Effective field theory for phonons, cosmological solids and dark energy

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Catania - June 13, 2019

Table of contents

Gravity

In physical cosmology, we assume

- General Relativity: the unique local, Lorentz invariant, low energy effective theory of a single massless spin 2 particle
  - solar system tests
  - table-top experiments
  - gravitational waves

Cosmology

- Homogeneous and isotropic background FLRW spacetime
  - The CMB is isotropic and homogeneous $\Delta T/T \sim 10^{-5}$
  - The Universe is statistically homogeneous on scales larger 200 Mpc

> "The 6-parameter ΛCDM model provides an astonishingly accurate description of the Universe form times prior to 380,000 years after the Big Bang, defining the last-scattering surface observed via the Cosmic Microwave Background (CMB) radiation, to the present day at an age of 13.8 billion years."
>
> — Planck 2018 results

- The Universe is in a phase of accelerated expansion. Why?
- What about the initial conditions? What is the origin of the primordial fluctuations that are the seeds of the LSS of the Universe?

Dark energy

- Dark energy is around 70% of the energy budget of the Universe
- Cosmological constant or something else?

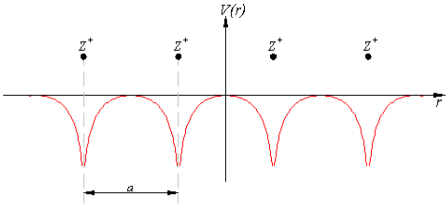
In the following, we consider dynamical dark energy and assume that

- at some scale, the low-energy excitations of the dark energy can be described by the hydrodynamics of a continuous medium
- condensed matter systems emerge as particular Lorentz-violating states subject to fundamentally relativistic laws
  - Lorentz invariance is spontaneously broken
- We need an EFT for (non-dissipative, relativistic) hydrodynamics

(Super-)Solids

- The starting point is the Schrödinger equation for electrons and ions
- The zero temperature configurations form generally an ideal lattice
- The potential $\mathcal{V}$ for the fluctuations is

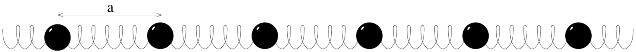
$$\mathcal{V} = \mathcal{V}^* + \frac{1}{2} \sum_{\vec{r},\vec{r}',\alpha,\beta} \frac{\partial^2 \mathcal{V}}{\partial q_{\vec{r},\alpha} \partial q_{\vec{r}',\beta}} u_\alpha(\vec{r}) u_\beta(\vec{r}\,') + \mathcal{O}(u^3) \tag{1}$$



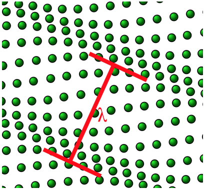
- Supersolid: superfluid behavior in solids, observed recently in laboratory!

Phonons

- Typical excitations at low temperatures have wavelengths $\lambda > \lambda(T) \approx \frac{\hbar v}{K_b T} \gg a$



- Eliminate the unimportant short wavelength modes by coarse graining
- The phonons are Goldstone modes, corresponding to the breaking of translation and rotation symmetries by a crystal structure.



- In supersolid: an additional Goldstone mode associated to the superfluid phase

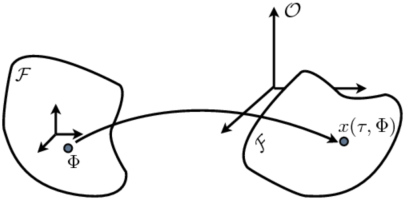
Effective description of homogeneous and isotropic medium

The effective field theory description of a generic homogeneous and isotropic continuous medium is characterized by

- The fields appearing in the effective Lagrangian, $\pi^A(\vec{x}, \tau)$, $A = 0, \dots, 3$, are the four Goldstone bosons which appear due to the spontaneous breaking of the U(1) particle number symmetry and the translational symmetry along three spatial directions
- The fluctuations π^0 around the ground state $\langle \varphi^0 \rangle = \mu t$ represents the Goldstone boson for the non-linearly realized time-translation
- The fluctuations π^i around the ground state $\langle \varphi^j \rangle = x^j$ are the Goldstone boson for the non-linearly realized space-translation, i.e. the phonons

Effective description of homogeneous and isotropic medium

- Eulerian description of the dynamics:
  - A system of coordinates φ^a is frozen in the body of the solid.
  - The time history of the solid is completely characterized by three functions $\varphi^a(x,t)$ which give the coordinates, in the comoving frame, of the material point that is located at the position x at time t.
  - The fluctuations around the equilibrium configuration are given by $\varphi^a(\vec{x},t) = x^a + \pi^a(\vec{x},t)$



