Eternity and the Cosmological Constant

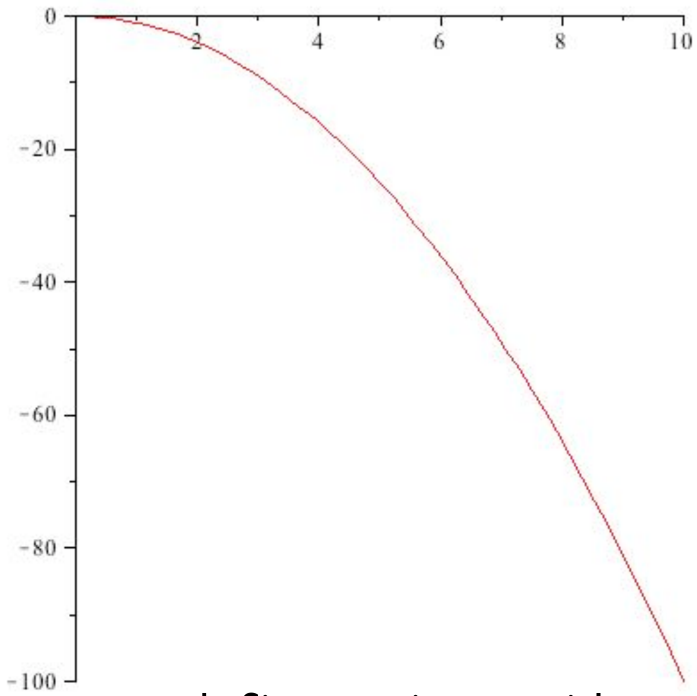
E.A. and Roberto Vidal

Old dream:

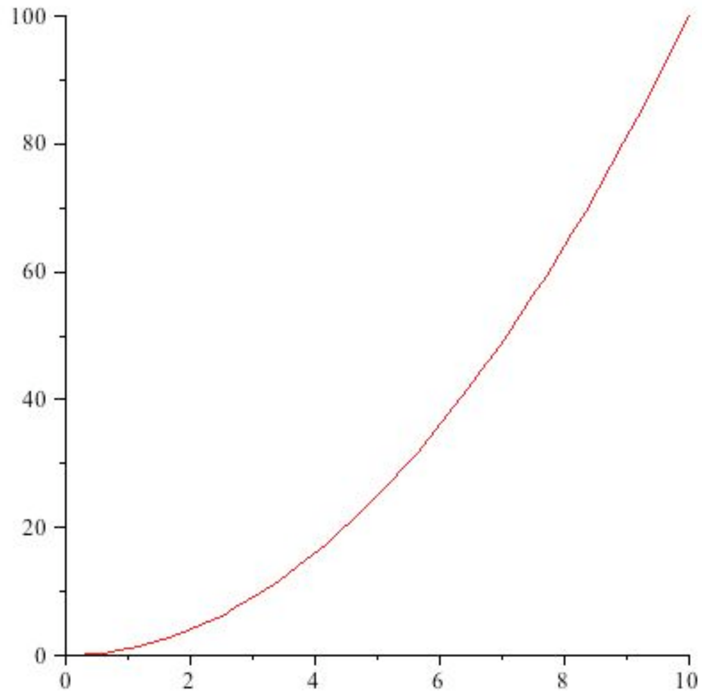
C.C.= vacuum energy why cannot decay?

To flat space?

Gravitational field looks very different depending on the sign of cc:



de Sitter static potential



anti de Sitter static potential

Abbott & Deser: classical stability outside horizons

(Many people in between...)

Tsamis &Woodard: two loop instability of positive c.c. (de sitter space) in ordinary quantum gravity

SUSY dislikes positive c.c. i.e., de Sitter.

(Killing energy is not globally defined because horizons.)

Maldacena: negative c.c. (AdS) consistent vacua of quantum gravity.

KKLT: metastable positive c.c. (de sitter) solutions in string theory.

A.M. Polyakov: Eternity and de Sitter

Claims matter instability in the presence of a positive cosmological constant (de sitter fails the "eternity test" =absence of spiders)

Proposes a novel way of selecting the correct propagator, claimed to be equivalent to unitarity.

Integrating the propagator, claims a non vanishing imaginary part for the effective potential

Consistent with the dual CFT being non-unitary?

In the presence of a gravitational background, vacuum is non unique

Besides, 0+ is not the same as 0- (CTP, Schwinger-Keldysh, BV)

Chernikov & Tagirov: One parameter family of dS invariant vacuum states.

Relations between the sphere, dS, adS EsdS etc confusing...

LOOKS LIKE A MESS...

AP'S composition principle:

$$\int d^n z\, G(x,z)G(z,y) = -\frac{\partial}{\partial m^2}G(x,y)$$

EA&RV: Actually this is automatic if the propagator stems from the heat kernel

Heat equation $K(\tau) \equiv e^{\frac{\tau}{\mu^2}\Delta}$

$$\Delta K(x,y;\tau) - \mu^2 \frac{\partial K(x,y;\tau)}{\partial \tau} = 0$$

Kac: Can one hear the shape of a drum?

FSRHE $\lim_{\tau \to 0+} K(x,y;\tau) = \delta(x,y).$

FSRHE is UNIQUE in the Riemannian case

Usually the heat equation is used in the small time expansion (de Witt-Schwinger) to get divergences .

$$K(x,y;\tau) = K_0(x,y,\tau)\sum_{n=0}^{\infty} a_n(x,y)\tau^n$$

This is because it is usually difficult to solve exactly the heat equation.

