R}} \mathrm{Tr}\ (g^{-1}\dot{g})^2 dt$$

  $g : t \in \mathbb{R} \to SU(2)$, with $g^{-1}dg = i\alpha^k\sigma_k$ the Maurer-Cartan left-invariant (Lie algebra-valued) one-form, α^k are the basic left-invariant one-forms, $*$ the Hodge star operator on the source space $\mathbb{R}$, $*dt = 1$, Tr the trace over the Lie algebra $\to$ group-valued field theory, reduction of Principal Chiral Model to 0+1 dimensions.

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- Invariance under both left and right actions of $SU(2)$.

The Langrangian Formalism and the Tangent Bundle $TSU(2)$

- Parametrization with $\mathbb{R}^4$ coordinates:
  $g = y^0\sigma_0 + iy^i\sigma_i \equiv 2(y^0 e_0 + iy^i e_i)$ with $(y^0)^2 + \sum_i (y^i)^2 = 1$
  $\sigma_0 \to \mathbb{I}$ matrix ; $\sigma_i \to$ the Pauli matrices.
  $y^i = -\frac{i}{2}\mathrm{Tr}(g\sigma_i), \;\; y^0 = \frac{1}{2}\mathrm{Tr}(g\sigma_0), \quad i = 1,..,3$

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- Coordinates on the tangent bundle $TSU(2)$: $(Q^i, \dot{Q}^i)$ with the Q^i's implicitly defined.
- Equations of motion: $\ddot{Q}^i = 0 \qquad$ or $\qquad \frac{d}{dt}\left(g^{-1}\frac{dg}{dt}\right) = 0$
  implying the conservation of the left generalized velocities.

The Hamiltonian Formalism and the cotangent bundle $T^*SU(2)$

- $T^*SU(2)$ is a trivial bundle $\simeq SU(2) \times \mathbb{R}^3$ with the cotangent space at the identity $T_e^*SU(2)$ dual to the tangent space $T_eSU(2) \simeq su(2)$.
- Coordinates: (Q^i, I_i), being the I_i's the left momenta:

$$I_i = \frac{\partial \mathcal{L}_0}{\partial \dot{Q}^i} = \delta_{ij}\dot{Q}^j \quad ; \quad I = iI_i e^{i*} \text{ dual basis } (e^{i*}) \quad < e^{i*}|e_j >= \delta^i_j.$$

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- The base space coordinates (y^0, y^i) are associated with the orientation of the rotator while the fiber coordinates I_i are associated with the angular momentum components.
- The Legendre transform from $TSU(2)$ to $T^*SU(2)$ yields the Hamiltonian function: $\mathcal{H}_0 = \frac{1}{2}\delta^{ij} I_i I_j$

Canonical Poisson brackets

- Canonical Poisson brackets:

$$\{y^i, y^j\} = 0 \quad \{I_i, I_j\} = \epsilon_{ij}{}^k I_k \quad \{y^i, I_j\} = \delta^i_j y^0 + \epsilon^i{}_{jk} y^k$$

from the first-order formulation of the action

$$S_1 = \int < I | g^{-1}\dot{g} > dt - \int \mathcal{H}_0 dt \equiv \int \theta - \int \mathcal{H}_0 dt$$

θ canonical one-form defining the symplectic form $\omega = d\theta = dI_i \wedge \delta^i_j \alpha^j - \frac{1}{2} I_i \delta^i_j \epsilon^j{}_{kl} \alpha^k \wedge \alpha^l$ with $d\alpha^k = \frac{i}{2}\epsilon^k_{ij}\alpha^i \wedge \alpha^j$.

- The Poisson brackets on the fibers I_i are the Kirillov-Soriau-Konstant brackets defined by a natural symplectic form existing on the orbits of the coadjoint representation of $SU(2)$ on the dual space of its algebra.

Equations of motion

- e.o.m.: $\dot{I}_i = 0, \quad g^{-1}\dot{g} = iI_i\delta^{ij}\sigma_j \to I_i$ are constants of motion while g undergoes a uniform precession.
- Since the Lagrangian and the Hamiltonian are invariant under right and left $SU(2)$ action, right momenta are conserved as well, being the model super-integrable.