Effective description of homogeneous and isotropic medium

- $\varphi^0 = \mu t + \pi^0$ represents the phase of the superfluid condensate
- Alternatively, φ^0 represents a clock in the internal medium space $\mathcal{F}$
- Lorentzian metric in the medium space $\mathcal{F}$

$$C_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \varphi^A} \frac{\partial x^\nu}{\partial \varphi^B} \tag{2}$$

Effective description of homogeneous and isotropic medium

We assume

- homogeneity, i.e. invariance under

$$\varphi^A \to \varphi^A + \lambda^A, \qquad \partial_\mu \lambda^A = 0 \tag{3}$$

- the leading order operators are built from

$$C^{AB} = g^{\mu\nu}\partial_\mu\varphi^A\partial_\nu\varphi^B, \qquad A, B = 0, \ldots, 3 \tag{4}$$

- isotropy, i.e. invariance under

$$\varphi^i \to R^i_j\varphi^j, \qquad i, j = 1, 2, 3 \qquad R \in SO(3), \qquad \partial_\mu R^i_j = 0 \tag{5}$$

Effective description of homogeneous and isotropic medium

- The definition of the φ^i as comoving coordinates implies that the 4-velocity u^μ is the vector field along which all the φ^i stay constant

$$\frac{d\varphi^i}{dt} = u^\mu \partial_\mu \varphi^i = 0 \tag{6}$$

and so

$$u^\mu = -\frac{\epsilon_{abc}\epsilon^{\mu\alpha\beta\gamma}}{6\sqrt{-g}\sqrt{\det B^{ab}}}\partial_\alpha\varphi^a\partial_\beta\varphi^b\partial_\gamma\varphi^c \tag{7}$$

- The 4-velocity associated to the superfluid phase is

$$\mathcal{V}^\mu = -\frac{\partial_\mu\varphi^0}{\sqrt{-g^{\alpha\beta}\partial_\alpha\varphi^0\partial_\beta\varphi^0}} \tag{8}$$

Effective description of homogeneous and isotropic medium

- The leading order action is

$$S = \int d^4x \sqrt{-g}[R + U(X, Y, Z^{ab}, B^{ab}, W^{ab})] \tag{9}$$

where the operators are contracted in rotational invariant way, with

- $X = C^{00}$
- $B^{ab} = C^{ab}$
- $Y = u^{\mu}\partial_{\mu}\varphi^{0}$
- $Z^{ab} = C^{0a}C^{0b}$
- $W^{ab} = B^{ab} - Z^{ab}/X$

$T \neq 0$ solid

- Consider the additional internal symmetry

$$\varphi^0 \to \varphi^0 + f(\varphi^a) \tag{10}$$

- Not all the operators are compatible, and we have

$$S = \int d^4x \sqrt{-g}\, U(B^{ab}, Y) \tag{11}$$

- The energy-momentum tensor can be written as

$$T_{\mu\nu} = \rho\, u_\mu\, u_\nu + p\, h_{\mu\nu} + \pi_{\mu\nu} \tag{12}$$

$T \neq 0$ fluid

- Consider the additional internal symmetry

$$\varphi^a \to \psi^a(\varphi^b) \qquad \det \left| \frac{\partial \psi^a}{\partial \varphi^b} \right| = 1 \tag{13}$$

- The selected leading operators are

$$S = \int d^4x \sqrt{-g}\, U(b, Y) \tag{14}$$

where $b^2 = \det B^{ab}$

- The energy-momentum tensor can be written as

$$T_{\mu\nu} = (\rho + p)\, u_\mu\, u_\nu + p\, g_{\mu\nu} \tag{15}$$

with $\rho = -u + YU_Y$ and $p = U - bU_b$

$T \neq 0$ fluid

- The conserved currents are

$$s^\mu = \; b \; u^\mu = n \; u^\mu \qquad\qquad n^\mu = \; U_Y \; u^\mu = s \; u^\mu \tag{16}$$

and we have

$$p + \rho = YU_Y - bU_b = sY - nU_b = sT + \mu n \tag{17}$$

$$T = Y, \quad \mu = -U_b \tag{18}$$

- The conservation of the two currents implies that the entropy per particle $\sigma = \frac{s}{n}$ is conserved along the flow lines

