Closed string amplitudes from single-valued correlation functions

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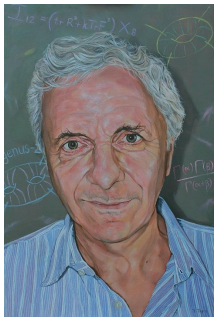
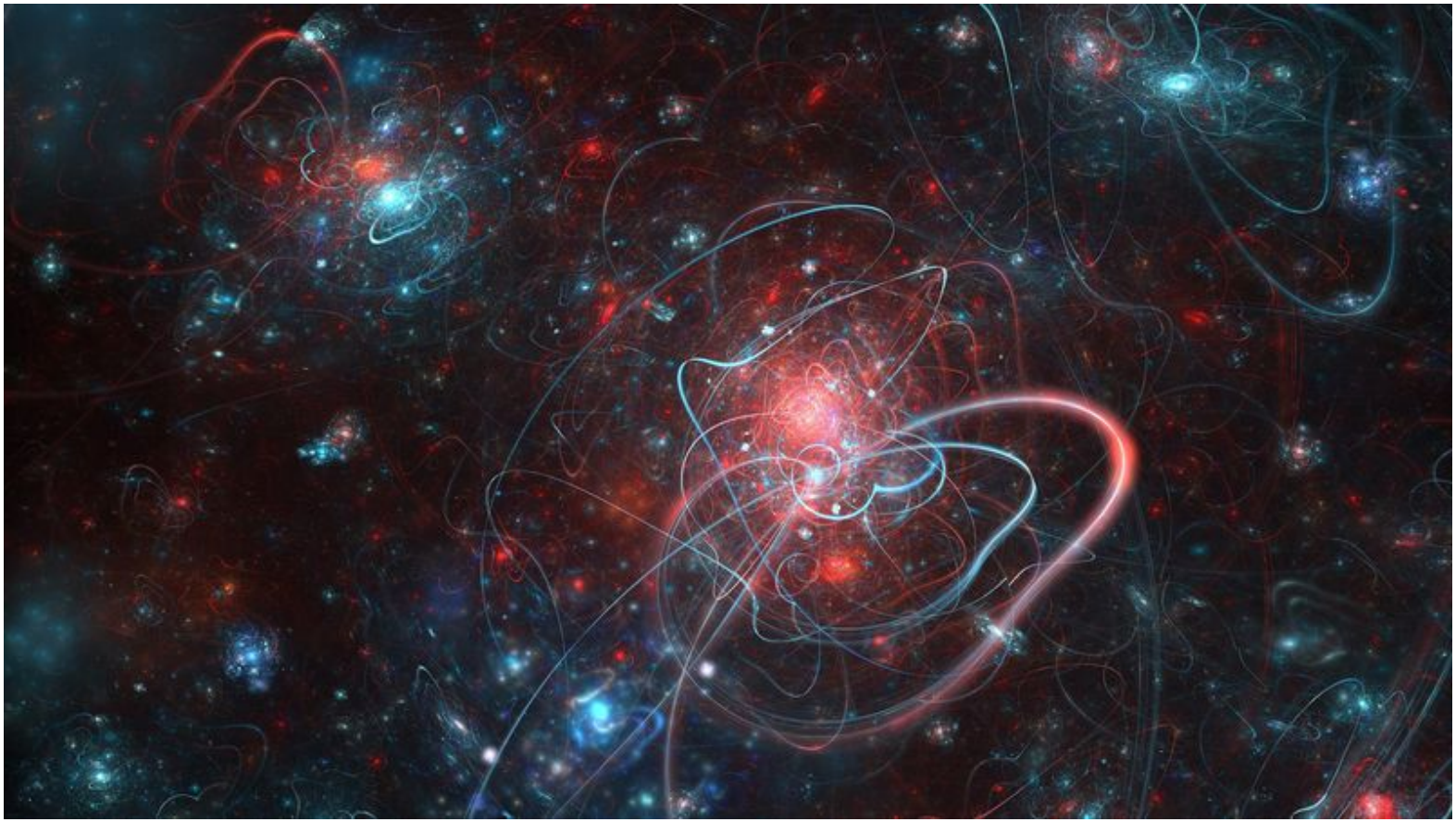
String Field Theory and String Perturbation Theory,
Galileo Galilei Institute for Theoretical Physics, Florence

[1502.06698], [1509.00363], [1512.06779], [1812.03018]



Part I

String theory effective action



Derivative expansion in string theory

String theory low-energy effective action (α' is the inverse tension of the string) in $D \leqslant 10$ dimensions

$$\mathcal{L} = \int d^D x \, |-g|^{\frac{1}{2}} \left(\frac{e^{-2\phi}}{\alpha'^4} \mathcal{R} + \sum_{k \geqslant 0} f_k(\vec{\varphi}) (\alpha' \partial^2)^k \alpha'^3 \mathcal{R}^4 + \cdots \right)$$

The coefficients of the higher derivative terms are automorphic functions of the moduli $\vec{\varphi} \in E_d(\mathbb{R})/K_d$

$$f_k(\gamma \cdot \vec{\varphi}) = f_k(\varphi) \qquad \gamma \in E_d(\mathbb{Z})$$

with $E_d(\mathbb{R})$ the duality group

Derivative expansion in string theory

The structure of the effective action is constrained by

- unitarity : constraints both the perturbative and the nonperturbative spectrum
- supersymmetry leading to differential equations on $f_k(\vec{\varphi})$
- duality symmetry acting on the scalars $\vec{\varphi}$

Derivative expansion in string theory

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Determining these coefficients is extremely important for

- UV divergences of maximal supergravity [Green, Russo, Vanhove; Pioline]
- flux compactification of type II and M-theory and the and de Sitter space vacua
- Flat space limit $AdS_5 \times S^5$ and (discontinuities) of $\mathcal{N} = 4$ correlators [Alday, Bissi, Perlmutter]
- Flat space limit $AdS_4 \times S^7/\mathbb{Z}_k$ and ABJM amplitude [Binder, Chester, Pufu]
- ...