$T^*SU(2)$ as a Lie Group

- As a group, $T^*SU(2) \simeq SU(2) \ltimes \mathbb{R}^3$ (semi-direct product) with the Lie algebra given by the semidirect sum of $\mathfrak{su}(2)$ and $\mathbb{R}^3$ with Lie brackets:

$$[L_i, L_j] = \epsilon_{ij}^k L_k \qquad [T_i, T_j] = 0 \qquad [L_i, T_j] = \epsilon_{ij}^k T_k$$

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- NOTICE here that $\mathbb{R}^3$ has vanishing Lie brackets as a Lie algebra while $\mathfrak{su}(2)$ has non trivial Lie brackets, in both cases differently from their corresponding Poisson brackets.
- The linearization of the Poisson structure at the unit e of $SU(2)$ provides a Lie algebra structure over the dual algebra $\mathfrak{su}(2)^* \simeq \mathbb{R}^3$.

Kirillov-Soriau-Konstant and Lie brackets

- The Kirillov-Soriau-Konstant $\{I_i, I_j\} = \epsilon_{ij}{}^k I_k$ on the fibers of the bundle can also be understood in terms of the coadjoint action of the group $SU(2)$ on $\mathfrak{su}(2)^* \simeq \mathbb{R}^3$.

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- Summarizing:
  1. The carrier space of the Hamiltonian dynamics is represented by the semi-direct product of a non-Abelian group, $SU(2)$ and the Abelian group $\mathbb{R}^3$, dual of its Lie algebra.
  2. The Poisson brackets governing the dynamics are the Kirillov-Soriau-Konstant brackets induced by the coadjoint action.
- $T^*SU(2)$ is the first example of *doubled group* since it is obtained by exponentiating the semidirect sum of two Lie algebras.
- More general structures could be obtained by *deforming* $\mathbb{R}^3$ to a non-Abelian Lie algebra.

$SL(2,\mathbb{C})$, $SU(2)$ and $SB(2,\mathbb{C})$

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- Non-degenerate invariant scalar products:

$$< u, v >= 2\mathrm{Im}[\mathrm{Tr}(uv)] \ ; \ (u, v) = 2\mathrm{Re}[\mathrm{Tr}(uv))] \quad \forall u, v \in \mathfrak{sl}(2,\mathbb{C})$$

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- The scalar product $< u, v >$ (Cartan-Killing) defines two maximally isotropic subspaces:

$$\begin{aligned} < e_i, e_j > &= \; < \tilde{e}^i, \tilde{e}^j >= 0, \\ < e_i, \tilde{e}^j > &= \; \delta_i^j \text{ with } \tilde{e}^i = \delta^{ij}(b_j + \epsilon_j^{k3} e_k). \end{aligned}$$

An explicit example of Manin triple

- $\{\tilde{e}^i\}$ dual basis of $\{e_i\}$ with respect to $< u, v >$ in the dual vector space $\mathfrak{su}(2)^*$, the special Borel algebra $\mathfrak{sb}(2, \mathbb{C})$ with Lie brackets:

$$[\tilde{e}^1, \tilde{e}^2] = 0, \ \ [\tilde{e}^1, \tilde{e}^3] = -i\tilde{e}^1, \ \ [\tilde{e}^2, \tilde{e}^3] = -i\tilde{e}^2$$

- $\{e_i\}$ and $\tilde{e}^i$ span the Lie algebra:

$$[e_i, e_j] = i\epsilon_{ij}^k e_k, \qquad [\tilde{e}^i, e_j] = i\epsilon^i_{\ jk}\tilde{e}^k + ie_k f^{ki}{}_j, \qquad [\tilde{e}^i, \tilde{e}^j] = if^{ij}_k \tilde{e}^k$$

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- The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ can be split into two maximally isotropic dual Lie subalgebras with respect to a bilinear, symmetric, non-degenerate form on it.
- The couple $(\mathfrak{su}(2), \mathfrak{sb}(2, \mathbb{C}))$, with the dual structure, is a Lie bialgebra with an interchangeable role of $\mathfrak{su}(2)$ and $\mathfrak{sb}(2, \mathbb{C})$.
- The triple $(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{su}(2), \mathfrak{sb}(2, \mathbb{C}))$ provides an example of *Manin triple*.

