AdS/CFT and condensed matter physics

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Outline

- Review AdS/CFT
- Application to condensed matter physics
- Finite temperature in AdS/CFT and hydrodynamics
- Charge transport and superconductivity
- Non-relativistic field theories: anisotropic conformal symmetry
- Reviews:
  - Hartnoll, arXiv:0903.3246
  - Rangamani, arXiv:0905.4352
  - McGreevy, arXiv:0909:0518

Review of AdS/CFT

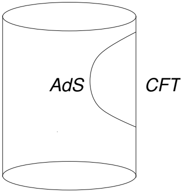
Maldacena

Conjecture a class of strongly-coupled field theories have gravity duals:

- Identify global symmetries with asymptotic spacetime isometries. For conformal field theories, $SO(d,2) \leftrightarrow$ isometries of AdS_{d+1}. In Poincare coordinates,

$$ds^2 = r^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dr^2}{r^2},$$

  Dilatation is $D : x^\mu \to \lambda x^\mu, r \to \lambda^{-1} r$.
- Energy scale $\leftrightarrow$ radial position.



Review of AdS/CFT

- $\langle e^{\int \phi_0 \mathcal{O}} \rangle = Z_{string}[\phi_0] \approx e^{-S[\phi_0]}$. Witten; Gubser Klebanov Polyakov
  - Choice of state $\leftrightarrow$ initial conditions.
  - Correlation functions $\leftrightarrow$ variation of on-shell action wrt boundary conditions.
  - Boundary stress tensor $\langle T_{\mu\nu} \rangle = \frac{1}{\sqrt{\hat{h}}} \frac{\delta \mathcal{S}}{\delta \hat{h}^{\mu\nu}}$ Henningson Skenderis; Balasubramanian Kraus
  - Conserved currents $\langle J_\mu \rangle = \frac{1}{\sqrt{\hat{h}}} \frac{\delta \mathcal{S}}{\delta A^\mu}$
- Examples: $\mathcal{N} = 4$ SYM $\leftrightarrow$ IIB string theory on asymp $\mathrm{AdS}_5 \times S^5$.
  - Classical gravity valid at strong 't Hooft coupling.
  - Many other explicit examples: $\mathrm{AdS}_5 \times X^5$, $AdS_4 \times S^7/\mathbb{Z}_n, \ldots$
  - Focus on universal subsector: pure gravity in bulk.
- Reasons for believing such a duality exists more generally:
  - Field theories local in energy scale: RG flow is a local equation in energy scale
  - Holographic principle: $S_{BH} = A/4G_N$ suggests gravity is related to a lower-dimensional theory.

Application to condensed matter physics

Application to QCD long studied. Why condensed matter?

- Rich system:
  - CFTs arise as IR desc near critical points; often strongly coupled.
  - Many different theories; can tune Hamiltonian in some settings.
- AdS/CFT provides a useful new perspective:
  - Few other methods for calculation at strong coupling.
  - Gravity dual calculates observables like transport coefficients directly, no quasiparticle picture.
- Prompts new questions:
  - Charge transport, phase transitions
  - Can have theories with an anisotropic scaling symmetry D: $x^i \to \lambda x^i, t \to \lambda^z t$.

Application to condensed matter physics

However, this is a big extension of AdS/CFT.

- Known examples involve large N gauge theories. To apply these ideas to condensed matter, need to assume other field theories will have holographic duals.
- Proceed phenomenologically: construct bulk theories with qualitative features of interest.
- Big question: for what types of theories can we construct a bulk dual in the classical approximation?
  - Strong coupling
  - $N_{dof} \gg 1$
  - Hierarchy in spectrum: $\Delta_{s>2} \gg \Delta_{s\leq 2}$.

Quantum critical point

Critical point at zero temperature.