UV structure of maximal supergravity

In $D = 4 + 6/L$ with $L = 1, 2, 3$ the counter-term of the UV divergences of maximal supergravity have been extracted from the constant Fourier mode of the automorphic couplings. As well as the $L = 2$ $D = 8$ divergence with counter-term $D^6 R^4$

[Green,Russo,Vanhove; Pioline]

In integer dimensions the counter-terms to UV divergences found by [Bern et al.] are zeta values.

Determining the coupling from $D^8 \mathcal{R}^4$ onward will give a direct proof of the UV behaviour $\mathcal{N} = 8$ supergravity in four dimensions

Therefore it is important to understand the systematics of these couplings. The most up-to-date description of this approach has been given in the book "Eisenstein series and automorphic representations" by [P. Fleig, H. P. A. Gustafsson, A. Kleinschmidt, D. Persson]

Type IIB superstring in $D = 10$

The vacuum parametrized by $\Omega = C^{(0)} + ie^{-\phi}$ in the duality group coset $SL(2,\mathbb{R})/SO(2)$

The Einstein frame effective action $\ell_P^8 = \alpha'^4 \exp(2\phi)$

$$\mathcal{L}_{IIB} = \int d^{10}x\,|-g|^{\frac{1}{2}} \left(\frac{1}{\ell_P^8}\mathcal{R} + \sum_{k\geqslant 0} f_k^0(\Omega)\,(\ell_P^2\partial^2)^k \ell_P^6\,\mathcal{R}^4 + \cdots \right)$$

The coefficients are modular forms for $SL(2,\mathbb{Z})$ with a $U(1)$ weight w set by the R-symmetry charge of the coupling

$$f_k^w\left(\frac{a\Omega+b}{c\Omega+d}\right) = \left(\frac{c\Omega+d}{c\bar{\Omega}+d}\right)^w f_k(\Omega) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Supersymmetry protected couplings

The $\frac{1}{2}$- and $\frac{1}{4}$-BPS protected terms satisfy [Green, Vanhove; Green, Sethi; Sinha]

$$4(\Im m\Omega)^2 \partial_\Omega \bar{\partial}_{\bar{\Omega}} f_k^w(\Omega) = \lambda_k^w f_k^w(\Omega); \quad k = 0, 2$$

Given by Eisenstein series $f_k^w(\Omega) = E^w_{\frac{3}{2}+k}(\Omega)$

$$E^w_{\frac{3}{2}+k}(\Omega) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{(\Im m\Omega)^{\frac{3+k}{2}}}{(m\Omega+n)^{\frac{3+k}{2}+w}(m\bar{\Omega}+n)^{\frac{3+k}{2}-w}}$$

Supersymmetry protected couplings

- $\mathcal{R}^4$: $\frac{1}{2}$-BPS term one-loop exact

$$\Omega_2^{\frac{1}{2}} E^0_{\frac{3}{2}}(\Omega) = 2\zeta(3)\Omega_2^2 + 4\zeta(2) + \text{non-pert.}$$

- $D^4\mathcal{R}^4$: $\frac{1}{4}$-BPS term two-loop exact

$$\Omega_2^{-\frac{1}{2}} E^0_{\frac{5}{2}}(\Omega) = 2\zeta(5)\Omega_2^2 + 0 + \frac{8}{3}\zeta(4)\Omega_2^{-2} + \text{non-pert.}$$

Only a finite number of perturbative term:

tree-level, one-loop, two-loop

Supersymmetry protected couplings

The $\frac{1}{8}$-BPS is not an Eisenstein series [Green, Vanhove; Green, Miller, Vanhove]

$$4\Im\mathrm{m}(\Omega)^2\partial_\Omega\bar{\partial}_{\bar{\Omega}}f_3^0(\Omega) = 12f_3^0(\Omega) - 6(f_0^0(\Omega))^2$$

with $f_0^0(\Omega) = E^0_{\frac{3}{2}}(\Omega)$ the $\mathcal{R}^4$ coupling.

- $D^6\mathcal{R}^4$: $\frac{1}{8}$-BPS three-loop exact

$$\Omega_2^{-1}f_3^0(\Omega) = \frac{2}{3}\zeta(3)^2\Omega_2^2 + \frac{4\zeta(2)\zeta(3)}{3} + 4\zeta(4)\Omega_2^{-2} + \frac{4\zeta(6)}{27}\Omega_2^{-4} + \text{non-pert.}$$

Only a finite number of perturbative term:

tree-level, one-loop, two-loop, three-loop

Weak coupling expansion

The weak coupling expansion reads

$$\Omega_2^{-\alpha_k} f_k^w(\Omega) = \sum_{g \geqslant 0} a_g^k \Omega_2^{2-2g} + \text{non-pert.}$$

The power behaved terms are the perturbative contributions given by the analytic contribution from genus-g four gravitons amplitudes in string theory

- The tree level coefficients a_0^k are polynomial in odd zeta values
- At genus one $a_1^k/\zeta(2)$ are polynomial in odd zeta values. $\zeta(2)$ is an overall normalisation of the genus one amplitude
- At genus two $a_2^k/\zeta(4)$ are polynomial in odd zeta values.

Part II

Tree amplitudes and MZV

$$\zeta(s_1, \cdots, s_r) = \sum_{n_r > \cdots > n_1 > 0} \prod_{i=1}^{r} \frac{1}{n_i^{s_i}}$$

Tree-level closed string amplitudes

Any closed string tree-level amplitudes can be decomposed

$$M_{N+3}(\mathbf{s},\boldsymbol{\epsilon}) = \sum_r c_r(\mathbf{s},\boldsymbol{\epsilon}) M_{N+3}(\mathbf{s},\mathbf{n}^r,\bar{\mathbf{n}}^r)$$