- The total algebra $\mathfrak{d} = \mathfrak{su}(2) \bowtie \mathfrak{sb}(2,\mathbb{C})$ is the Lie algebra of the Drinfeld double $D \equiv SL(2, C)$ of $SU(2)$ and $SB(2,\mathbb{C})$.
- D generalizes both the cotangent bundles of $SU(2)$ and of $SB(2,\mathbb{C})$.
- The construction can be generalized to any Lie group G.
- Given $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$, the group D with Lie algebra $\mathfrak{d}$ is the Drinfel'd double and G, G^* are dual groups.
- For $f_k^{ij} = 0 \quad D \to T^*G$, while for $c_{ij}^k = 0 \;\; D \to T^*G^*$, with c_{ij}^k the structure constants of $\mathfrak{g}$ and f_k^{ij} the structure constants of $\mathfrak{g}^*$.
- D generalizes both the cotangent bundle of G and of G^*.
- The bialgebra structure induces Poisson structures on the double group manifold (I will come back to these in a while).

The $O(d,d)$ invariant metric

- Introduce the *doubled* notation

$$e_I = \begin{pmatrix} e_i \\ \tilde{e}^i \end{pmatrix}, \qquad e_i \in \mathfrak{su}(2), \quad \tilde{e}^i \in \mathfrak{sb}(2,\mathbb{C}),$$

The scalar product $< u, v >= 2\mathrm{Im}(\mathrm{Tr}(uv))$ yields

$$< e_I, e_J >= \eta_{IJ} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix}$$

This is the $O(3,3)$ invariant metric.

The Generalized Metric

- Through the scalar product $(u, v) = 2\mathrm{Re}[\mathrm{Tr}(uv)]$

$$(e_i, e_j) = -(b_i, b_j) = \delta_{ij}, \qquad (e_i, b_j) = 0$$

it is possible to define a non-degenerate scalar product $((,))$, giving rise to a Riemannian metric:

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- Computing also $((\tilde{e}^i, \tilde{e}^j))$ and $((e_i, \tilde{e}^j))$ yields:

$$((e_I, e_J)) = \mathcal{H}_{IJ} = \begin{pmatrix} \delta_{ij} & {\epsilon_{3i}}^j \\ -{\epsilon^i}_{j3} & \delta^{ij} + {\epsilon^i}_{l3}\delta^{lk}{\epsilon^j}_{k3}) \end{pmatrix}$$

satisfying the relation: $\mathcal{H}^T \eta \mathcal{H} = \eta$.

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- $\mathcal{H}$ is an $O(3,3)$ matrix having the same structure as the $O(d,d)$ *generalized metric* of DFT with δ_{ij} playing the role of G^{ij} and ϵ_{ij3} playing the role of B_{ij}.

Geometry of the Dual Model

- A suitable action for the dual system is the following:

$$\tilde{S}_0 = -\frac{1}{4}\int_{\mathbb{R}} \mathcal{T}r[\tilde{g}^{-1}d\tilde{g}\wedge *\tilde{g}^{-1}d\tilde{g}] = -\frac{1}{4}\int_{\mathbb{R}} \mathcal{T}r[(\tilde{g}^{-1}\dot{\tilde{g}})(\tilde{g}^{-1}\dot{\tilde{g}})]dt$$

with $\tilde{g} : t \in \mathbb{R} \to SB(2, \mathbb{C})$, the group-valued target space coordinates, so that

$$\tilde{g}^{-1}d\tilde{g} = i\beta_k\tilde{e}^k$$

is the Maurer-Cartan left invariant one-form on the group manifold, with β_k the left-invariant basic one-forms, $*$ the Hodge star operator on the source space $\mathbb{R}$, such that $*dt = 1$.

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- The trace $\mathcal{T}r$ is defined by the scalar product $((,))$ that makes the dual Lagrangian only left-$SB(2,\mathbb{C})$ invariant, differently from the Lagrangian of the rigid rotator which is invariant under both left and right actions of both groups.