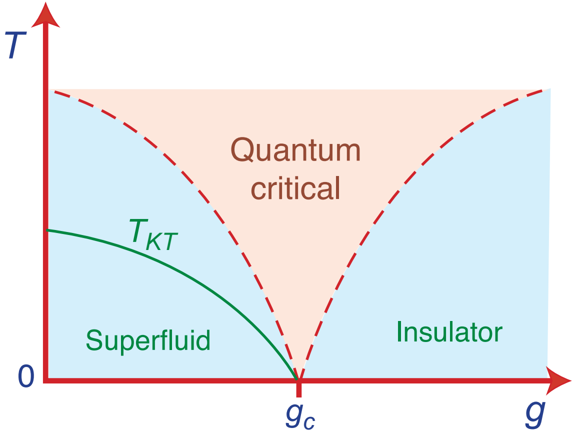


Figure from Sachdev, arXiv:0711.3015

- Critical point described by a CFT;
  finite region described by finite-temperature CFT.

Finite temperature in AdS/CFT

Field theory at finite temperature: partition function $Z = \mathrm{tr} e^{-\beta H} = e^{-\beta F}$.

$\triangleright$ Bulk saddle-point Euclidean black hole Witten

$$ds^2 = r^2 \left[\left(1 - \frac{r_+^d}{r^d} \right) d\tau^2 + d\mathbf{x}^2 \right] + \left(1 - \frac{r_+^d}{r^d} \right)^{-1} \frac{dr^2}{r^2}$$

- Temperature $T = \frac{dr_+}{4\pi}$.
- Entropy $S = \frac{1}{4G_5} r_+^{d-1} V = \frac{(4\pi)^{d-1}}{4d^{d-1} G_5} V T^{d-1}$
- Stress tensor $\langle T_{\mu\nu} \rangle = \frac{r_+^d}{16\pi G_5} (\eta_{\mu\nu} + d u_\mu u_\nu)$, $u^\mu = \delta^\mu_t$.

Hydrodynamics

- Effective description of long-wavelength perturbations

$$T_{\mu\nu} \sim T^d(\eta_{\mu\nu} + d u_\mu u_\nu) - 2\eta\sigma_{\mu\nu} + \dots$$

- Transport coefficients calculated in linearised theory on black hole background:

Policastro
Son
Starinets

Linear response, $\delta\langle T_{xy}\rangle = G^R_{xy,xy}(\omega, k)\delta g_{xy}$, gives Kubo formula

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\, G^R_{xy,xy}(\omega, 0).$$

Universal result $\frac{\eta}{s} = \frac{1}{4\pi}$

Bhattacharyya
Hubeny
Minwalla
Rangamani

- More direct approach:
  - Consider black hole solution with $T(t, \mathbf{x})$, $u^\mu(t, \mathbf{x})$.
  - Correct order by order in derivative expansion.
  - Hydrodynamic parameters determined by bulk dynamics:

$$16\pi G_5 T_{\mu\nu} = r_+^d(\eta_{\mu\nu} + d u_\mu u_\nu) - 2 r_+^{d-1}\sigma_{\mu\nu} + \dots$$

Charge transport

Consider thermodynamics with chemical potential μ.

▷ Bulk dual is Reissner-Nördstrom-AdS black hole,

$$ds^2 = r^2 \left[f(r) d\tau^2 + d\mathbf{x}^2 \right] + \frac{dr^2}{r^2 f(r)},$$

$$f(r) = 1 - \left(1 + \frac{\mu^2}{r_+^2 \gamma^2} \right) \frac{r_+^d}{r^d} + \frac{\mu^2}{r_+^2 \gamma^2} \frac{r_+^{2(d-1)}}{r^{2(d-1)}},$$

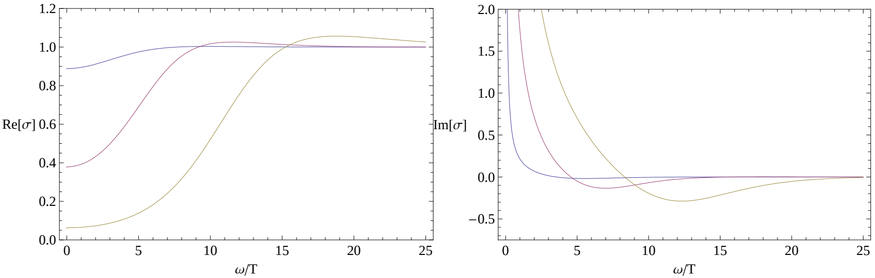
$$A = \mu \left(1 - \frac{r_+^{d-2}}{r^{d-2}} \right) dt.$$

Constant term in A_μ is a source for J_μ in dual CFT; subleading term gives expectation value of J_μ.