If we knew it, we would also know the effective action

$$\boxed{W = \frac{1}{2}\int_0^{\infty}\frac{d\tau}{\tau}\int d^n x\sqrt{|g|}tr\,K(x,x|\tau)}$$

FSRHE defines a propagator

$$G \equiv -\Delta^{-1} \equiv \int_0^\infty K(\tau) d\tau$$

This yields in flat space: $K_0(x,y;\tau) = \frac{\mu^{n-2}}{(4\pi\tau)^{n/2}} e^{-\frac{\mu^2(x-y)^2}{4\tau} - \frac{m^2}{\mu^2}\tau}$

$$G_0(x,y) \equiv \int_0^\infty d\tau K_0(x,y;\tau) = \frac{1}{2\pi}\left(\frac{m}{2\pi|x-y|}\right)^{n/2-1} K_{n/2-1}(m|x-y|)$$

$$K_{m=0}(\tau) = K(\tau)\, e^{\frac{m^2}{\mu^2}\tau}$$

$$G_m(x) = \int_0^\infty K_{m=0}(\tau)\, e^{\frac{m^2}{\mu^2}\tau} \frac{d\tau}{\mu^2}$$

$$K_{m=0}(\tau) = \int_{c-i\infty}^{c+i\infty} \frac{dm^2}{\mu^2} e^{\frac{m^2}{\mu^2}\tau} G_m(x)$$

Constant curvature manifolds= spheres and their analytic continuations

$$\sum_{A=0}^{n} X_A^2 \equiv \delta_{AB} X^A X^B = l^2 \qquad ds^2 = \delta_{AB} dX^A dX^B$$

antipodal mapping: $\mathbb{Z}_2 : X^A \to -X^A$

projective space: $\mathbb{RP}^n = S_n/\mathbb{Z}_2$

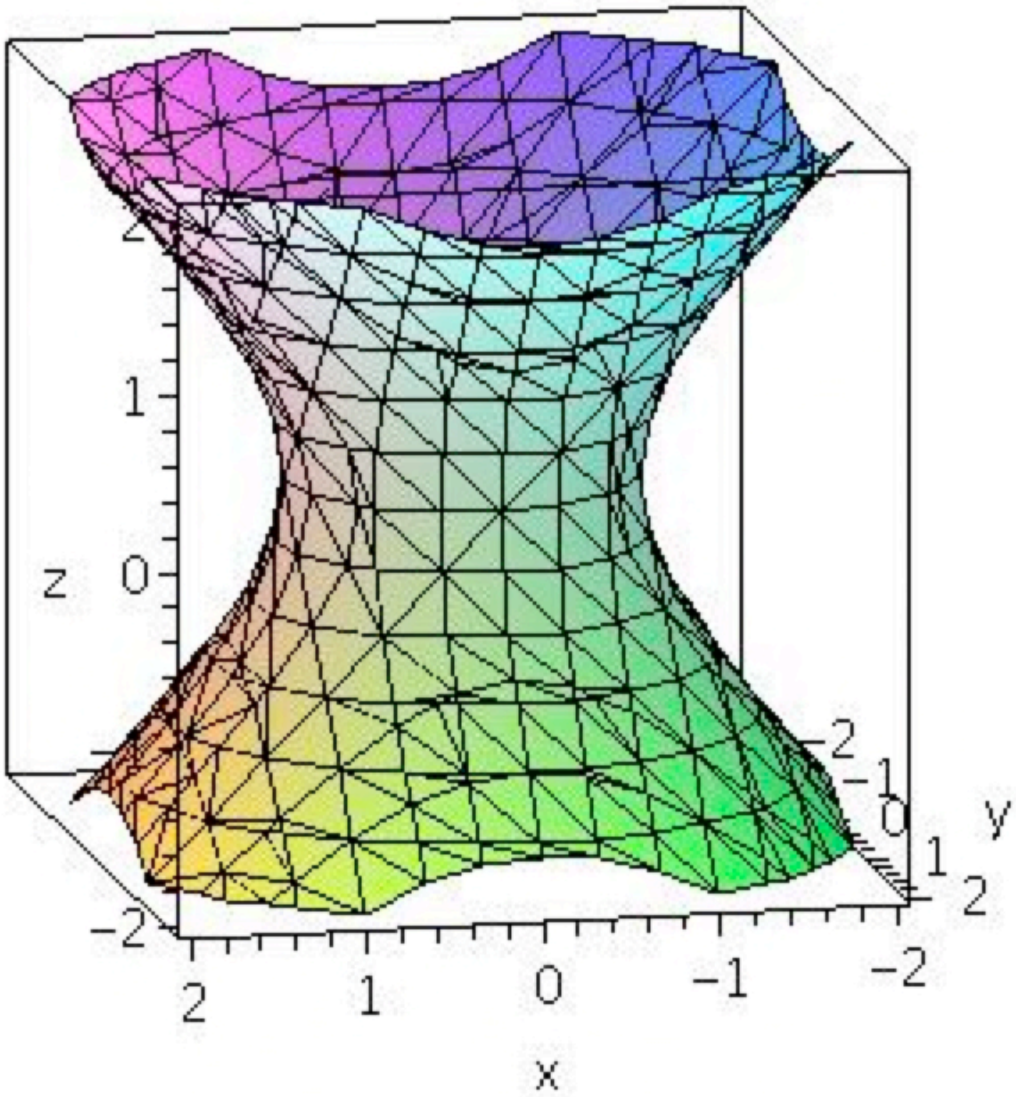
$$X^2 = \sum_A \epsilon_A (X^A)^2 = \pm l^2 \qquad ds^2 = \sum_A \epsilon_A (dX^A)^2$$

$$S_s^n : X \in \mathbb{R}_s^{n+1},\ X^2 = l^2$$
$$H_s^n : X \in \mathbb{R}_{s+1}^{n+1},\ X^2 = -l^2$$