The Tangent Bundle $TSB(2,\mathbb{C})$

- Parametrization: the group manifold $SB(2,\mathbb{C})$ can be parametrized with $\mathbb{R}^4$ coordinates: $\tilde{g} = 2(u_0\tilde{e}^0 + iu_i\tilde{e}^i)$, being $u_0^2 - u_3^2 = 1$ and $\tilde{e}^0 = \mathbb{I}/2$ with

$$u_i = \frac{1}{4}((i\tilde{g},\tilde{e}^i)),\ i = 1,2, \qquad u_3 = \frac{1}{2}((i\tilde{g},\tilde{e}^3)), \qquad u_0 = \frac{1}{2}((\tilde{g},\tilde{e}^0))$$

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- In terms of the left generalized velocities $\dot{\tilde{Q}}_i = u_0\dot{u}_i - u_i\dot{u}_0 + f_i^{\ jk}u_j\dot{u}_k$ the Lagrangian $\tilde{L}_0$ becomes:

$$\tilde{L}_0 = \dot{\tilde{Q}}_i\dot{\tilde{Q}}_r h^{ir}$$

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- Equations of motion: $(\delta^{ij} + \epsilon^i{}_{k3}\epsilon^j{}_{l3}\delta^{kl})\ddot{\tilde{Q}}_j = 0$ reflecting the non-invariance of the Lagrangian under the right action.

The cotangent bundle $T^*SB(2,\mathbb{C})$

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- The linear combination over the dual basis is introduced:

$$\tilde{I} = i\tilde{I}^j \tilde{e}^*_j \quad \text{with} \quad \langle \tilde{e}^*_j | \tilde{e}^i \rangle = \delta^i_j$$

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- Poisson brackets:

$$\{u_i, u_j\} = 0 \quad ; \quad \{\tilde{I}^i, \tilde{I}^j\} = f^{ij}{}_k \tilde{I}^k \quad ; \quad \{u_i, \tilde{I}^j\} = \delta^j_i u_0 - f_i{}^{jk} u_k$$

which are derived from the first order formulation of the action.
- Hamilton equations:

$$\dot{\tilde{I}}^j = \{\tilde{I}^j, \tilde{H}\} = f^{jk}_l \tilde{I}^l \tilde{I}^r h^{-1}_{kr}$$

non-invariance of the Hamiltonian under right action.

- As a group, $T^*SB(2,\mathbb{C}) \simeq SB(2,\mathbb{C}) \ltimes \mathbb{R}^3$, with Lie algebra;

$$\begin{aligned} [B_i, B_j] &= i f_{ij}{}^k B_k \\ [S_i, S_j] &= 0 \\ [B_i, S_j] &= i f_{ij}{}^k S_k . \end{aligned}$$

  The non-trivial Poisson bracket on the fibers of the bundle $\{\tilde{I}^i, \tilde{I}^j\} = \delta_{ib} f^j_{bc} \tilde{I}^c$ can be understood in terms of the coadjoint action of the group $SB(2,\mathbb{C})$ on its dual algebra $\mathfrak{sb}(2,\mathbb{C})$.
- The two models can be obtained from the same *parent action* defined on the *whole* $SL(2,\mathbb{C}) \rightarrow$ they are dual.
- Goal: to define a dynamical model symmetric under $SU(2) \leftrightarrow SB(2,\mathbb{C})$ on the Drinfeld double $SL(2,\mathbb{C})$.

So far...

- Two dynamical models have been introduced with configuration spaces being Lie groups which are dually related.
- The Poisson algebras on the respective cotangent bundles, $T^*SU(2)$, $T^*SB(2,\mathbb{C})$ are:

$$\begin{aligned} \{g,g\} &= 0, \quad \{I_i,I_j\} = \epsilon_{ij}{}^k I_k, \quad \{g,I_j\} = 2ige_j \\ \{\tilde g,\tilde g\} &= 0, \quad \{\tilde I^i,\tilde I^j\} = f^{ij}{}_k \tilde I^k, \quad \{\tilde g,\tilde I^j\} = 2i\tilde g\tilde e^j \end{aligned}$$