$$u^\mu \nabla_\mu \sigma = 0 \tag{19}$$

- Non-barotropic fluid

$$\delta p = c_s^2 \delta\rho + \Gamma \delta\sigma \tag{20}$$

K-essence and Irrotational perfect fluids

- Consider shift-symmetric K-essence

$$S = \int d^4x \sqrt{-g}\, U(X) \tag{21}$$

- The stress energy tensor is

$$T_{\mu\nu} = p\, g_{\mu\nu} + (p+\rho)\mathcal{V}_\mu \mathcal{V}_\nu \tag{22}$$

where $p = U$, $\rho = 2XU_X - U$, with $\delta p = c_s^2 \delta\rho$, and $\mathcal{V}_\mu = -\frac{\partial_\mu \varphi^0}{\sqrt{-X}}$.

- Conserved current

$$J^\mu = -2\sqrt{-X}\, U_X \mathcal{V}^\mu = n\mathcal{V}^\mu \qquad \mu = \sqrt{-X} \tag{23}$$

so that

$$P + \rho = 2XU_X = \mu n \tag{24}$$

- $U(X)$ describes an irrotational perfect fluid: $T = 0$ superfluid.

Internal symmetries

Four-dimensional media		
Symmetries	**LO operators**	**Type of medium**
$SO(3)_s \quad \& \quad \varphi^A \to \varphi^A + f^A$	$X,\ Y,\ B^{ab}, W^{ab},\ Z^{ab}$	supersolids
$\varphi^0 \to \varphi^0 + f(\varphi^a)$	$Y,\ B^{ab}$	solids
$\varphi^a \to \varphi^a + f^a(\varphi^0)$	$X,\ W^{ab}$	irrotational
$\varphi^0 \to \varphi^0 + f(\varphi^0)$	W^{ab}	solids
V_sDiff: $\varphi^a \to \Psi^a(\varphi^b)$	$\det B,\ Y,\ X$	superfluids
$\varphi^0 \to \varphi^0 + f(\varphi^a) \quad \& \quad V_s$Diff	$\det B,\ Y$	perfect fluid
$\varphi^A \to \Psi^A(\varphi^B)$	$b\,Y$	C. C.

Cosmology

- In unitary gauge, the function U can be expanded up to second order (Son 2005; Dubovsky, Gregoire, Nicolis, Rattazzi 2005)

$$\sqrt{-g}\,U = \frac{\sqrt{-\bar{g}}}{2}(\rho\delta^{\mu}_{0}\delta^{\nu}_{0} + p\delta^{\mu}_{i}\delta_{ij}\delta^{\nu}_{j})h^{(ug)}_{\mu\nu} + \frac{M_{\rm Pl}^2}{4}\left[M_0^2 h^{(ug)2}_{00} + 2M_1^2 h^{(ug)2}_{0i} - 2M_4^2 h^{(ug)}_{00} h^{(ug)}_{ii} + M_3^2 h^{(ug)2}_{ii} - M_2^2 h^{(ug)2}_{ij}\right] \tag{25}$$

- Imposing Lorentz invariance, instead of $SO(3)$, we get Fierz-Pauli massive gravity

$$\sqrt{-g}\,U = \frac{M_{Pl}^2\, m_g^2}{4}\left(h^{(ug)2}_{\mu\nu} - h^{\mu(ug)2}_{\mu}\right) \tag{26}$$

Cosmology

- In Lorentz breaking massive gravity (Rubakov 2004, Dubovsky 2004)
  - no vDVZ discontinuity
  - agreement with solar system test without any screening mechanism
  - cut-off of the effective field theory $\Lambda_2^{-1} = (m_g M_{Pl})^{-1/2} \sim 0.01$ mm
  - agreement with table-top experiments

Cosmology: Mass parameters

Lagrangian	Medium	Masses
$U(b,\ Y)$	Perfect fluid	$M_{1,\,2} = 0$
$U(b,\ Y,\ X)$	Superfluid	$M_1 \neq 0,\ M_2 = 0$
$U(B^{ab},\ Y)$	Solid	$M_1 = 0,\ M_2 \neq 0$
$U(W^{ab}, X)$	Irrotational solid	$\tilde{M}_1 = M_1 + \frac{p+\rho}{M_{Pl}^2} = 0$ $M_2 \neq 0$
$U(B^{ab}, W^{ab}, Y,\ X)$	Supersolid	$M_{1,2} \neq 0$