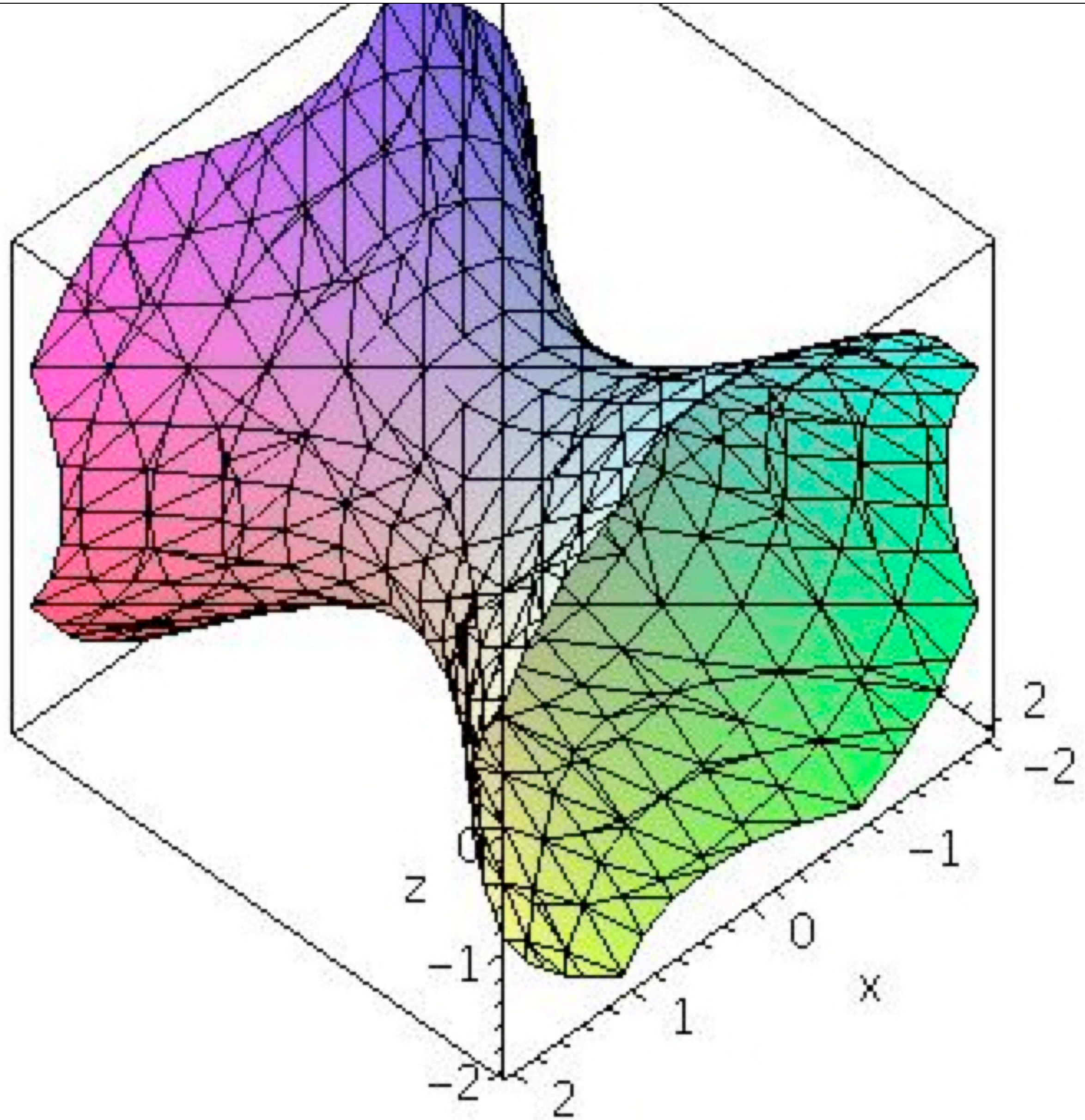
$$\Rightarrow \quad R = \pm \frac{n(n-1)}{l^2}$$