- Define $g(\lambda) = e^{i\lambda J^i e_i}$ and $\tilde g(\lambda) = e^{i\mu \tilde J_i \tilde e^i}$. The expansion of the group variables $g, \tilde g$

$$g \simeq \mathbb{1} + i\lambda J^i e_i + O(\lambda^2), \quad \tilde g \simeq \mathbb{1} + i\mu \tilde J_i \tilde e^i + O(\mu^2)$$

leads to:

$$\begin{aligned} \{J^i,J^j\} &= 0, \quad \{I_i,I_j\} = \epsilon_{ij}{}^k I_k, \quad \{J^i,I_j\} = -\epsilon^i{}_{jk} J^k \\ \{\tilde J_i,\tilde J_j\} &= 0, \quad \{\tilde I^i,\tilde I^j\} = f^{ij}{}_k \tilde I^k, \quad \{\tilde J_i,\tilde I^j\} = -\tilde J^k f_{ki}{}^j . \end{aligned}$$

- $J^i, \tilde J_i$ will be identified with $\tilde I^i, I_i$ by unifying the cotangent bundles $T^*SU(2)$, $T^*SB(2,\mathbb{C})$ into the Drinfel'd double $SL(2,\mathbb{C})$

The doubled action

- Introduce an action on $TSL(2,\mathbb{C})$ (doubled coordinates) as a $(0+1)$-dimensional field theory which is group-valued with $\gamma : t \in \mathbb{R} \to \gamma(t) \in SL(2,\mathbb{C})$.

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- Left invariant one-form on the group manifold:

$$\gamma^{-1}\mathrm{d}\gamma = \gamma^{-1}\dot{\gamma}\; dt \equiv \dot{\mathbf{Q}}^I e_I \; dt \tag{1}$$

with $e_I = (e_i, \tilde{e}^i)$ the $\mathfrak{sl}(2,\mathbb{C})$ *doubled* basis and $\dot{\mathbf{Q}}^I$ the left generalized velocities.

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- Defining the decomposition $\dot{\mathbf{Q}}^I \equiv (A^i, B_i)$ implies:

$$\gamma^{-1}\dot{\gamma} \; = (A^i e_i + B_i \tilde{e}^i).$$

  where both components are tangent bundle coordinates for $SL(2,\mathbb{C})$.
- The components of the generalized velocity can be explicitly obtained:

$$A^i = 2\mathrm{Im}\mathrm{Tr}(\gamma^{-1}\dot{\gamma}\tilde{e}^i); \quad B_i = 2\mathrm{Im}\mathrm{Tr}(\gamma^{-1}\dot{\gamma}e_i).$$

The doubled action

- Proposed action:

$$S = \frac{1}{2}\int_{\mathbb{R}} (k_1 < \gamma^{-1}\mathrm{d}\gamma \stackrel{\wedge}{,} *\gamma^{-1}\mathrm{d}\gamma > + k_2((\gamma^{-1}\mathrm{d}\gamma \stackrel{\wedge}{,} *\gamma^{-1}\mathrm{d}\gamma))),$$

k_1, k_2 are real parameters and

$$\begin{aligned} < \gamma^{-1}\mathrm{d}\gamma \stackrel{\wedge}{,} *\gamma^{-1}\mathrm{d}\gamma > &= \dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J < e_I, e_J > = \dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J\eta_{IJ} \\ ((\gamma^{-1}\mathrm{d}\gamma \stackrel{\wedge}{,} *\gamma^{-1}\mathrm{d}\gamma)) &= \dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J((e_I, e_J)) = \dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J\mathcal{H}_{IJ}. \end{aligned}$$

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- Explicitly

$$\hat{L} = \frac{1}{2}(k\,\eta_{IJ} + \mathcal{H}_{IJ})\dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J$$

$k_1/k_2 \equiv k$.

- Strictly reminding the double action by Tseytlin!

Recovering both the actions

- Fix a local decomposition for the elements of the double group $SL(2,\mathbb{C})$: $\gamma = \tilde{g}g$, $g \in SU(2)$ and $\tilde{g} \in SB(2,\mathbb{C})$.

Recovering both the actions

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- $\hat{L}$ is invariant under left and right action of the group $SU(2)$ but only after left action of the group $SB(2,\mathbb{C})$, given by:

$$SB(2,\mathbb{C})_L : \gamma \to \tilde{h}\gamma = \tilde{h}\tilde{g}g \quad \forall \tilde{h} \in SB(2,\mathbb{C})$$

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- Promote the $SB(2,\mathbb{C})_L$ invariance to a gauge symmetry for getting the usual description of the rotator.
- Promote the global invariance of $\hat{L}$ under right action of the group $SU(2)$ for getting the dual rotator.