Charge transport: $\langle \mathbf{J} \rangle = \sigma \mathbf{E}$. Electrical conductivity given by

$$\sigma = -\frac{1}{g^2} \frac{i}{\omega} \frac{\delta A_x^{(1)}}{\delta A_x^{(0)}} = -\frac{1}{g^2} \frac{i}{\omega} G^R_{A_x, A_x}(\omega, 0).$$

Conductivity



Different curves are different values of μ. Figure from Hartnoll, arXiv: 0903.3264

- Pole in Im(σ) at $\omega = 0$ implies δ-function in Re(σ). Consequence of translation invariance.
- Broad features similar to graphene, which is described by a $2 + 1$ relativistic field theory.

Superconductivity

Gubser

Hartnoll
Herzog
Horowitz

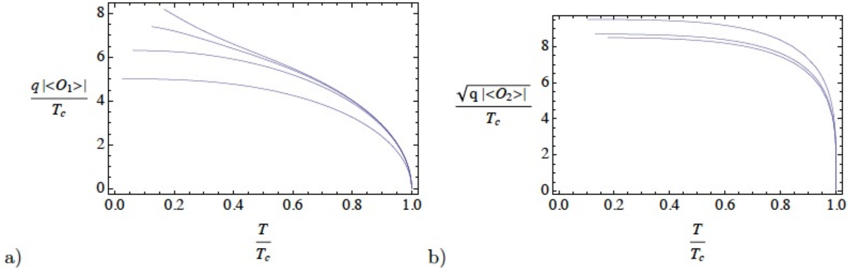
⋆ Model superconducting transition by condensation of a charged operator, breaking global $U(1)$ symmetry.

Simple bulk dual: add a scalar of charge q, mass m to RN-AdS background. Consider $d = 3$.

- If $q^2\gamma^2 \geq 3 + 2m^2$, scalar is unstable for $T \leq T_c$ for some T_c.
- Two mechanisms for instability:
  - Effective negative mass^2 in region near horizon from charge coupling.
  - AdS_2 near horizon region in RN-AdS for small T: if scalar has $-\frac{9}{4} < m^2 < -\frac{3}{2}$, unstable in AdS_2 but stable in AdS_4.
- Condensation produces a hairy black hole; solutions obtained numerically.

▷ This phenomenological theory can be embedded in string theory.

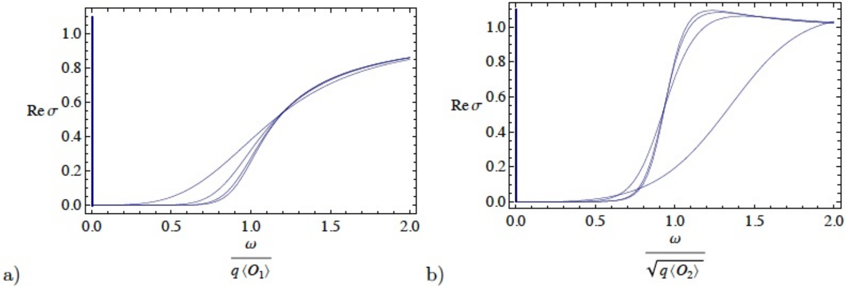
Superconductivity



Condensate as a function of temperature. Different curves are different values of q. Figure from Hartnoll, Herzog & Horowitz, arXiv:0810.1563.

- Second-order phase transition

Superconductivity



Real part of the conductivity as a function of frequency. Different curves are different values of q. Figure from Hartnoll, Herzog & Horowitz, arXiv:0810.1563.

- Delta-function at $\omega = 0$; size jumps at transition.
- Gap at low temperatures: $\mathrm{Re}(\sigma) \approx 0$ for $\omega < \omega_g$.

Non-relativistic holography

In condensed matter, have theories with an anisotropic scaling symmetry D: $x^i \to \lambda x^i, t \to \lambda^z t$.