Constant curvature= symmetric spaces=cosets

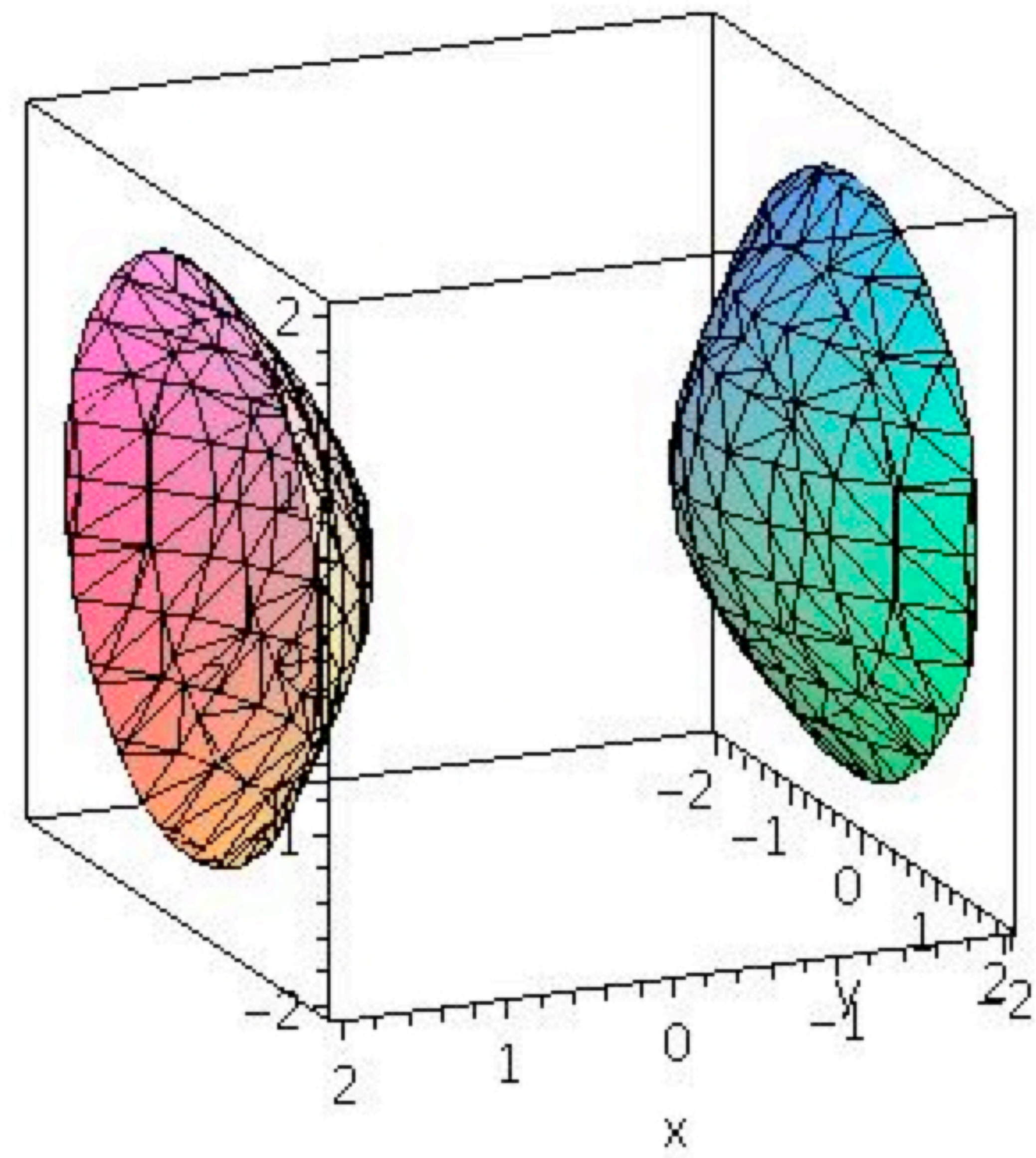
$$AdS_n = SO(2, n-1)/SO(1, n-1)$$
$$EAdS_n = SO(1, n)/SO(n)$$
$$dS_n = SO(1, n)/SO(1, n-1)$$
$$EdS_N = SO(n, 1)/SO(n)$$



de Sitter space



Anti-de Sitter space



Euclidean anti-de Sitter space

"Global" coordinates for S_s^n

$$X^A = l\,(cosh\,\tau\,\vec{n}_t\,(\Omega)\,, sinh\,\tau\,\vec{n}_s\,(\Omega'))$$

Global coordinates for H_s^n

$$X^A = l\,(sinh\,\tau\,\vec{n}_{t-1}\,(\Omega)\,, cosh\,\tau\,\vec{n}_{s+1}\,(\Omega'))$$

$$\vec{n}\,(\Omega) \equiv (cos\,\theta_1, \quad sin\,\theta_1\,cos\,\theta_2, \quad \ldots, sin\,\theta_1 \ldots \quad sin\,\theta_{n-1})$$

$$N\,(S_s^n) = \left(l, \vec{0}\right)$$

$$N\,(H_s^n) = \left(\vec{0}^{t-1}, l, \vec{0}\right)$$

Invariant distance

$$z\,(X,Y) \equiv \pm\frac{X.Y}{l^2}$$

$$S_n : X \equiv l\vec{n}; \quad z = \cos\theta_1$$

$$EadS_n : X = l\,(\cosh\tau, \sinh\tau\,\vec{n}_{n-1})\,; \quad z = \cosh\tau$$

$$dS_n : X = l\,(\sinh\tau, \cosh\tau\,\vec{n}_{n-1})\,; \quad z = \cosh\tau\cos\theta_1$$