Tensor perturbations

The quadratic Lagrangian for the tensor perturbations in the Fourier basis

$$L_t^{(2)} = \frac{M_{\rm Pl}^2}{2}\left[a^2\,\chi_{ij}'^2 - \chi_{ij}^2\,\left(k^2\,a^2 + M_2^2\right)\right]\,. \tag{27}$$

- For fluids and superfluids ($M_2 = 0$) the spin 2 modes are standard.
- Bounds on the graviton mass from GW observations $M_2 \leq 10^{-22}$ eV.
- In order to explain the expansion of the Universe $M_2 \sim 10^{-33}$ eV.
- We have $c_T \approx 1$
  - agreement with GW observations

Scalar perturbations: stability

The perturbations of the Stückelberg fields are

$$\varphi^0 = \phi(t) + \pi^0(\vec{x}, t), \qquad \varphi^a = x^a + \partial^a \pi_L(\vec{x}, t) \tag{28}$$

In the $k \to \infty$, the total energy in the scalar sector is given by

$$E_s = M_0^2 \pi_0'^{\,2} - \frac{k^2}{2} M_1^2 \, \pi_0^2 + \frac{k^2}{2} \tilde{M}_1^2 \pi_L'^{\,2} + k^4 \left(M_2^2 - M_3^2 \right) \pi_L^2 \tag{29}$$

- If $p + \rho > 0$ for $M_0 > 0$, $M_1 < 0$ but $\tilde{M}_1 = M_1 + \frac{p+\rho}{M_{Pl}^2} > 0$
  - we can have 6 healthy degrees of freedom
- If $p + \rho = 0$ the sixth mode is unstable

Scalar perturbations: Einstein equations

- The Einstein equations are

$$
\begin{aligned}
a^2\,\delta\rho &= 4\,M_{\rm Pl}^2\,\left[k^2\,\Phi + 3\,\mathcal{H}\,(\Phi' + \mathcal{H}\,\Psi)\right] \\
\delta p &= -\frac{4 M_{\rm Pl}^2}{3\,a^2}\left\{k^2\Phi - \Psi\left[9\,w\,\mathcal{H}^2 + k^2\right] + 3\mathcal{H}(\Psi' + 2\,\Phi') + 3\,\Phi''\right\} \\
\Phi - \Psi &= \frac{M_2}{a^2}\,\pi_L \\
\Phi' + \mathcal{H}\,\Psi &= \frac{3}{2}(w+1)\mathcal{H}^2 v + \frac{\phi'}{4\,a^2\,M_{Pl}^2 k^2}\delta\sigma'
\end{aligned}
\tag{30}
$$

Dark energy

"Combining Planck data with Pantheon supernovae and BAO data, the equation of state of dark energy is tightly constrained to $w_0 = -1.03 \pm 0.03$, consistent with a cosmological constant."

— Planck 2018 results

Λ-medium with $w = -1$

- Assume $w(z) = p_\Lambda/\rho_\Lambda = -1$
- Suppose that dark energy can be described by a fluid with $w = -1$

$$T^{\Lambda}_{\mu\nu} = p_\Lambda \, g_{\mu\nu} \tag{31}$$

- The conservation of the EMT implies

$$\nabla^\mu T^{\Lambda}_{\mu\nu} = 0 = g_{\mu\nu}\nabla^\mu p_\Lambda \Longrightarrow p_\Lambda = const. = -\rho_\Lambda \tag{32}$$

- Suppose that dark energy can be described by a solid with $w = -1$

$$T^{\Lambda}_{\mu\nu} = p_\Lambda \, g_{\mu\nu} + \Pi_{\mu\nu} \tag{33}$$

- The conservation of the EMT implies

$$\nabla_\nu p_\Lambda = -\nabla^\mu \Pi_{\mu\nu} = -\nabla_\nu \rho_\Lambda \neq 0 \tag{34}$$

Special super solids

- Consider $\mathcal{L} = U(X, w_n)$, $w_n = Tr[W^n]$ protected by

$$\varphi^a \to \varphi^a + f^a(\varphi^0) \tag{35}$$

corresponding to a special supersolid solid $\Pi_{\mu\nu} \neq 0$

- the longitudinal scalar mode π_L and the vector modes do not propagate
- gravitational waves feature a mass term
- π_0 propagates
- pressure and energy density in general are given by

$$p = U - 2/3 \sum_m m w_m U_{w_m} \qquad \rho = -U + 2XU_X \tag{36}$$

Λ-super solids

- Imposing $p = -\rho$, we have the solution

$$U_\Lambda = U\left(X w_1^3, \frac{w_2}{w_1^2}, \frac{w_3}{w_1^3}\right) \tag{37}$$