$c_r(\mathbf{s},\boldsymbol{\epsilon})$ rational functions of kinematic invariant, polarisation tensors, and colour factors

$$M_{N+3}(\mathbf{s},\mathbf{n}^r,\bar{\mathbf{n}}^r) := \int_{\mathbb{C}^N} \prod_{i=1}^{N} d^2 w_i \times$$

$$\times \prod_{1\leqslant i<j\leqslant N} |w_i - w_j|^{2\alpha' s_{ij}} \prod_{i=1}^{N} |w_i|^{2\alpha' s_{N+1\,i}} |w_i - 1|^{2\alpha' s_{N+2\,i}}$$

$$\prod_{i=1}^{N} w_i^{n_i} \bar{w}_i^{\bar{n}_i} (1-w_i)^{m_i} (1-\bar{w}_i)^{\bar{m}_i} \prod_{1\leqslant i<j\leqslant N} (w_i - w_j)^{u_{ij}} (\bar{w}_i - \bar{w}_j)^{\bar{u}_{ij}}$$

with n_i, $\bar{n}_i$, m_i, $\bar{m}_i$, u_{ij}, $\bar{u}_{ij}$ integers

Tree-level expansion : closed vs open

The four-graviton amplitude has the expansion

$\sigma_n = \alpha'^n(s^n + t^n + u^n)$

$$M_{4g} = \frac{\mathcal{R}^4}{\sigma_3} \exp\left(-\sum_{n\geqslant 0} \frac{2\zeta(2n+1)}{2n+1}\sigma_{2n+1}\right)$$

$$M_{4g} = \mathcal{R}^4\Big(\frac{3}{\sigma_3} + 2\zeta(3) + \zeta(5)\,\sigma_2 + \frac{2\zeta(3)^2}{3}\sigma_3 + \frac{\zeta(7)}{2}\sigma_2^2 + \frac{2}{3}\zeta(5)\zeta(3)\sigma_2\sigma_3 + \dots\Big)$$

Whereas open string amplitudes have all zeta values in its low-energy expansion

$$A_4 = \left(1 - \alpha'^2\zeta(2)\partial^4 - \alpha'^3\zeta(3)\partial^6 - \frac{\zeta(4)}{4}\alpha'^4\partial^8 + O(\alpha')^5\right)\frac{F^4}{st}$$

Brown's single valued multiple zetas

[Stieberger] conjectured that the low-energy expansion of closed string tree-level amplitudes involve only single-valued multiple zeta values,

$$\zeta(s_1, \ldots, s_r) = \sum_{n_r > \cdots > n_1 \geqslant 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}$$

They are the value at 1 of the multiple polylogarithms (MPL) of

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{n_r > \cdots > n_1 > 0} \frac{z^{n_r}}{\prod_{i=1}^{r} n_i^{k_i}}$$

This function has monodromies around $z = 0$ and $z = 1$, e.g.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\mathrm{Li}_1(z) & 2i\pi & & \\ -\mathrm{Li}_2(z) & 2i\pi \log z & (2i\pi)^2 & \\ \vdots & \cdots & & \end{pmatrix}$$

Zagier single valued polylogarithms

[Zagier] introduced $D_{a,a}(x)$ single valued on $\mathbb{C}\backslash\{0,1\}$

$$D_{a,a}(x) = 2\mathfrak{Re}\Big((-1)^{a-1}\sum_{k=0}^{a-1}\binom{k+a-1}{a-1} \times \frac{(-2\log|x|)^{a-1-k}}{(a-1-k)!}\mathrm{Li}_{a+k}(x)\Big)$$

e.g.

$$D_{1,1}(z) = \mathrm{Li}_1(z) + (\mathrm{Li}_1(z))^* = G_{\text{tree}}(1-z)$$

$$D_{2,2}(z) = 2\mathfrak{Re}\Big(\log|z|\,\mathrm{Li}_2(z) - \mathrm{Li}_3(z)\Big)$$

Brown's single valued multiple zetas

By cancelling the monodromies at $z = 0$ and $z = 1$ one can defined a single valued function on $\mathbb{C}\backslash\{0, 1\}$

$$sv(\mathrm{Li}_{k_1,\ldots,k_r}(z))$$

[Oliver Schnetz] and [Francis Brown] defined the single valued MZV as their value at $z = 1$ of the single-valued version of the MPL e.g. for $n \in \mathbb{N}$

$$\zeta_{sv}(2n) = 0$$

$$\zeta_{sv}(2n+1) = 2\zeta(2n+1)$$

There is no unique way of removing the monodromies (after weight 2) but [Francis Brown] introduced a canonical definition for the sv-MPL.

Brown's single valued multiple zetas

The dimension d_w^{sv} of the subspace of weight w in the ring over $\mathbb{Q}$ of single-valued multiple-zeta values is smaller [Brown]

At weight 11 a basis of MZVs has dimension 9 is

$$\zeta(3,5,3),\ \zeta(3,5)\zeta(3),\ \zeta(3)^2\zeta(5),\ \zeta(11),$$

$$\zeta(2)\zeta(3)^3,\ \zeta(2)^4\zeta(3),\ \zeta(2)^3\zeta(5),\ \zeta(2)^2\zeta(7),\ \zeta(2)\zeta(9)\,.$$

Since $\zeta_{sv}(2) = 0$ and $\zeta_{sv}(3,5) = -10\zeta_{sv}(3)\zeta_{sv}(5)$ the basis at weight 11 has dimension 3 [Brown; Schnetz]

$$\zeta_{sv}(3,5,3),\ \zeta_{sv}(3)^2\zeta_{sv}(5),\ \zeta_{sv}(11)$$

with $\zeta_{sv}(3,5,3) = 2\zeta(3,5,3) - 2\zeta(3)\zeta(3,5) - 10\zeta(3)^2\zeta(5)$

Using the canonical definition by [Francis Brown]

sv-map on amplitudes

Schlotterer, Stieberger; Stieberger, Taylor conjectured that

$$M^{\text{closed}} = \mathrm{sv}A^{\text{open}}$$

$$A_4 = \left(1 - \alpha'^2\zeta(2)\partial^4 - \alpha'^3\zeta(3)\partial^6 - \frac{\zeta(4)}{4}\alpha'^4\partial^8 + O(\alpha'\partial^2)^5\right)\frac{F^4}{st}$$

$$\Big\downarrow \text{sv-map}$$

$$M_{4g} = \frac{\mathcal{R}^4}{\sigma_3}\left(3 + 2\zeta(3)(\alpha'\partial^2)^3 + \zeta(5)\,(\alpha'\partial^2)^4 + O(\alpha'\partial^2)^5\right)$$

Proven recently [Schlotterer, Schnetz;Brown, Dupont; Vanhove, Zerbini].