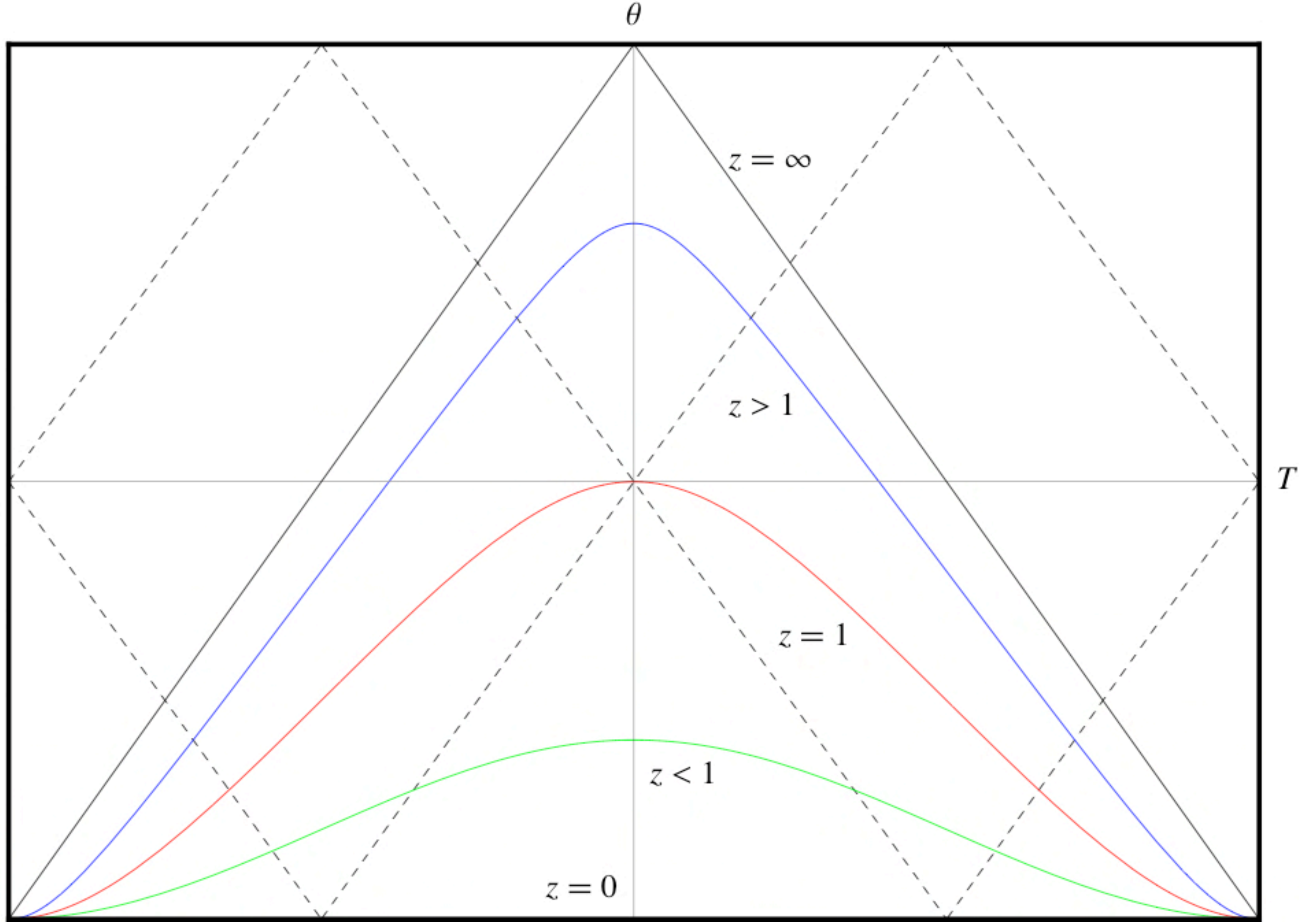
$$adS_n : X = l\,(\cosh\tau\cos\theta, \cosh, \tau\sin\theta, \sinh\tau\,\vec{n}_{n-2})\,; \quad z = \cosh\tau\cos\theta$$

Poincare (Horospheric coordinates)

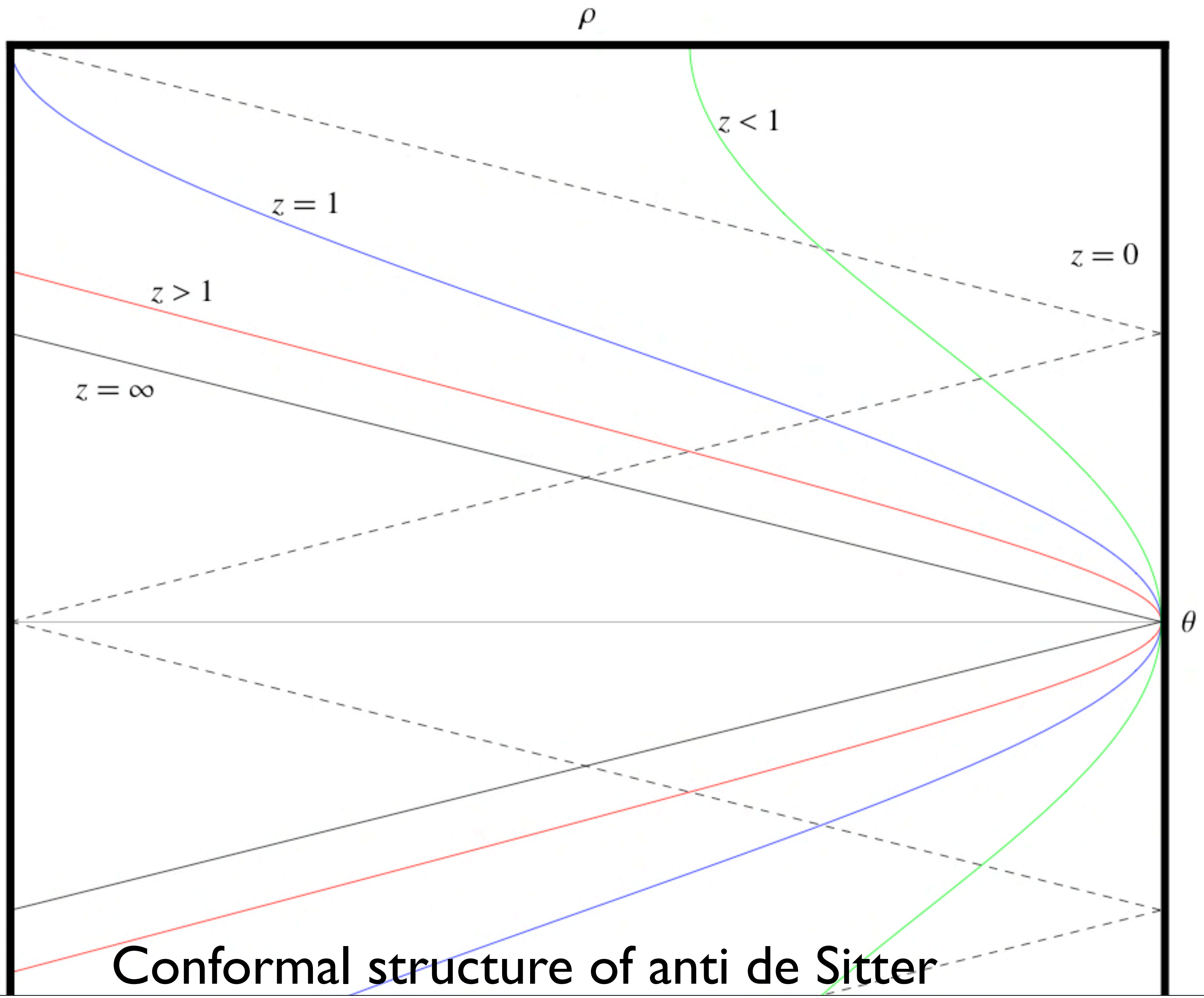
$$\frac{l}{z} = X^{-} \equiv X^{n} - X^{0}$$