- Protected by the scaling symmetry

$$\varphi^0 \to \lambda^{-3}\,\varphi^0\,, \qquad \varphi^j \to \lambda\,\varphi^j \tag{38}$$

- π_0 propagates with $c_s^2 = 0$
- Since $w = -1$ we have $\delta p_\Lambda(x,t) = -\delta\rho_\Lambda = -\delta\sigma_\Lambda \neq 0$ if $U_X \neq 0$
- The entropy per particle and the energy density are constant in time

$$\frac{d\delta\sigma_\Lambda}{dt} = \frac{d\delta\rho_\Lambda}{dt} = 0 \tag{39}$$

Λ-super solids and dark matter

- For $\bar{\rho}_m \gg \bar{\rho}_\Lambda$, the leading terms as dark matter dominated universe

$$\Phi = \bar{\Phi}\,; \qquad \delta_m = \frac{2\,a\,k^2\,\bar{\Phi}}{3\,H_0^2} + \bar{\delta}\,; \tag{40}$$

- For $\bar{\rho}_\Lambda \gg \bar{\rho}_m$, neglecting sub-leading decreasing modes

$$\Phi = \frac{a^2}{4}\,\frac{\delta\rho_\Lambda(k)}{k^2\,M_{\rm Pl}^2} \tag{41}$$

$$\delta_m = \bar{\delta} - \left[\frac{3\,a^2}{4} + \frac{k^2\,\log(a)}{4\,H_0^2}\right]\frac{\delta\rho_\Lambda(k)}{k^2\,M_{\rm Pl}^2}\,. \tag{42}$$

- Gravitational slip

$$\Phi - \Psi = \frac{3}{4}\,a^2\,\frac{\delta\rho_\Lambda(k)}{k^2\,M_{\rm Pl}^2}\,. \tag{43}$$

- Strong constraints on $\delta\rho_\Lambda(k)$ from matter power spectrum

Single field inflation

- Inflation is probably the most successful way to explain the horizon and the curvature problems, suggesting a quantum origin for the primordial perturbations.
- In the case of standard single field, inflationary predictions are basically independent on the details of reheating.
- According to the the Weinberg theorem, there is always an adiabatic mode, $\mathcal{R}$, or equivalent ζ, constant on super-horizon scales. The other mode is a decreasing mode.
- Violation of the Weinberg theorem: the would-be decreasing mode of $\mathcal{R}$ becomes growing; $\mathcal{R}$ and ζ grow super- horizon and are different. This is the case, for instance, for fluid and solid inflation.

Single field inflation

- Single field inflation

$$\text{diffs} = \begin{cases} t \to t + \xi^0(t, \vec{x}) & \text{broken} \\ x^i \to x^i + \xi^i(t, \vec{x}) & \text{unbroken} \end{cases} . \tag{44}$$

- The symmetry breaking pattern is rather constraining
  - conservation of the physical curvature perturbation on super-horizon scales (Weinberg theorem)
  - "consistency relations" between different n-point functions that stem from the associated Ward identities
  - the unbroken spatial diffeomorphisms forbid a small "mass" term for the graviton, which would affect the tilt of the tensor spectrum

Fluid inflation (Chen, Firouzjahi, Namjoo, Sasaki 2013)

- Violation of the Weinberg theorem: Fluid inflation $\mathcal{L} = \sqrt{-g}\, U(b)$
- Define the slow-roll parameters

$$\epsilon = -\frac{\dot{H}}{H^2} \qquad \eta = \frac{\dot{\epsilon}}{H\epsilon} \tag{45}$$

with

$$0 = 1 + c_s^2 - 2/3\epsilon + 1/3\eta \tag{46}$$

- Power spectrum for fluid inflation (MC, Comelli, Pilo, Rollo, to appear)

$$\mathcal{P}_{\mathcal{R}} = A_{\mathcal{R}} k^{6+\eta} \qquad \mathcal{P}_{\zeta} = A_{\zeta} k^{10+\eta} \tag{47}$$

The requirement of a scale-free spectrum for $\mathcal{R}$ and ζ, e.g. $n_s = 1$, gives very different values of η.