Closed string amplitudes from correlators

$$M_{N+3}(\mathbf{s},\mathbf{n},\bar{\mathbf{n}}) = \mathcal{G}_N(1,1)$$

is $z = 1$ value of CFT correlators ($\tilde{V}$ special polarisation strings vop)

$$\mathcal{G}_N(z,\bar{z}) :=$$

$$\left\langle \widetilde{V}_1(0) \prod_{r=2}^{N+1} \int d^2 z_i \widetilde{\mathcal{V}}(w_r,\bar{w}_r) \widetilde{V}_{N+2}(1) \widetilde{V}_{N+3}(\infty) : e^{ik_*\cdot X(z,\bar{z})} : \right\rangle$$

$$= \int_{\mathbb{C}^N} \prod_{1\leqslant i<j\leqslant N} (w_i - w_j)^{g_{ij}} (\bar{w}_i - \bar{w}_j)^{\bar{g}_{ij}} \prod_{i=1}^{N} d^2 w_i$$

$$\prod_{i=1}^{N} w_i^{a_i} (w_i - 1)^{b_i} (w_i - z)^{c_i} \bar{w}_i^{\bar{a}_i} (\bar{w}_i - 1)^{\bar{b}_i} (\bar{w}_i - \bar{z})^{\bar{c}_i}$$

It is important that difference between the holomorphic and anti-holomorphic exponent are integer (spins)

$$\mathbf{a} - \bar{\mathbf{a}}, \mathbf{b} - \bar{\mathbf{b}}, \mathbf{c} - \bar{\mathbf{c}}, \mathbf{g} - \bar{\mathbf{g}} \in \mathbb{Z}$$

Holomorphic factorisation

CFT correlator decompose on conformal blocks

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) I_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$$

The conformal block are the ordered integrals

$$I_{(\sigma,\rho)}(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) = \int_{\Delta(\sigma,\rho)} \prod_{j=1}^{N} dw_j$$

$$\prod_{m,n} |w_m - w_n|^{g_{mn}} \prod_m w_m^{a_m} (w_m - 1)^{b_m} (w_m - z)^{c_m} ,$$

integrated along the real line

$$\Delta(\sigma,\rho) := \{0 \leqslant w_{\rho(1)} \leqslant \cdots \leqslant w_{\rho(s)} \leqslant z \leqslant 1 \leqslant w_{\sigma(1)} \leqslant \cdots \leqslant w_{\sigma(r)}\}$$

Their value at $z = 1$ are open string amplitudes

Aomoto-Gel'fand integrals

$$I_{(\sigma,\rho)}(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) = \int_{\Delta(\sigma,\rho)} \prod_{j=1}^{N} dw_j$$

$$\prod_{m,n} |w_m - w_n|^{g_{mn}} \prod_{m} w_m^{a_m} (w_m - 1)^{b_m} (w_m - z)^{c_m} \,,$$

integrated along the real line

$$\Delta(\sigma,\rho) := \{0 \leqslant w_{\rho(1)} \leqslant \cdots \leqslant w_{\rho(s)} \leqslant z \leqslant 1 \leqslant w_{\sigma(1)} \leqslant \cdots \leqslant w_{\sigma(r)}\}$$

The integrals $I_{(\sigma,\rho)}(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z)$ are

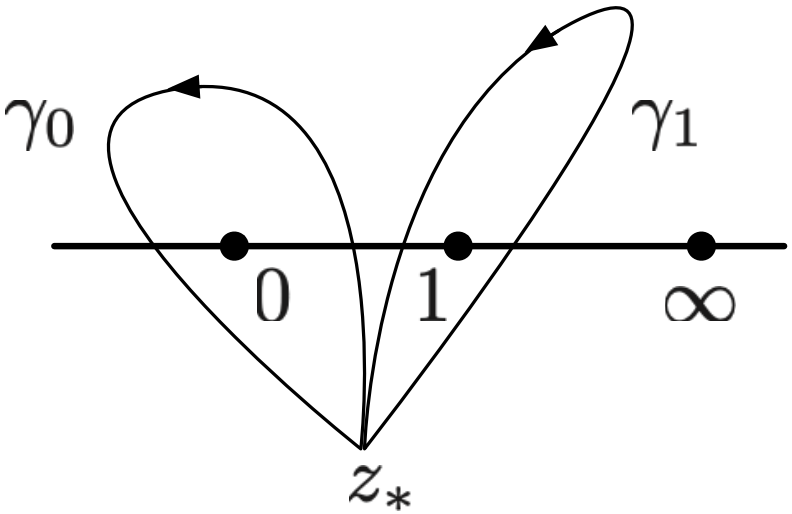
- Aomoto-Gel'fand (hypergeometric) integrals are the multivalued solution of the Knizhnik-Zamolodchikov equations
- conformal blocks for the CFT

Monodromies

As CFT correlator $\mathcal{G}_N(z,\bar{z})$ is single-valued in $\mathbb{C}$

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) I_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$$

The integrals $I_r(\cdots;z)$ have monodromies



$$I_r(\cdots;z) \xrightarrow{\gamma_0} \sum_s (g_0)_r{}^s I_s(\cdots;z)$$

$$I_r(\cdots;z) \xrightarrow{\gamma_1} \sum_s (g_1)_r{}^s I_s(\cdots;z)$$

The monodromy matrices g_0 and g_1 are the *same* for $I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z)$ and $I_r(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$ because $\mathbf{a}-\bar{\mathbf{a}} \in \mathbb{Z}^N$, $\mathbf{b}-\bar{\mathbf{b}} \in \mathbb{Z}^N$, $\mathbf{c}-\bar{\mathbf{c}} \in \mathbb{Z}^N$, $\mathbf{g}-\bar{\mathbf{g}} \in \mathbb{Z}^{\frac{N(N+1)}{2}}$

Monodromies around $z = 0$

$I_r(\cdots\,;z)$ have diagonal monodromies around $z = 0$

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) I_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$$