$$y^{i} = zX^{i}$$

$$ds^2 = \frac{\sum_1^{n-1} \epsilon_1 dy_i^2 \mp l^2 dz^2}{z^2}$$



Conformal structure of de Sitter



Conformal structure of anti de Sitter

Heat kernel on the sphere

$$K(\theta,\theta';T) \equiv \int \mathcal{D}\Omega\, e^{i\int_0^T d\tau\left(\frac{ml^2}{2}\dot{\Omega}^2+\frac{n(n-2)}{8ml^2}\right)} = e^{iT\frac{n(n-2)}{8ml^2}} \int \mathcal{D}\Omega\, e^{i\int_0^T d\tau\,\frac{ml^2}{2}\dot{\Omega}^2} \equiv$$

$$e^{iT\frac{n(n-2)}{8ml^2}} Z(\theta,\theta';T) \qquad (\text{:}$$

$$S = \frac{ml^2}{2}\sum_{i=1}^{n}\left(\vec{\Omega}_i-\vec{\Omega}_{i-1}\right)^2 = ml^2\sum_{i=1}^{n}\left(1-\cos\psi_{i-1}\right) \qquad \cos\psi_{i-1}\equiv\vec{\Omega}_i\cdot\vec{\Omega}_{i-1}$$

$$e^{z\cos\psi} = \left(\frac{z}{2}\right)^{-\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\sum_{j=0}^{\infty}\left(j+\frac{n-1}{2}\right)I_{j+\frac{n-1}{2}}(z)\,C_j^{\frac{n-1}{2}}(\cos\psi) \qquad (3.8)$$

$$\sum_{\vec{m}} Y^*_{j\ldots\vec{m}}(\Omega_z)Y_{j\ldots\vec{m}}(\Omega) = \frac{1}{V(S_n)}\frac{n-1+2j}{n-1}C_j^{\frac{n-1}{2}}(\cos\theta_n)$$

$$\int d\Omega_1\ldots d\Omega_{n-1}\sum_{j_1\vec{m}_1}\sum_{j_2\vec{m}_2}\ldots Y_{j_1\vec{m}_1}(\Omega_1)Y^*_{j_1\vec{m}_1}(\Omega_0)Y_{j_2\vec{m}_2}(\Omega_2)Y^*_{j_2\vec{m}_2}(\Omega_1)\ldots$$

$$\ldots\sum Y_{j_n\vec{m}_n}(\Omega_n)Y^*_{j_n\vec{m}_n}(\Omega_{n-1}) = \sum_{j\vec{m}} Y_{j\vec{m}}(\Omega_n)Y^*_{j\vec{m}}(\Omega_0) \sim \sum_j C_j^{\frac{n-1}{2}}(\cos\psi)$$

$$K(\Omega,\Omega';T) = \frac{\Gamma(n/2)}{2\pi^{n/2}}\sum_{j=0}^{\infty}\frac{2j+n-1}{n-1}C_j^{\frac{n-1}{2}}(\Omega.\Omega')e^{-\frac{iT}{2mR^2}j(j+n-1)}$$

Scalar field on the Sphere

$$(z^2-1)G'' + nzG' \pm m^2l^2 = 0$$

$z = \cos\theta.$

Look for solutions with a single source at z=1

$$G(z) = F_{\pm}(z) = F\left(\frac{1 \pm z}{2}\right) \equiv F\left(i\mu + \frac{n-1}{2}, -i\mu + \frac{n-1}{2}; \frac{n}{2}; \frac{1 \pm z}{2}\right)$$

$$m^2l^2 = \mu^2 + \tfrac{(n-1)^2}{4}.$$

This Green function is unique and it obeys the composition law

$$G(\Omega \cdot \Omega') = \sum_{j\vec{k}} \frac{Y_{j\vec{k}}(\Omega) Y_{j\vec{k}}(\Omega')^*}{j(j+n-1)+m^2}$$

Scalar field on de Sitter

$$\frac{1}{\cosh\tau^{n-1}}\partial_\tau\left(\cosh\tau^{n-1}\partial_\tau G\right)-\frac{1}{\cosh\tau^2\sin\theta^{n-2}}\partial_\theta\left(\sin\theta^{n-2}\partial_\theta G\right)+m^2l^2G=0$$

$$(z^2-1)G''+nzG'+m^2l^2G=0\ \boxed{,\ z=\cosh\tau\cos\theta} \tag{4.8}$$

$\therefore$ must specify the values in the branch cuts

Matching flat space singularities gives the Bunch-Davies (Euclidean) vacuum.

Ambiguities in the GF

$$G(z)=G_{BD}(z)+\alpha\,\mathrm{Re}\,F_+(z)+\beta\,\mathrm{Re}\,F_-(z)$$

Not all GF correspond to a vev

Subset: Chernikov-Tagirov family of vacua

$$G_\alpha(z)=\frac{i|\Gamma\left(i\mu+\frac{n-1}{2}\right)|^2}{2(4\pi)^{\frac{n}{2}}\{-\Gamma(2-\frac{n}{2})|\Gamma(\frac{n}{2})\}}\left\{\cosh 2\alpha\,\mathrm{Re}F\left(\frac{1+z}{2}\right)+\right.$$

$$\left.+\sinh 2\alpha\,\mathrm{Re}F\left(\frac{1-z}{2}\right)-i\,\mathrm{Im}F\left(\frac{1+z}{2}\right)\right\}$$

Scalar field on EadS

In this case, the naive analytic continuation diverges at infinity

$$\therefore$$

One has got to combine this with an homogeneous solution

$$G(z) = \left(z^2 - 1\right)^{\frac{1-n}{4}} \, Q^{\frac{n-1}{2}}_{\mu-\frac{1}{2}}(z) \;\; \equiv R^{\frac{n-1}{2}}_{\mu-1/2}$$

Scalar field on AdS

G must vanish at the well defined spacial infinity

The EadS expression must be continued to the full real axis.