Hamiltonian Formalism for the doubled action

- In the doubled description the left generalized momenta are given by:

$$\mathbf{P}_I = \frac{\partial L}{\partial \dot{\mathbf{Q}}^{\mathbf{I}}} = (\eta_{IJ} + k\mathcal{H}_{IJ})\dot{\mathbf{Q}}^{\mathbf{J}}$$

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- Hamiltonian: $\hat{H} = (\mathbf{P}_I \dot{\mathbf{Q}}^I - \hat{L})_{\mathbf{P}} = \frac{1}{2}[(\eta + k\mathcal{H})^{-1}]^{IJ}\mathbf{P}_I\mathbf{P}_J$

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- The Hamiltonian equations on the Drinfeld double are obtained through the determination of Poisson brackets from the first order action:

$$\widehat{\mathcal{S}} = \int \langle \mathbf{P} | \gamma^{-1} d\gamma \rangle - \int \widehat{H} dt \equiv \int \boldsymbol{\theta} - \int \widehat{H} dt$$

with

$$\begin{aligned} \mathbf{P} &= i\, \mathbf{P}_I e^{I^*} = i\, (I_i e^{i^*} + \tilde{I}^i \tilde{e}_i^*) \\ \gamma^{-1} d\gamma &= i\, \boldsymbol{\alpha}^J e_J = (\alpha^k e_k + \beta_k \tilde{e}^k)\ . \end{aligned}$$

- $\mathbf{P}_\mathbf{I}$, $\boldsymbol{\alpha}^J$ are respectively generalized momenta and basis one-forms on the doubled configuration space $SL(2, \mathbb{C})$.

Poisson Brackets on $T^*SL(2,\mathbb{C})$

- The symplectic form $\omega = d\theta$ on $T^*SL(2,\mathbb{C}) \simeq SL(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})^*$ yields for the generalized momenta the Poisson brackets to:

$$
\begin{aligned}
\{I_i, I_j\} &= \epsilon_{ij}{}^k I_k \\
\{\tilde{I}^i, \tilde{I}^j\} &= f^{ij}{}_k \tilde{I}^k \\
\{I_i, \tilde{I}^j\} &= \epsilon^j{}_{il} \tilde{I}^l - I_l f^{lj}{}_i \quad \{\tilde{I}^i, I_j\} = -\epsilon^i{}_{jl} \tilde{I}^l + I_l f^{li}{}_j
\end{aligned}
$$

while the Poisson brackets between momenta and configuration space variables $g, \tilde{g}$ are unchanged with respect to $T^*SU(2)$, $T^*SB(2,\mathbb{C})$.

POISSON STRUCTURES ON THE DOUBLE GROUP $SL(2,C)$

- The bi-algebra structure induces Poisson structures on the group manifold of the double D (Heisenberg double) which generalize both those of T^*G and of T^*G^* and reproduce the KSK brackets on coadjoint orbits of G, G^* when $f_k^{ij} = 0, c_{ij}^k = 0$ respectively.
- For $\gamma \in D$ and $r = \lambda \tilde{e}^i \otimes e_i, \lambda \in \mathbb{R}$ the classical Yang-Baxter matrix, the brackets [[Semenov-Tyan-Shanskii '91, Alekseev-Malkin '94]]

$$\{\gamma_1, \gamma_2\} = -\gamma_1\gamma_2 r^* - r\gamma_1\gamma_2 \tag{2}$$

with $\gamma_1 = \gamma \otimes 1, \gamma_2 = 1 \otimes \gamma$, $r^* = -\lambda e_i \otimes \tilde{e}^i$, can be shown to define a Poisson structure on the group manifold.
- By writing γ as $\gamma = \tilde{g}g$ it can be shown that these brackets are compatible with

$$\{\tilde{g}_1, \tilde{g}_2\} = -[r, \tilde{g}_1\tilde{g}_2],$$
$$\{\tilde{g}_1, g_2\} = -\tilde{g}_1 r g_2, \qquad \{g_1, \tilde{g}_2\} = -\tilde{g}_2 r^* g_1$$
$$\{g_1, g_2\} = [r^*, g_1 g_2].$$