This imposes that the matrix G_{rs} has the bloc diagonal form

$$G_N = \begin{pmatrix} G_N^{(1)} & 0 & 0 \\ 0 & G_N^{(2)} & 0 \\ 0 & 0 & G_N^{(3)} \end{pmatrix}$$

- $G_N^{(i)}$ with $i = 1, 3$ are real square matrices of size $N!$
- $G_N^{(2)}$ are diagonal matrix of size $(N-1)\,N!$

Monodromies around $z = 1$

The monodromies of $I_r(\cdots\,;z)$ around $z = 1$ are not diagonal but $J_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) := I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};1-z)$ have diagonal monodromies around $z = 1$

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r,s=1}^{(N+1)!} \hat{G}_{r,s} J_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) J_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$$

therefore

$$\hat{G}_N = \begin{pmatrix} \hat{G}_N^{(1)} & 0 & 0 \\ 0 & \hat{G}_N^{(2)} & 0 \\ 0 & 0 & \hat{G}_N^{(3)} \end{pmatrix}$$

- $\hat{G}_N^{(i)}$ with $i = 1, 3$ are real square matrices of size $N!$
- $\hat{G}_N^{(2)}$ are diagonal matrix of size $(N-1)\,N!$

Monodromies constraints

The two sets of integral are related by linear relations derived using the contour deformation method of [Bjerrum-Bohr, Damgaard, Vanhove]

$$I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) = \sum_{r=1}^{(N+1)!} S(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{G})_r{}^s J_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z)$$

We need to solve the linear system

$$\hat{G}_N = S(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{G}) \begin{pmatrix} G_N^{(1)} & 0 & 0 \\ 0 & G_N^{(2)} & 0 \\ 0 & 0 & G_N^{(3)} \end{pmatrix} S(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{G})$$

must have the above block diagonal form of $\hat{G}_N$

Monodromies constraints

- The linear system has *unique solution* up to an scale
- Matching the closed string partial amplitude *determines* the scale factor, *therefore there is no ambiguities*
- The coefficients of the matrices G_N and $\hat{G}_N$ are rational functions $\sin(\pi\alpha' x)$ where x are linear combination of kinematic invariants. This is a non-local version of the momentum kernel
- The small α' expansion of the $I_r(\cdots;z)$ and $J_r(\cdots;z)$ are on multiple polylogarithm with coefficients polynomials of MZV's and $2\pi i$.
- The proof is constructive as it is for any CFT minimal models [Dotsenko, Fateev]

Matching closed string amplitudes

At $z = 1$ we get the colour-ordered open string amplitudes

$$J_{(\sigma,\emptyset)}(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};1) = A_{N+3}(\sigma(1,\ldots,N+1),1,N+2,N+3;\mathbf{n})$$
$$J_{(\sigma,\rho)}(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};1) = 0$$

$$\begin{aligned} M_{N+3}(\mathbf{s},\mathbf{n},\bar{\mathbf{n}}) = \sum_{\sigma,\rho\in\mathfrak{S}_N} & \hat{G}_{\sigma,\rho} \\ & \times A_{N+3}(\sigma(2,\ldots,N+1),1,N+2,N+3;\mathbf{n}) \\ & \times \bar{A}_{N+3}(\rho(2,\ldots,N+1),1,N+2,N+3;\bar{\mathbf{n}}) \,, \end{aligned}$$

The α' has only single-valued multiple zeta values as the valuation at $z = 1$ of combination of single-valued multiple polylogarithms

Remarks

- It is not necessary that the total amplitude is given by the special value at $z = 1$ of a single-valued correlation function. It is enough that each partial amplitude arises this way
- a given order in the α'-expansion can mix single-valued multiple zeta values of different way (due to tachyonic pole in the kinematic coefficients $c_r(\mathbf{s}, \boldsymbol{\epsilon})$ for heterotic-string amplitudes)
- We mathematically prove that each of these convergent integrals, i.e. the coefficients of the small α'-expansion, is a single-valued multiple-zeta value. This is done by repeatedly applying Lemma D.1 of part II of the paper [arXiv:1812.03018] with Zerbini

Four point case

The partial amplitudes of four points tree-level closed string amplitudes are (n_{ij}, $\bar{n}_{ij}$ in $\mathbb{Z}$)

$$M_4(\mathbf{s},\mathbf{n},\bar{\mathbf{n}}) = \int_{\mathbb{C}} d^2w |w|^{2\alpha' k_1\cdot k_2} |1-w|^{2\alpha' k_2\cdot k_3} \times w^{n_{12}} \bar{w}^{\bar{n}_{12}} (1-w)^{n_{23}} (1-\bar{w})^{\bar{n}_{23}} \,.$$

and the single-valued correlator

$$\mathcal{G}_1(z,\bar{z}) := \int_{\mathbb{C}} w^{a_1}(w-1)^{b_1}(w-z)^{c_1}\bar{w}^{\bar{a}_1}(\bar{w}-1)^{\bar{b}_1}(\bar{w}-\bar{z})^{\bar{c}_1} d^2w$$

such that $\mathcal{G}_1(1,1) = M_4(\mathbf{s},\mathbf{n},\bar{\mathbf{n}})$

Four point case: holomorphic factorisation

$$\mathcal{G}_1(z,\bar{z}) = \left(J_{((1),\emptyset)}(\bar{a},\bar{b},\bar{c};\bar{z}) \quad J_{(\emptyset,(1))}(\bar{a},\bar{b},\bar{c};\bar{z})\right) \hat{G}_1 \begin{pmatrix} J_{((1),\emptyset)}(a,b,c;z) \\ J_{(\emptyset,(1))}(a,b,c;z) \end{pmatrix}$$

with

$$\hat{G}_1 = \begin{pmatrix} -\frac{\sin(\pi(A_1+B_1+C_1))\sin(\pi A_1)}{\sin(\pi(B_1+C_1))} & 0 \\ 0 & -\frac{\sin(\pi C_1)\sin(\pi B_1)}{\sin(\pi(B_1+C_1))} \end{pmatrix}$$