The delta singularities lie in the imaginary part of the F functions; and the homogeneous pieces can be taken real.

We have to eliminate the imaginary part of $F\left(\frac{1-z}{2}\right)$

$$\therefore$$

$$\tilde{R}_\nu^{\frac{n-2}{2}}(z) \equiv e^{i\pi\nu} R_\nu^{\frac{n-2}{2}}(z+i\epsilon) + e^{-i\pi\nu} R_\nu^{\frac{n-2}{2}}(z-i\epsilon)$$

Projective spaces

Lift to a symmetric function defined on the full space.

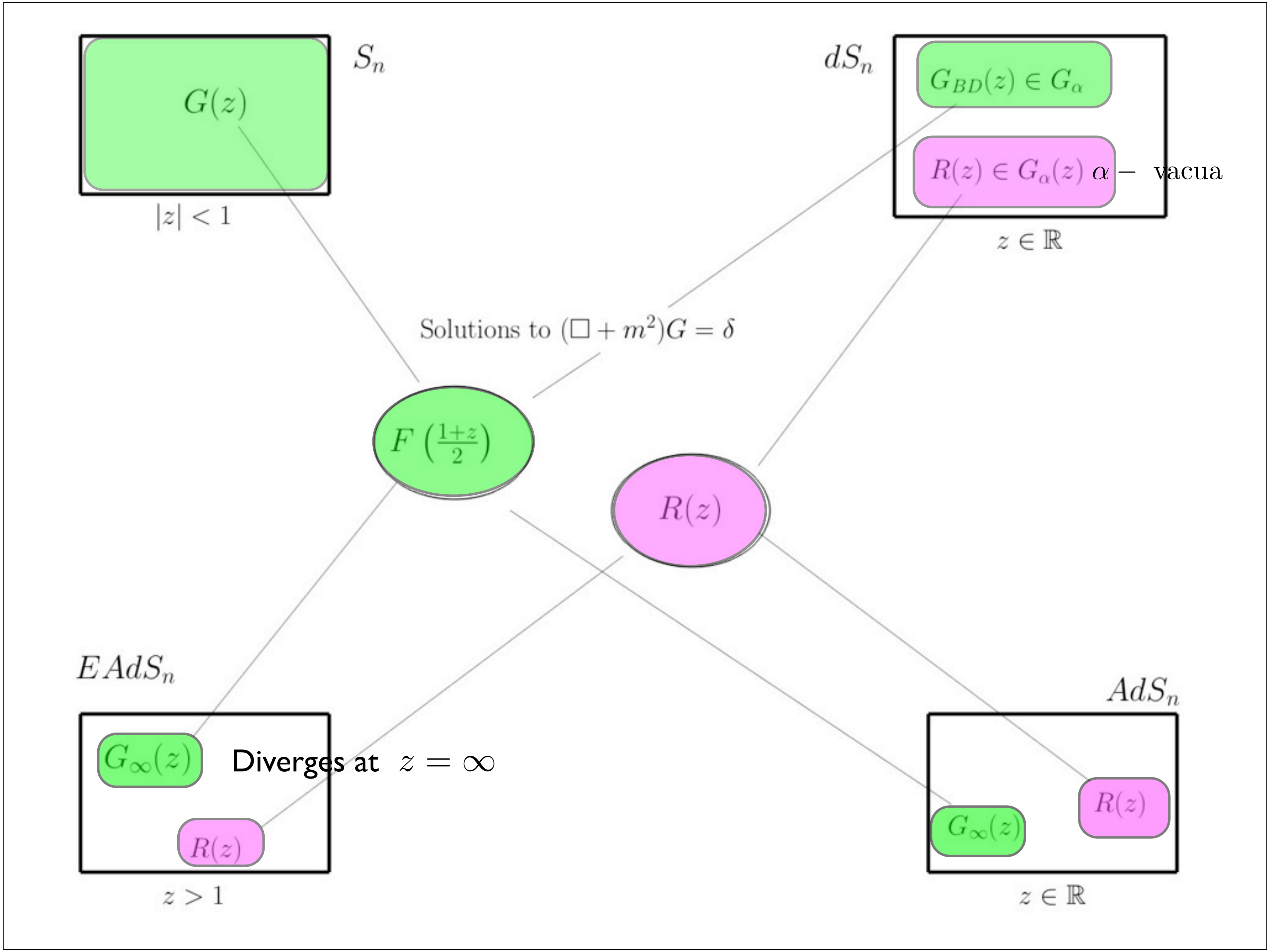
$$\mathbb{R}P_n \equiv S_n/\mathbb{Z}_2$$

$$G(z) + G(-z)$$

$$dS_n/\mathbb{Z}_2$$

$$adS_n/\mathbb{Z}_2$$

$$G_P(z) = G_{BD}(z) + G_{BD}(-z) + \alpha\left(Re\,F_+(z) + Re\,F_-(z)\right)$$



S_n
$G(z)$
$|z| < 1$
dS_n
$G_{BD}(z) \in G_\alpha$
$R(z) \in G_\alpha(z)$ $\alpha-$ vacua
$z \in \mathbb{R}$
Solutions to $(\Box + m^2)G = \delta$
$F\left(\frac{1+z}{2}\right)$
$R(z)$
$EAdS_n$
$G_\infty(z)$
Diverges at $z = \infty$
$R(z)$
$z > 1$
AdS_n
$G_\infty(z)$
$R(z)$
$z \in \mathbb{R}$

The imaginary part of the effective potential

We can see the heat kernel formally as

$$K(\tau) \equiv e^{-\tau \bar{M}^2} \tag{3.14}$$

where $\bar{M}^2$ is the positive definite operator acting on quadratic fluctuations around the background field, id est,

$$\bar{M}^2 \equiv -\Delta + \partial^2 V(\bar{\phi}) \tag{3.15}$$

and we include masses in the potential.