Two cases of interest:

- Lifshitz-like theories: D, H, $\vec{P}$, M_{ij}
- Schrödinger symmetry: add Galilean boosts $\vec{K}$. $z = 2$ special.

Can these also have a dual geometrical description?

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Lifshitz:

- Simple deformation of AdS:

Kachru
Liu
Mulligan

$$ds^2 = -r^{2z} dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2}.$$

Lifshitz geometry

- Solution of a theory with a massive vector:

$$S = \int d^4x\sqrt{-g}(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu),$$

with $\Lambda = -\frac{1}{2}(z^2 + z + 4)$, $m^2 = 2z$. Lifshitz solution has

$$A = \alpha r^z dt, \quad \alpha^2 = \frac{2(z-1)}{z}.$$

- No conformal boundary: can define boundary conditions by working in terms of frame fields. $e^{(0)} = r^z \hat{e}^{(0)}$, $e^{(i)} = r\hat{e}^{(i)}$, $e^{(3)} = \frac{dr}{r}$.
- Finite temperature black hole solutions obtained numerically.

  Danielsson Thorlacius   Mann   Bertoldi Burrington Peet
- Analytic black holes in a theory with an extra scalar field. Balasubramanian McGreevy
- Not known how to embed these solutions in string theory

Action & stress tensor

SFR
Saremi

AdS/CFT dictionary is based on an action: $\langle e^{\int \phi_0 \mathcal{O}} \rangle \approx e^{-S_{on-shell}[\phi_0]}$.

Need an action principle for asymptotically Lifshitz spacetimes; naive bulk action divergent. Regularize by adding boundary terms:

$$S_{ct} = bulk + \frac{1}{16\pi G_4} \int d^3\xi \sqrt{-h}(2K - 4 - \sqrt{2z(z-1)}\sqrt{-A_\alpha A^\alpha}) + S_{deriv}$$

$\star$ S_{ct} satisfies $\delta S = 0$ for variations, $S_{on-shell}$ finite.

In relativistic case, stress tensor defined by $\delta S = \int d^3\xi\, T_{\alpha\beta} \delta \hat{h}^{\alpha\beta}$.

Non-relativistic theory: Stress-energy complex satisfying conservation equations $\partial_t \mathcal{E} + \partial_i \mathcal{E}^i = 0$, $\partial_t \mathcal{P}_j + \partial_i {\Pi^i}_j = 0$.

Define by

$$\delta S = \int [-\mathcal{E} \delta \hat{e}_t^{(0)} - \mathcal{E}^i \delta \hat{e}_i^{(0)} + \mathcal{P}_i \delta \hat{e}_t^{(i)} + \Pi_i^j \delta \hat{e}_j^{(i)} + s_A \delta A^A].$$

- Results finite
- Bulk equations of motion imply conservation equations, $z\mathcal{E} = \Pi^i_i$.

Schrödinger symmetry

Galilean symmetry: rotations M_{ij}, translations P_i, boosts K_i, $(i = 1, \ldots, d)$,
Hamiltonian H, particle number N.
Extended by the dilatation D,

$$[D, P_i] = iP_i, [D, H] = ziH, [D, K_i] = (1 - z)iK_i, [D, N] = i(2 - z)N.$$

Dynamical exponent z determines scaling of H under dilatations.

For $z = 2$, N is central, and there is a special conformal generator C: $[D, C] = -2iC, [H, C] = -iD$.
Schrödinger algebra: Symmetry of free Schrödinger equation.

$\star$ Isometries of a gravitational dual?

Geometrical dual

Embed Galilean symmetry in $ISO(d+1,1)$ by light-cone quant: $H = \tilde{P}_+$, $N = \tilde{P}_-$, $K_i = \tilde{M}_{-i}$. Extend to embed $Sch(d)$ in $SO(d+2,2)$ by

$$D = \tilde{D} + (z-1)\tilde{M}_{+-}.$$

⋆ $Sch(d)$ is a subgroup of $SO(d+2,2)$

Geometrical dual

Son   Balasubramanian McGreevy

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Gravitational dual: deform AdS_{d+3} to

$$ds^2 = -\sigma^2 r^4 (dx^+)^2 + r^2(-2dx^+dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2}.$$

- Solution of a theory with a massive vector, $A^- = \sigma$.
- N discrete implies x^- periodic. Compact null direction?