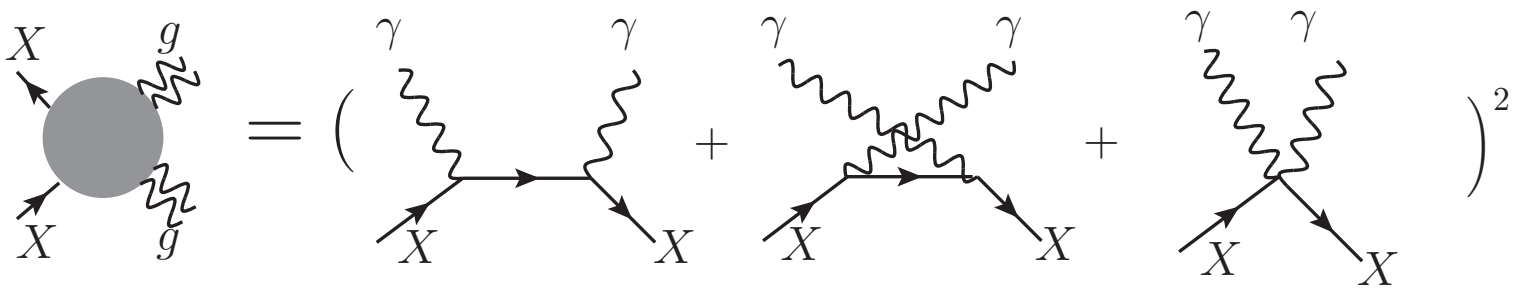
The J_r integrals map at $z = 1$ to the open string amplitudes

$$J_{((1),\emptyset)}(a,b,c;1) = A_4(2,1,3,4;\mathbf{n});$$
$$J_{(\emptyset,(1))}(a,b,c;1) = 0\,.$$

The value at $z = 1$ gives $M_4(\mathbf{s},\mathbf{n},\bar{\mathbf{n}}) = \mathcal{G}_1(1,1)$ gives the non-local version of the KLT relations given in [Bjerrum-Bohr, Damgaard, Vanhove]

$$M_4(\mathbf{s},\mathbf{n},\bar{\mathbf{n}}) = \frac{\sin(2\pi\alpha' k_1\cdot k_2)\sin(2\pi\alpha' k_2\cdot k_4)}{\sin(2\pi\alpha' k_2\cdot k_3)}|A_4(2,1,3,4;\mathbf{n})|^2$$

Gravitational Compton scattering



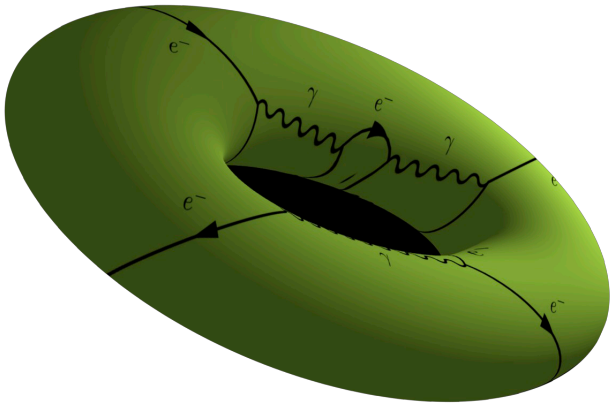
The gravity Compton scattering is a product of two abelian QED Compton amplitudes [Bjerrum-Bohr, Donoghue, Vanhove]

$$\mathfrak{M}(X^s g \to X^s g) = G_N \frac{(p_1 \cdot k_1)(p_1 \cdot k_2)}{k_1 \cdot k_2} \mathcal{A}_s(1324)\tilde{\mathcal{A}}_0(1324)$$

This expression was particular useful when evaluation the post-Minkowskian correction from one-loop amplitude

Part III

Loop amplitudes and elliptic polylogarithms



$$\mathrm{Li}_{s_1,\ldots,s_r}(q,\bar{q},\zeta) = \sum_{n_i,m_i} \mathrm{Li}_{s_1,\ldots,s_r}(q^{n_1}\bar{q}^{m_i}\zeta,\cdots)$$

Genus-one amplitude

By unitarity the special properties of the α' expansion at tree-level amplitude will reappear in some way at one-loop

$$A_{1-\text{loop}}(\alpha' s_{ij}) = \int_{\mathcal{F}} \mathcal{B}_N(\alpha' s_{ij}|\tau) \frac{d^2\tau}{\tau_2^2}$$

$\mathcal{F}$ is a fundamental domain for $SL(2,\mathbb{Z})$

$$\mathcal{B}_N(s_{ij}|\tau) = \prod_{n=1}^{N} \int_{\Sigma} \frac{d^2 z_n}{\tau_2} \exp\left(\sum_{1\leqslant i<j\leqslant N} \alpha' s_{ij}\, G_{1-\text{loop}}(z_i - z_j|\tau) \right)$$

Genus-one amplitude

The analytic part (i.e. not on the logarithmic thresholds which start from $\alpha'^7 D^8 \mathcal{R}^4$) [Green, Vanhove; Green, Russo, Vanhove; Green, d'Hoker, Vanhove]

$$\mathcal{B}_1(s,t,u|\tau) = \sum_{p,q=0}^{\infty} j^{(p,q)}(\tau)\, \sigma_2^p\, \sigma_3^q$$

$j^{(p,q)}(\tau)$ are $SL(2,\mathbb{Z})$ modular functions of weight zero

$$j^{(p,q)}(\gamma\cdot\tau) = j^{(p,q)}(\tau) \qquad \gamma \in SL(2,\mathbb{Z})$$

Integrating these modular functions lead to a_g in ten or lower-dimensions when multiplied by the appropriate lattice factor [Green, Vanhove, Russo; Angelantonij, Florakis, Pioline, ...].