Let us mention that whenever the full eigenvalue problem for the operator $\bar{M}^2$ is known, there is a formal FSRHE. Using the discrete notation,

$$\bar{M}^2 u_n(x) = \lambda_n u_n(x) \tag{3.16}$$

with eigenfunctions which can be chosen to obey

$$(u_n, u_m) \equiv \int d\mu(x)\, u_n^*(x) u_m(x) = \delta_{nm} \tag{3.17}$$

(where the measure $d\mu(x)$ is usually $\sqrt{|g|}d^n x$) as well as a completeness relationship of the type

$$\sum_n u_n^*(x) u_n(y) = \delta(x-y) \tag{3.18}$$

then the following is the sought for FSRHE

$$\boxed{K(x, y|\tau) = \sum_n e^{-\lambda_n \tau} u_n^*(x) u_n(y)} \tag{3.19}$$

If the eigenvalues are real

$\Rightarrow$

Then the heat kernel is also real

$\Rightarrow$

and the imaginary part of the free energy vanishes

Sphere, S_n

Heat kernel

$$K(\tau; z) = \frac{1}{V(S_n)} \sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(z) e^{-\tau\left(m^2 l^2 + V''(\bar{\phi}) + j(j+n-1)\right)}$$

Free Energy

$$W = \frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\int d^n x \sqrt{|g|} K(\tau; x, x) = \frac{\mathrm{Vol}_{dS}}{2}\int_0^\infty \frac{d\tau}{\tau} K(\tau; 1) =$$

$$= \frac{\mathrm{Vol}_{dS}}{2V(S_n)}\int_0^\infty \frac{d\tau}{\tau}\sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(1) e^{-\tau\left(m^2 l^2 + V''(\bar{\phi}) + j(j+n-1)\right)}$$

de Sitter & anti de Sitter

Discrete spectrum

$$-\frac{L\,(L+n-1)}{l^2}$$

$$L=-[\frac{n}{2}]+1,-[\frac{n}{2}]+2,\ldots,-[\frac{n}{2}]+j$$

Continuous spectrum

$$\frac{\Lambda^2+\frac{(n-1)^2}{4}}{l^2}$$

$$\Lambda\in[0,\infty)$$

Euclidean anti de Sitter

Continuous spectrum only

In all cases, we have closure

$$\sum_L Y_L(x)^* Y_L(y) + \int d\Lambda Z_\Lambda(x)^* Z_\Lambda(y) = \delta\,(x, y)$$

Useful approximation (blind to the differences)

$$\sum 2je^{-\tau(j+n-1)j} \sim 2\int_0^\infty djje^{-\tau\left(j+\frac{n-1}{2}\right)^2+\tau\frac{(n-1)^2}{4}} = e^{\frac{(n-1)^2}{4}\tau}\left(\frac{1}{\tau}+1-n\right)$$

$$W \sim \int_{\Lambda^{-2}}^\infty \frac{d\tau}{\tau} e^{-\left(\frac{m^2+V''(\bar{\phi})}{\mu^2}-\frac{(n+1)^2}{4}\right)\tau}\left(\frac{1}{\tau}+1-n\right) =$$

$$\frac{\Lambda^2}{\mu^2} - \left(\frac{m^2+V''\left(\bar{\phi}\right)}{\mu^2}-\frac{(n+1)^2}{4}\right)\left(log\frac{\Lambda^2}{\mu^2}+log\left(\frac{m^2+V''\left(\bar{\phi}\right)}{\mu^2}-\frac{(n+1)^2}{4}\right)\right)$$

The only imaginary part appears when

$$\boxed{\frac{m^2 + V''\left(\bar{\phi}\right)}{\mu^2} - \frac{(n+1)^2}{4} \leq 0}$$

This corresponds exactly in flat space to

SSB (Weinberg and Wu)

What are the conditions for an imaginary part to appear?

Schwinger-Dyson

$$0 = \int \mathcal{D}g_{\mu\nu}\mathcal{D}b\mathcal{D}c\mathcal{D}\phi \frac{\delta}{\delta g^{\mu\nu}} e^{i(S_g(g_{\mu\nu})+S_{gf}(g_{\mu\nu},\phi)+S_{gh}(g_{\mu\nu},b,c)+S_{count}(g_{\mu\nu},\phi))}$$

Einstein-Hilbert:

$$\langle\chi|\sqrt{|g|}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \kappa^2 T_{\mu\nu}\right)|\psi\rangle = 0$$

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \left(S_m + S_{gf} + S_{gh} + S_{count} \right)$$

Einstein-Hilbert, neglecting counterterms

$$\langle \chi | \sqrt{|g|} \left(\frac{2-n}{2} R + \kappa^2 T \right) | \psi \rangle = 0$$

$$S_{count.grav.} = \int \sqrt{|g|} d^n x \left(c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \ldots \right)$$

$$S_{count.matt.} = \int \sqrt{|g|} d^n x \left(d_1 R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \ldots \right)$$

Proper analysis of Schwinger-Dyson quite complicated...

To conclude:

It seems possible to build up models
with decaying cosmological constant, but we still
are shy of a workable one