Fluid inflation

- $n_s = 1$ for $\mathcal{R}$ gives $\eta = -6$ with $c_s^2 = 1$, Ultra-slow roll
- $n_s = 1$ for ζ implies $\eta = -10$ and $c_s^2 = 7/3 > 1$, Ultra-slow roll
- No standard slow roll regime is allowed
- $\mathcal{R}$ and ζ grow on super-horizon scales and are different
- The reheating phase needs to be analyzed in detail
- Generally, ζ is continuous on the reheating hyper-surface while $\mathcal{R}$ sharply jumps in order to reach the ζ values

Solid inflation (Endlich, Nicolis, Wang 2012)

- For solid inflation $\mathcal{L} = \sqrt{-g}\,U(B^{ab})$
- "mass" term for the graviton, tilt of the tensor spectrum

$$\frac{M_{Pl}^2 M_2^2}{p+\rho} = \frac{3}{4}(1 + c_L^2 - 2/3\epsilon + 1/3\eta) \tag{48}$$

- Slow roll is possible and $\epsilon \ll 1$ and $\eta \ll 1$
- Power spectrum for solid inflation

$$\mathcal{P}_{\mathcal{R}} = A_{\mathcal{R}} k^{3-2\nu} \qquad \mathcal{P}_{\zeta} = A_{\zeta} k^{3-2\nu} \tag{49}$$

with

$$\frac{A_\zeta}{A_{\mathcal{R}}} = \frac{1}{c_L^4}, \qquad \nu = \frac{3}{2} + \epsilon + \frac{\eta}{2} - \frac{4}{3}\frac{M_{Pl}^2 M_2^2}{p+\rho}\epsilon \tag{50}$$

- The spectral index n_s is the same but the amplitude is different.
- Superfluid and supersolid? Work in progress (also Bartolo, Cannone, Ricciardone, Tasinato 2015)

Conclusions

- The models we have considered cannot really explain why the cosmological constant is small compared to M_{Pl}.
- Dark sector might be described as a condensed matter system, whose low-energy excitations are described by EFT for hydrodynamics
- Connections with modified gravity (Lorentz breaking massive gravity)
- Stability of the effective theory (ghosts, gradient)
- Symmetries and thermodynamical properties of the medium
- Some extreme model describing dark energy with $w = -1$
- Interesting inflationary features very different from single-field (EFT of Inflation)

...and perspectives

- Theoretical aspects:
  - Higher derivative corrections?
  - UV completion?
  - Relations between symmetry breaking patters and primordial non-Gaussianity?
- Phenomenological aspects:
  - Anisotropic stress tensor $\pi_{\mu\nu}$ for solids and supesolids is responsible for gravitational slip $\Phi \neq \Psi$ that can be measured using weak lensing and integrated Sachs-Wolfe effect
  - Entropy perturbations $\delta\sigma \neq 0$ changes structure formation
  - Primordial tensor modes detectable by LISA?

Thank you for your attention!

$T = 0$ superfluid: From UV to IR

- Consider a relativistic superfluid, described by the Lagrangian

$$\mathcal{L}_0 = -\partial_\mu \Phi \partial^\mu \Phi^* + m^2 |\Phi|^2 - \lambda |\Phi|^4, \tag{51}$$

- Finite chemical potential:
  - it is possible to show that the chemical potential looks like the temporal component of a non-dynamical gauge field

$$\mathcal{L}_\mu = |(\partial_0 - i\mu)\Phi|^2 - \partial_i \Phi \partial^i \Phi^* + m^2 |\Phi|^2 - \lambda |\Phi|^4, \tag{52}$$

- Writing $\Phi = \frac{\sigma e^{i\pi}}{\sqrt{2}}$, we have

$$\mathcal{L}_\mu = -\frac{1}{2}\nabla_\mu \sigma \nabla^\mu \sigma - \frac{1}{2}\sigma^2(\delta^0_\mu \mu + \nabla_\mu \pi) g^{\mu\nu} (\delta^0_\nu \mu + \nabla_\nu \pi) + \frac{m^2}{2}\sigma^2 - \frac{\lambda}{4}\sigma^4 \tag{53}$$

$T = 0$ superfluid: From UV to IR

- Studying the fluctuations around the ground state, describing the theory at finite charge density, and defining

$$\sigma(x) = v + s(x)\ , \qquad\qquad \theta(x) = \mu t + \pi(x) \tag{54}$$

the field $s(x)$ has mass $M = 2(m^2 + \mu^2)$.

- Integrating out, we get at leading order

$$\mathcal{L}_{eff} = \frac{1}{2}(\dot{\pi}_c^2 - c_\pi^2 \nabla \pi_c^2) \tag{55}$$

where $c_\pi^2 = \frac{1}{(1+4\mu^2/M^2)}$ and $\pi_c = \mu\pi/c_\pi$