The one-loop Green function

The one-loop green function satisfies

$$4\partial_z\bar{\partial}_{\bar{z}}G_{1-\text{loop}}(z|\tau) = -4\pi\delta^{(2)}(z)+\frac{4\pi}{\tau_2}; \qquad \int_\Sigma d^2z\, G_{1-\text{loop}}(z|\tau) = 0$$

solved by the modular invariant expression $z = v + \tau u$

$$G_{1-\text{loop}}(z|\tau) = -\ln\left|\frac{\vartheta_1(z|\tau)}{\eta(\tau)}\right|^2 - \frac{\pi}{2\tau_2}(z-\bar{z})^2$$

it has the lattice sum expansion

$$G_{1-\text{loop}}(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi}\frac{e^{2i\pi(mv-nu)}}{|m\tau+n|^2}$$

Elliptic polylogarithm

This Green function is Zagier's a single valued elliptic 1-log $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z} = q^u e^{2i\pi v}$

$$G_{1-\text{loop}}(z|\tau) = \sum_{n\geqslant 0} D_{1,1}(q^n\zeta) + \sum_{n\geqslant 1} D_{1,1}(q^n/\zeta) + 2\pi\tau_2\left(u^2 - u + \frac{1}{6}\right)$$

This expression is single valued in $\zeta \in \mathbb{C}^\times/q^{\mathbb{Z}}$ on the torus

Elliptic polylogarithm

This Green function is Zagier's a single valued elliptic 1-log $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z} = q^u e^{2i\pi v}$

$$G_{1-\text{loop}}(z|\tau) = 2\Re\mathrm{e}\Big(\sum_{n\geqslant 0} \mathrm{Li}_1\left(q^n\zeta\right) + \sum_{n\geqslant 1} \mathrm{Li}_1\left(q^n/\zeta\right)\Big) + 2\pi\tau_2\left(u^2 - u + \frac{1}{6}\right)$$

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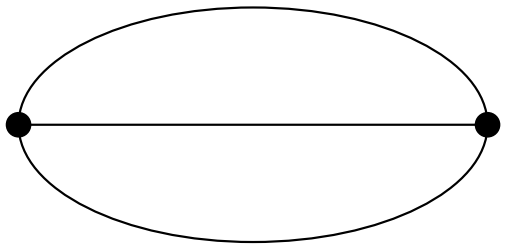
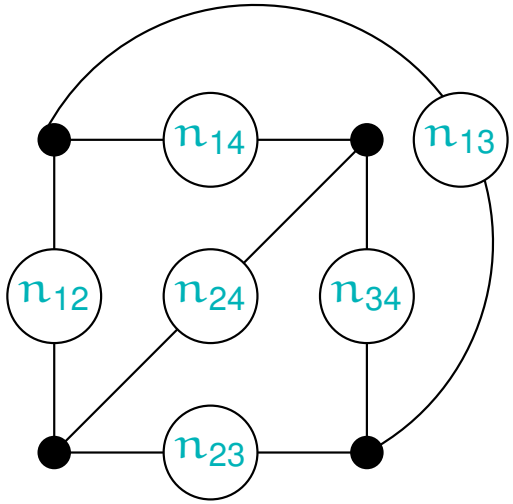
Part IV

Modular graph functions

 $= \sum_{n,m\geqslant 0} a_{n,m} q^n \bar{q}^m.$

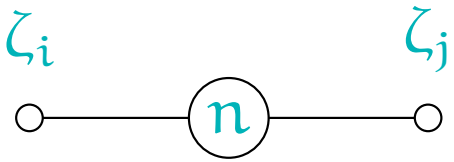
Genus-one world-sheet graphs

$I_\Gamma(q) =$



$$I_\Gamma(q) = \prod_{k=2}^{4} \int_\Sigma \frac{d^2 z_k}{2\pi\tau_2} \prod_{1\leqslant i<j\leqslant 4} G_{1-\text{loop}}(z_j - z_i|\tau)^{n_{ij}}$$

The propagator is one-loop propagator $G_{1-\text{loop}}$

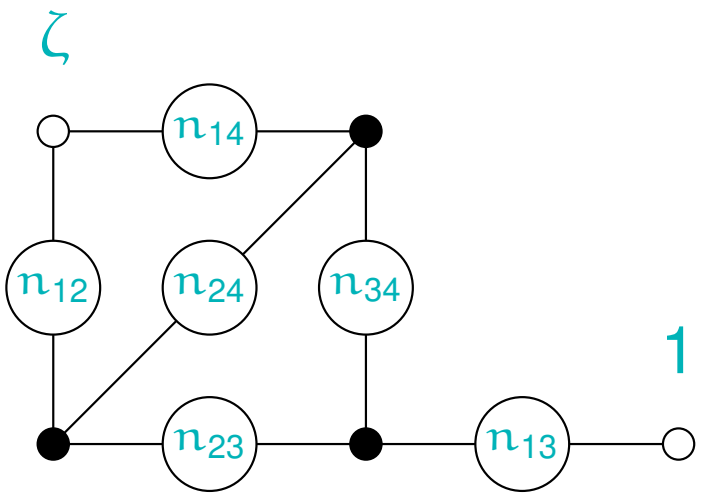


$= G(z_i - z_j|\tau)^n.$

Recall $\zeta = e^{2i\pi z} \in \mathbb{C}^\times/q^{\mathbb{Z}}$

Modular graph functions as svEMZ

$I_\Gamma(q;\zeta) \;=$

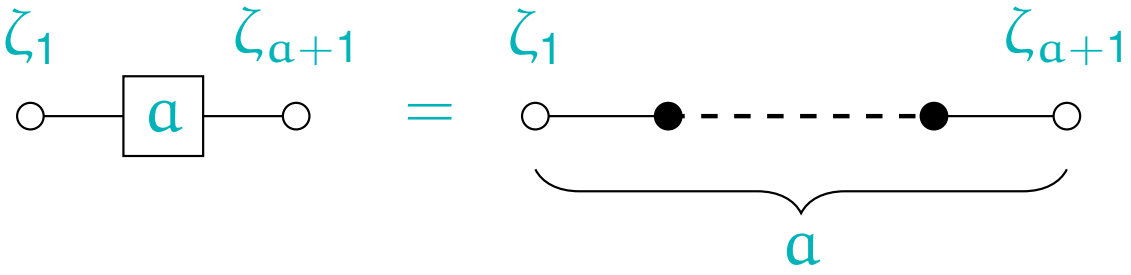


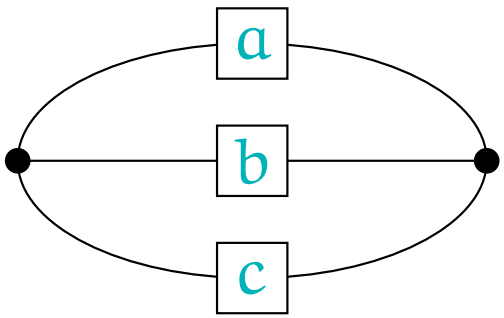
$$I_\Gamma(q;\zeta) = \prod_{k=2}^{4} \int_\Sigma \frac{d^2 \log \zeta_k}{4\pi^2 \tau_2} \prod_{\substack{1\leqslant i<j\leqslant 4 \\ (i,j)\neq(1,3)}} D_{1,1}\left(q;\frac{\zeta_j}{\zeta_i}\right)^{n_{ij}} \times D_{1,1}\left(q;\frac{\zeta_1\zeta}{\zeta_3}\right)^{n_{13}}$$