Recovering the PB's for $T^*SU(2)$ and $T^*SB(2,\mathbb{C})$

- With $G = SU(2)$ and $G^* = SB(2,\mathbb{C})$, upon expanding $\tilde{g} \in SB(2,\mathbb{C})$ as a function of the parameter λ, $\tilde{g}(\lambda) = 1 + i\lambda I_i \tilde{e}^i + \mathcal{O}(\lambda^2)$, while keeping $g = y^0\sigma_0 + iy^i\sigma_i$ we obtain, in the limit $\lambda \to 0$,

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\begin{aligned}
\{I_i, I_j\} &= \epsilon_{ij}^{k} I_k \\
\{I_i, y^0\} &= -y^j \delta_{ij} \qquad \{I_i, y^j\} = y^0 \delta_i^j - \epsilon_{ik}^{j} y^k \\
\{y^0, y^j\} &= \{y^i, y^j\} = 0 + O(\lambda)
\end{aligned}
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  which reproduce correctly the canonical Poisson brackets on the cotangent bundle of $SU(2)$, $T^*SU(2)$.
- Consider now $r^* = -\mu e_k \otimes \tilde{e}^k$ as a solution of the Yang Baxter equation and expand $g \in SU(2)$ as a function of the parameter μ: $g = 1 + i\mu \tilde{I}^i e_i + O(\mu^2)$.

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- By repeating the same analysis as above one gets back the canonical Poisson structure on $T^*SB(2,C)$, with position coordinates and momenta now interchanged. In particular:

$$\{\tilde{I}^i, \tilde{I}^j\} = f^{ij}{}_k \tilde{I}^k$$

Connection with Generalized Geometry

- It is possible to consider a different Poisson structure on the double [Semenov]: $\{\gamma_1, \gamma_2\} = \frac{\lambda}{2}\left[\gamma_1(r^* - r)\gamma_2 - \gamma_2(r^* - r)\gamma_1\right]$
- This is the one that correctly dualizes the bialgebra structure on $\mathfrak{d}$ when evaluated at the identity of the group D.

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- This is the one that correctly dualizes the bialgebra structure on $\mathfrak{d}$ when evaluated at the identity of the group D.
- Expand $\gamma \in D$ as $\gamma = \mathbf{1} + i\lambda I_i \tilde{e}^i + i\lambda \tilde{I}^i e_i$ and rescale r, r^* by the same parameter $\lambda \Longrightarrow$

$$\begin{aligned} \{I_i, I_j\} &= \epsilon_{ij}{}^k I_k; \qquad \{\tilde{I}^i, \tilde{I}^j\} = f^{ij}{}_k \tilde{I}^k \\ \{I_i, \tilde{I}^j\} &= -f_i^{jk} I_k - \tilde{I}^k \epsilon_{ki}{}^j \end{aligned}$$

which are the Poisson brackets induced by the Lie bi-algebra structure of the double.

- The fiber coordinates I_i and $\tilde{I}^j$ play a symmetric role. Moreover, since the fiber coordinate $\tilde{I}^i$ appears in the expansion of g, it can also be thought of as the fiber coordinate of the *tangent* bundle $TSU(2)$, so that the couple $(I_i, \tilde{I}^i)$ identifies the fiber coordinate of the generalized bundle $T \oplus T^*$ over $SU(2)$.

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- Example. Right transformations of $SU(2)$ on $SL(2,\mathbb{C})$:

$$\gamma \to \gamma h, \quad h \in SU(2) \;,, \;\; \gamma \in SL(2,\mathbb{C})$$

and left action of $SB(2,\mathbb{C})$ on $SL(2,\mathbb{C})$:

$$\gamma \to \tilde{h}\gamma \quad \tilde{h} \in SB(2,\mathbb{C}) \;, \;\; \gamma \in SL(2,\mathbb{C})$$

.

- In terms of the coordinates $(\tilde{g}, g)$ such that $\gamma = \tilde{g}g$ this implies:

$$g \to gh \quad , \quad \tilde{g} \to \tilde{g}$$

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- They preserve the Poisson brackets *only if* the paramaters of the transformation $h = h_1 h_2$ satisfy the Poisson brackets:

$$\{h_1, h_2\} = [r^*, h_1 h_2]$$

and zero Poisson brackets with g and $\tilde{g}$. The $SU(2)$ right multiplication becomes a Poisson-Lie group transformation.