Dual in string theory

Herzog
Rangamani
SFR

Adams
Balasubramanian
McGreevy

Maldacena
Martelli
Tachikawa

Take $AdS_5 \times S^5$,

$$ds^2 = r^2(-2dx^+dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2} + (d\psi + P)^2 + d\Sigma_4^2,$$

and apply a TsT transformation:

Maldacena
Martelli
Tachikawa

- T-dualize the Hopf fiber coordinate ψ to $\tilde{\psi}$,
- Shift $x^- \to \tilde{x}^- = x^- + \tilde{\psi}$,
- T-dualise $\tilde{\psi}$ to ψ at fixed $\tilde{x}^-$.

Resulting solution is

$$ds^2 = -r^4(dx^+)^2 + r^2(-2dx^+dx^- + d\mathbf{x}^2) + \frac{dr^2}{r^2} + (d\psi + P)^2 + d\Sigma_4^2,$$

$$B = r^2 dx^+ \wedge (d\psi + P).$$

Heating up the non-relativistic theory

Apply the TsT transformation to the Schwarzschild-AdS solution,

$$ds^2 = r^2(-f(r)dt^2 + dy^2 + d\mathbf{x}^2) + \frac{dr^2}{r^2 f(r)} + ds^2_{S^5},$$

where $f(r) = 1 - r_+^4/r^4$.
Resulting solution in 5d is

$$\begin{aligned} ds^2 &= r^2 k(r)^{-\frac{2}{3}} \left(\left[\frac{r_+^4}{4\beta^2 r^4} - r^2 f(r) \right] (dx^+)^2 \right. \\ &\left. + \frac{\beta^2 r_+^4}{r^4}(dx^-)^2 - [1 + f(r)] dx^+ dx^- \right) + k(r)^{\frac{1}{3}} \left(r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2 f(r)} \right). \end{aligned}$$

$$A = \frac{r^2}{k(r)} \left(\frac{1 + f(r)}{2} dx^+ - \frac{\beta^2 r_+^4}{r^4} dx^- \right),$$

$$e^\phi = \frac{1}{\sqrt{k(r)}}; \quad k(r) = 1 + \frac{\beta^2 r_+^4}{r^2}.$$

▷ β param. choice of x^- coord in TsT; define $\gamma^2 \equiv \beta^2 r_+^4$.

Action principle

Herzog
Rangamani
SFR

SFR
Saremi

Similarly construct action by adding boundary counterterms:

$$\mathcal{S} = bulk + \int d^4\xi \sqrt{-h}(2K - 6 + (1 + c\phi)A^\mu A_\mu + (2c + 3)\phi^2)$$

Finite, satisfies $\delta\mathcal{S} = 0$, but boundary conditions for variations more restrictive.

Stress tensor — proceed as in Lifshitz case: vary metric at fixed A^A, ϕ.

Differences:

- Conserved particle number. Particle number density ρ, flux ρ_i.
- No natural choice of frame.

Write

$$\delta S = \int d^4\xi (s_{\alpha\beta}\delta h^{\alpha\beta} + s_\alpha \delta A^\alpha),$$

Define $\mathcal{E} = 2s^+{}_+ - s^+ A_+, \quad \mathcal{E}^i = 2s^i{}_+ - s^i A_+$, etc.

- For black hole solution, $\mathcal{E} = r_+^4$, $\Pi_{xx} = \Pi_{yy} = r_+^4$, $\rho = 2\beta^2 r_+^4$.
- For solutions obtained by TsT from vacuum AdS solution, agrees with AdS stress tensor.

Discussion

- Holography gives a simple approach to calculating observables such as transport coefficients for some class of strongly-coupled field theories.
  - Includes relativistic and non-relativistic theories.
  - Doesn't rely on a quasiparticle picture.
  - Potential applications to QCD and condensed matter.
- Need to understand which theories have such duals
  - General conditions for dual, validity of classical approximation?
  - Embedding phenomenological Lagrangians in string theory.
- May not model specific theories of interest, but will at least get new insight into strongly-coupled theories. Useful testing bed for refining expectations.