- the integral evaluates to at $\zeta = 1$ as $I_\Gamma(q;1) = I_\Gamma(q)$
- $I_\Gamma(q;\zeta)$ is single-valued in ζ by construction
- By explicit computation the expression is a linear combination over $\mathbb{Q}$ of single valued elliptic multiple polylogarithms

Modular graph functions

They satisfy a lot of important algebraic relations





$$= C_{a,b,c}(q;\zeta_1/\zeta_1')$$

$$C_{a_1,\ldots,a_\rho}(q) = \sum_{(m_r,n_r)\neq(0,0)} \delta\Big(\sum_r (m_r\tau+n_r)\Big) \prod_{r=1}^{\rho} \left(\frac{\tau_2}{\pi|m_r\tau+n_r|^2}\right)^{a_r}$$

The $C_{a,b,c}$ satisfy differential equations
$(\Delta-\lambda)C_{a,b,c} = P(E_a,\zeta(a))$ and only contain ζ_{sv}

Modular graph functions

$$C_{1,1,1}(q) = E_3(q) + \zeta(3); \qquad C_{2,2,1}(q) = \frac{2}{5}E_5(q) + \frac{\zeta(5)}{30}$$

$$40\,C_{2,1,1,1}(q) = 300\,C_{3,1,1}(q) + 120\,E_2(q)E_3(q) - 276\,E_5(q) + 7\,\zeta(5)$$

$$C_{1,1,1,1,1}(q) = 60\,C_{3,1,1}(q) + 10\,E_2(q)E_3(q) - 48\,E_5(q) + 10\,\zeta(3)E_2(q) + 16\,\zeta(5)$$

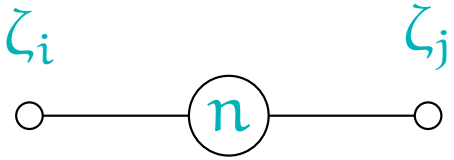
These functions have a mixed q and $\bar{q}$ expansion

$$F(q) = \sum_{n\geqslant 0, m\geqslant 0} c_{n,m}\, q^n \bar{q}^m$$

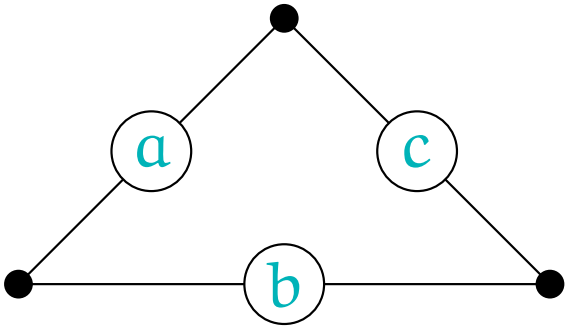
The constant term $c_{0,0}$ is a Laurent polynomial in $y = \pi\tau_2$

$$C_{2,1,1}(q)\Big|_{0,0} = \frac{2y^4}{14175} + \frac{\zeta(3)y}{45} + \frac{5\zeta(5)}{12y} - \frac{\zeta(3)^2}{4y^2} + \frac{9\zeta(7)}{16y^3}$$

Modular graph functions and MZV



$$= G(z_i - z_j|\tau)^n.$$



$$= D_{a,b,c}(q)\,.$$

Federico Zerbini computed that

$$\frac{D_{1,1,5}(q)}{4^7}\Big|_{0,0} = \frac{62\,y^7}{10945935} + \frac{\zeta_{sv}(3)}{243}y^4 + \frac{119}{648}\zeta_{sv}(5)y^2$$
$$+ \frac{11}{54}\zeta_{sv}(3)^2 y + \frac{21}{32}\zeta_{sv}(7) + \frac{23}{}\frac{\zeta_{sv}(3)\zeta_{sv}(5)}{6y}$$
$$+ \frac{7115\zeta_{sv}(9) - 1800\zeta_{sv}(3)^2}{576y^2}$$
$$+ \frac{1245\zeta_{sv}(3)\zeta_{sv}(7) - 150\zeta_{sv}(5)^2}{64y^3}$$
$$+ \frac{288\zeta_{sv}(3,5,3) - 4080\zeta_{sv}(5)\zeta_{sv}(3)^2 - 9573\zeta_{sv}(11)}{256y^4}$$
$$+ \frac{2475\zeta_{sv}(5)\zeta_{sv}(7) + 1125\zeta_{sv}(9)\zeta_{sv}(3)}{128y^5} - \frac{1575}{64}\frac{\zeta_{sv}(13)}{y^6}$$

Conclusion

- The construction clarifies the role of the momentum kernel in the single-valued projection $\mathcal{S}_{\alpha'}$ is one block of $G_N S(\mathbf{A}, \mathbf{B}, \mathbf{C}; \mathbf{G})$
- Notice that the α' expansion does not need to have uniform weight: tree-level heterotic string from the tachyonic pole, or genus two type II expansion [Green, Vanhove]
- Closed string amplitude are special value single-valued CFT correlators, and open strings are multivalued conformal block extended to higher genus
- The low-energy expansion of genus one closed string amplitudes has single-valued modular graph functions

  [D'Hoker, Green, Gurdogan, Vanhove; Zerbini; Brown; Gerken, Kleinschmidt, Schlotterer]
- Single-valued modular graph functions in degeneration limits of genus-two amplitudes [D'Hoker, Green, Pioline]