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and zero Poisson brackets with g and $\tilde{g}$. The $SU(2)$ right multiplication becomes a Poisson-Lie group transformation.
- The left $SL(2, \mathbb{C})$ multiplication is a Poisson-Lie transformation if

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- In terms of the coordinates $(\tilde{g}, g)$ such that $\gamma = \tilde{g} g$ this implies:

$$g \to gh \quad , \quad \tilde{g} \to \tilde{g}$$

and

$$g \to g \quad , \quad \tilde{g} \to \tilde{h}\tilde{g}$$

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- Both of them become canonical in the limit $\lambda \to 0$.

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$$\{\tilde{g}_1, g_2\} = -\tilde{g}_1 r g_2, \qquad \{g_1, \tilde{g}_2\} = -\tilde{g}_2 r^* g_1$$
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are invariant under the simultaneous action of both $SU(2)$ and $SL(2,\mathbb{C})$ if $\{\tilde{h}_1, h_2\} = 0$.

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- The momentum variables of each model inherit their Poisson brackets from the Poisson-Lie structure of the dual group, which in turn exhibits Poisson-Lie symmetry in the sense elucidated above.

Conclusions

- The double formulation of a mechanical system in terms of dual configuration spaces has been discussed.
- The geometrical structures of DFT have been reproduced ($O(d, d)$-invariant metric and Generalized Metric).
- Poisson brackets for the generalized momenta (C-brackets) have been derived establishing a connection with Generalized Geometry.
- Poisson-Lie symmetries of the dual models have been revealed.
- The model is simple, but it is readily generalizable, for instance, to the Principal Chiral Model (work in progress); in fact, by adding one space dimension to the source space one has a 2-d field theory, modeled on the rigid rotator, which is duality invariant and has all the richness of the Double and Generalized Geometries.

Principal Chiral Model

- A Principal Chiral Model is a field theory with target space given by a Lie group G and base space given by the two-dimensional space $\mathbb{R}^2$ endowed with the metric $h_{\alpha\beta} = \mathrm{diag}(-1, 1)$.
- It describes the dynamics of two dimensional fields $g : \mathbb{R}^{1,1} = (\mathbb{R}^2, h) \to G$. The action may be written in terms of Lie algebra valued left invariant one forms

$$g^{-1}\mathrm{d}g = g^{-1}\partial_t g \mathrm{d}t + g^{-1}\partial_\sigma g \mathrm{d}\sigma$$

so to have

$$S = \frac{1}{2}\int_{\mathbb{R}^2} Tr(g^{-1}\mathrm{d}g \wedge *g^{-1}\mathrm{d}g),$$

where trace is understood as the scalar product on the Lie algebra $\mathfrak{g}$. The Hodge operator exchanges the time and space derivatives

$$*(g^{-1}\mathrm{d}g) = *(\dot{Q}^i dt + Q^{i\prime} d\sigma)e_i = (\dot{Q}^i d\sigma - Q^{i\prime} dt)e_i$$

with $\dot{Q}^i = \mathrm{Tr}\ g^{-1}\partial_t g e_i,\ Q^{i\prime} = \mathrm{Tr} g^{-1}\partial_\sigma g e_i$.

- The Hodge operator maps one-forms into one-forms while exchanging time and space derivatives.
- In Hamiltonian formalism the momenta $I_i = \dot{Q}^j \delta_{ji}$ and the space derivatives $J^i := Q^{i\prime}$ close a Poisson algebra, which results to be isomorphic to the Kac-Moody algebra $\widehat{\mathfrak{sl}(2,\mathbb{C})}$.
- It is therefore natural to conceive a dual model with the same underlying $\widehat{\mathfrak{sl}(2,\mathbb{C})}$ structure but with the role of I_i, J^i exchanged. The action of the dual model is the natural two-dimensional analogue the IRR dual, with $\tilde{g} = \tilde{g}(\sigma, t)$.
- A parent action encoding both models can be introduced. The symmetries of the two models under duality transformations are addressed as well. The structure is richer than the one exhibited by the particle dynamical systems considered here.

The End

Thank you for your attention.